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Fixed point of generalized F-Suzuki contraction mapping on complete extended b-metric spaces with application

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Abstract. The present article introduces the concept of generalized F-Suzuki type contraction in the setting of extended b-metric spaces, inspired by the existing concept in b-metric spaces. The newly introduced notion is further utilized to prove fixed point theorems for F-Suzuki type contraction which unified, extended and generalized many existing results in the literature. Finally, the established results are utilized to solve Fredholm integral equation.

1. Introduction

In 1906, French scholar Maurice Frechet [11] introduced the notion of metric space. Multiple approaches have been made to expand metric space since, including fuzzy metric spaces, probabilistic metric spaces and so on, via changing or eliminating certain axioms, shifting the metric function or removing some axioms entirely. These techniques are more used in fixed point research these days. Several beneficial findings have been produced in this being [16, 18–20].

Fixed point theory has swell as one of the most successful approaches in contemporary mathematical analysis. It is important in its own right way and has progressed considerably over the past century. Stefan Banach [3] was the first person who established fixed point theorem in the settings of metric spaces. Banach was able to extract the basic idea of fixed points from these outcomes and so began studying the subject of metric fixed-point theory. Following the fact that this branch grew independently and contributed to numerous advancements in a variety of fields of scientific research. The Banach contraction principle has been extended and generalized by several writers due to its relevance and simplicity (see to [4, 6, 7, 10, 13, 14]).

In this process, Wardowski [23] generalized Banach contraction in different manner by introducing F-contraction and proved a fixed point theorem. After that Wardowski and Dung [24] establish the notion of weak F-contraction by weaking the contraction condition and proved fixed point theorem. Hussein et

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al. extended the result of Wardowski by applying some weaker conditions on the self-map of a complete metric space and proved a fixed point result which generalizes the result of Wardowski.

Bakhtin [4] and Czerwik [8] initially put forward the ideas of b-metric spaces as a new type of metric space that weaken the triangle inequality and Czerwik [9] proved Banach contraction principle in the settings of b-metric spaces. After that, many authors have proved fixed point theorems in the context of a b-metric space [1, 17]. Piri et al. [1] introduced generalized F-Suzuki type contraction in b-metric space and proved fixed point theorem for the newly established contraction.

In 2017, Kamran et al. [12] introduced the concept of extended b-metric space as a generalization of b-metric space and proved some fixed point theorem. The main aim of this article is to establish generalized F-Suzuki type contraction in the settings of extended b-metric space.

2. Preliminaries

In this section, some basic definitions and examples are collected from background study for further utilize them in the main results.

Definition 2.1. *[12] Let E be a non-empty set and $\theta : E\times E \to [1,\infty)$ be a function. A mapping $m_\theta : E\times E \to [0,\infty)$ is said to be an extended b-metric if for all $\kappa, \lambda, \mu \in E$, the following conditions are satisfied:*

($m_\theta 1$) $m_\theta(\kappa,\lambda) = 0$ *iff* $\kappa = \lambda$,
($m_\theta 2$) $m_\theta(\kappa,\lambda) = m_\theta(\lambda,\kappa)$,
($m_\theta 3$) $m_\theta(\kappa,\lambda) \le \theta(\kappa,\lambda)[m_\theta(\kappa,\mu) + m_\theta(\mu,\lambda)]$.

The pair (E, m_θ) is called an extended b-metric space.

An extended b-metric is a b-metric on setting $\theta(\kappa,\lambda) = s$ and $\theta(\kappa,\lambda) = 1$ it become usual metric space. Hence the notion of extended b-metric is a real generalization of b-metric space.

Example 2.2. *Assume $E = \{1,2,3\}$. Define $\theta : E\times E \to [1,\infty)$ and $m_\theta : E\times E \to \mathbb{R}^+$ as:*

$$\theta(\kappa,\lambda) = 1+\kappa+\lambda, m_\theta(1,1) = m_\theta(2,2) = m_\theta(3,3) = 0,$$
$$m_\theta(1,2) = m_\theta(2,1) = 10, m_\theta(1,3) = m_\theta(3,1) = 100, m_\theta(2,3) = m_\theta(3,2) = 50.$$

The conditions ($m_\theta 1$) and ($m_\theta 2$) trivially hold. For ($m_\theta 3$) we have:

$$m_\theta(1,3) = 100, \theta(1,3)\,[m_\theta(1,2) + m_\theta(2,3)] = 5(10+50) = 300,$$
$$m_\theta(1,2) = 10, \theta(1,2)\,[m_\theta(1,3) + m_\theta(3,2)] = 4(100+50) = 600,$$
$$m_\theta(2,3) = 50, \theta(2,3)\,[m_\theta(2,1) + m_\theta(1,3)] = 6(10+100) = 660.$$

Hence, for all $\kappa, \lambda, \mu \in E$

$$m_\theta(\kappa,\mu) \le \theta(\kappa,\mu)\left[m_\theta(\kappa,\lambda) + m_\theta(\lambda,\mu)\right].$$

Hence, (E, m_θ) is an extended b-metric space but not a metric space as $m_\theta(1,3) \ge m_\theta(1,2) + m_\theta(2,3)$, triangle inequality does not hold.

Definition 2.3. *[12] Let (E, m_θ) be an extended b-metric space. A sequence $\{\kappa_n\}_{n=1}^\infty$ in E is called convergent sequence iff $\exists\ \kappa \in E$ such that $m_\theta(\kappa_n,\kappa) \to 0$ as $n\to\infty$ and in this case we write $\lim_{n\to\infty} \kappa_n = \kappa$.*

Definition 2.4. *[12] Let (E, m_θ) be an extended b-metric space. A sequence $\{\kappa_n\}_{n=1}^\infty$ in E is called Cauchy sequence iff $m_\theta(\kappa_n,\kappa_m) \to 0$ as $m, n\to\infty$.*

Definition 2.5. *[12] If every Cauchy sequence in E is convergent then (E, m_θ) is called complete extended b-metric space.*

Definition 2.6. *We define the continuity for extended b-metric space as follows: Let (E, m_{θ_E}) and (Y, m_{θ_Y}) are extended b-metric spaces. A mapping $f : E \to Y$ is called continuous at a point $\kappa \in E$, if $\forall$ sequence $\{\kappa_n\}_{n=1}^{\infty}$ in E such that $\kappa_n \to \kappa$, then $f(\kappa_n) \to f(\kappa)$. And mapping $f : E \to Y$ is called continuous on E, if it is continuous at each point on E.*

Definition 2.7. *[1] Let $\mathfrak{F}$ denote the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ is mapping satisfying the following conditions:*

($\mathcal{F}_1$) *For all $\kappa, \lambda \in \mathbb{R}^+$ such that $\kappa < \lambda \Rightarrow F(\kappa) < F(\lambda)$ i.e. F is strictly increasing,*

($\mathcal{F}_2$) *For every sequence $\{\kappa_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty}(\kappa_n) = 0 \Leftrightarrow \lim_{n\to\infty} F(\kappa_n) = -\infty$; where $\kappa_n > 0$ for all $n \in \mathbb{N}$.*

Example 2.8. *[23] Define $F : \mathbb{R}^+ \to \mathbb{R}$ as $F(\kappa) = \ln(\kappa) + \kappa$. Then, F hold the conditions ($\mathcal{F}_1$) and ($\mathcal{F}_2$). Hence, $F \in \mathfrak{F}$.*

3. Main results

Throughout this section motivated by the theme of generalized F-Suzuki type contraction in b-metric space, the idea of generalized F-Suzuki type contraction in extended b-metric space as follows.

Definition 3.1. *Let (E, m_θ) be an extended b-metric space and $H : E \to E$ be a mapping. If $\exists\, F \in \mathfrak{F}$ and $\tau > 0$ such that $\forall\ \kappa, \lambda \in E$ with $\kappa \neq \lambda$*

$$\frac{1}{2\theta(\kappa, H\kappa)} m_\theta(\kappa, H\kappa) < m_\theta(\kappa, \lambda)$$
$$\Rightarrow \tau + F(m_\theta(H\kappa, H\lambda)) \leq \alpha F(m_\theta(\kappa, \lambda)) + \beta F(m_\theta(\kappa, H\kappa)) + \gamma F(m_\theta(\lambda, H\lambda)). \tag{1}$$

Where, $0 \leq \alpha, \beta \leq 1$, $0 \leq \gamma < 1$ with $1 = \alpha + \beta + \gamma$. Then, H is called generalized F-Suzuki type contraction mapping on (E, m_θ) .

Theorem 3.2. *Let (E, m_θ) be a complete extended b-metric space, $H : E \to E$ be a mapping and $\theta : E \times E \to [1, \infty)$ be a bounded function. If H is a generalized F-Suzuki type contraction on E, then H has a fixed point $\kappa^* \in E$.*

Proof. Let fix $\kappa_0 \in E$. Consider an iterative sequence $\{\kappa_n\}$ as follows:

$$\kappa_1 = H\kappa_0, \kappa_2 = H\kappa_1 = H^2\kappa_0, \ldots, \kappa_{n+1} = H\kappa_n = H^{n+1}\kappa_0,\ \forall\ n \in \mathbb{N} \cup \{0\}.$$

If $\kappa_n = \kappa_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, therefore it is obvious that κ_n is a fixed point of H.

Now, consider that $\kappa_n \neq \kappa_{n+1}$ $\forall\ n \in \mathbb{N} \cup \{0\}$, this yields $m_\theta(\kappa_n, H\kappa_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$\frac{1}{2\theta(\kappa_n, H\kappa_n)} m_\theta(\kappa_n, H\kappa_n) < m_\theta(\kappa_n, H\kappa_n),\ \forall\ n \in \mathbb{N}. \tag{2}$$

Thus,

$$\tau + F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq \alpha F(m_\theta(\kappa_n, H\kappa_n)) + \beta F(m_\theta(\kappa_n, H\kappa_n)) + \gamma F(m_\theta(H\kappa_n, H^2\kappa_n))$$
$$\Rightarrow \tau + (1 - \gamma) F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq (\alpha + \beta) F(m_\theta(\kappa_n, H\kappa_n)).$$

Since, $\alpha + \beta + \gamma = 1$.

$$\Rightarrow F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq F(m_\theta(\kappa_n, H\kappa_n)) - \frac{\tau}{\alpha + \beta} < F(m_\theta(\kappa_n, H\kappa_n)). \tag{3}$$

From ($\mathcal{F}_1$), conclude that

$$m_\theta(\kappa_{n+1}, H\kappa_{n+1}) = m_\theta(H\kappa_n, H^2\kappa_n) < m_\theta(\kappa_n, H\kappa_n), \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Therefore, $\{m_\theta(\kappa_n, H\kappa_n)\}_{n=1}^{\infty}$ is a bounded below and monotonically decreasing sequence. Thus, sequence $\{m_\theta(\kappa_n, H\kappa_n)\}_{n=1}^{\infty}$ is convergent and

$$\begin{aligned} F(m_\theta(\kappa_n, H\kappa_n)) &= F(m_\theta(H\kappa_{n-1}, H^2\kappa_n)),\\ &\leq F(m_\theta(\kappa_{n-1}, H\kappa_{n-1})) - \frac{\tau}{\alpha+\beta},\\ &\leq F(m_\theta(\kappa_{n-2}, H\kappa_{n-2})) - 2\frac{\tau}{\alpha+\beta},\\ &\vdots\\ &\leq F(m_\theta(\kappa_0, H\kappa_0)) - n\frac{\tau}{\alpha+\beta}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} F(m_\theta(\kappa_n, H\kappa_n)) = -\infty \Leftrightarrow \lim_{n\to\infty} m_\theta(\kappa_n, H\kappa_n) = 0. \tag{5}$$

Our aims to demonstrate this in the subsequent stage,

$$\lim_{n,k\to\infty} m_\theta(\kappa_n, \kappa_k) = 0.$$

On the other hand, consider that there exists $\epsilon > 0$ and sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ satisfying,

$$r_n > s_n > n,\ \ m_\theta(\kappa_{r_n}, \kappa_{s_n}) \geq \epsilon,\ m_\theta(\kappa_{r_{n-1}}, \kappa_{s_n}) < \epsilon,\ \text{for all } n \in \mathbb{N}. \tag{6}$$

Then,

$$\begin{aligned} m_\theta(\kappa_{r_n}, \kappa_{s_n}) &\leq \theta(\kappa_{r_n}, \kappa_{s_n})\Big[m_\theta(\kappa_{r_n}, \kappa_{r_{n-1}}) + m_\theta(\kappa_{r_{n-1}}, \kappa_{s_n})\Big]\\ &\leq \theta(\kappa_{r_n}, \kappa_{s_n}) m_\theta(\kappa_{r_n}, \kappa_{r_{n-1}}) + \theta(\kappa_{r_n}, \kappa_{s_n})\epsilon\\ &= \theta(\kappa_{r_n}, \kappa_{s_n}) m_\theta(\kappa_{r_{n-1}}, H\kappa_{r_{n-1}}) + \theta(\kappa_{r_n}, \kappa_{s_n})\epsilon,\ \text{for all } n \in \mathbb{N}. \end{aligned} \tag{7}$$

From equation (5), there exists $N_2 \in \mathbb{N}$ such that

$$m_\theta(\kappa_{r_n}, H\kappa_{r_n}) < \epsilon,\ \text{for all } n > N_2. \tag{8}$$

Put the value in (7) find that

$$m_\theta(\kappa_{r_n}, \kappa_{s_n}) < 2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon,\ \text{for all } n > N_2.$$

So from $(\mathcal{F}_2)$, we obtain

$$F(m_\theta(\kappa_{r_n}, \kappa_{s_n})) < F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon),\ \text{for all } n > N_2. \tag{9}$$

On the other hand,

$$\frac{1}{2\theta(\kappa_{r_n}, H\kappa_{r_n})}(m_\theta(\kappa_{r_n}, H\kappa_{r_n})) < \frac{\epsilon}{2\theta(\kappa_{r_n}, H\kappa_{r_n})} < \epsilon \leq m_\theta(\kappa_{r_n}, \kappa_{s_n}),\ \text{for all } n > N_2. \tag{10}$$

Although, H is generalized F-Suzuki type contraction, for all $n > N_2$, we investigate

$$\begin{aligned} \tau + F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) \leq &\alpha F(m_\theta(\kappa_{r_n}, \kappa_{s_n})) + \beta F(m_\theta(\kappa_{r_n}, H\kappa_{r_n}))\\ &+ \gamma F(m_\theta(\kappa_{s_n}, H\kappa_{s_n})), \end{aligned} \tag{11}$$

By the (9),

$$\tau + F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) \leq \alpha F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon) + \beta F(m_\theta(\kappa_{r_n}, H\kappa_{r_n})) + \gamma F(m_\theta(\kappa_{s_n}, H\kappa_{s_n})).$$

There exist $M \in \mathbb{R}$ such that $F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon) \leq M$, because θ is bounded function. So from (5) and $(\mathcal{F}_2)$, obtain that,

$$\lim_{n\to\infty} F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) = -\infty.$$

From $(\mathcal{F}_2)$, we get

$$\lim_{n\to\infty}(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) = 0 \Leftrightarrow \lim_{n\to\infty}(m_\theta(\kappa_{r_{n+1}}, \kappa_{s_{n+1}})) = 0.$$

This is a contradiction with the relation in (6). Hence, $\lim_{n,k\to\infty} m_\theta(\kappa_n, \kappa_k) = 0$, that is $\{\kappa_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Because of completion of (E, m_θ), there exists $\kappa^* \in E$ as well as

$$\lim_{n\to\infty} m_\theta(\kappa_n, \kappa^*) = 0. \tag{12}$$

Further show that, for every $n \in \mathbb{N}$

$$\begin{aligned} &\text{Either } \frac{1}{2\theta(\kappa_n, H\kappa_n)} m_\theta(\kappa_n, H\kappa_n) < m_\theta(\kappa_n, \kappa^*), \\ &\text{or } \frac{1}{2\theta(\kappa_n, H\kappa_n)} m_\theta(H\kappa_n, H^2\kappa_n) < m_\theta(H\kappa_n, \kappa^*), \ \forall n \in \mathbb{N}. \end{aligned} \tag{13}$$

Assume, on the other hand, that there is $n_0 \in \mathbb{N}$ satisfying,

$$\begin{aligned} &\frac{1}{2\theta(\kappa_{n_0}, H\kappa_{n_0})} m_\theta(\kappa_{n_0}, H\kappa_{n_0}) \geq m_\theta(\kappa_{n_0}, \kappa^*) \\ &\text{and } \frac{1}{2\theta(H\kappa_{n_0}, H^2\kappa_{n_0})} m_\theta(H\kappa_{n_0}, H^2\kappa_{n_0}) \geq m_\theta(H\kappa_{n_0}, \kappa^*) \end{aligned} \tag{14}$$

From (4)

$$m_\theta(H\kappa_{n_0}, H^2\kappa_{n_0}) < m_\theta(\kappa_{n_0}, H\kappa_{n_0}). \tag{15}$$

It follows from (14) and (15) that

$$\begin{aligned} m_\theta(\kappa_{n_0}, H\kappa_{n_0}) &\leq \theta(\kappa_{n_0}, H\kappa_{n_0}) \left[m_\theta(\kappa_{n_0}, \kappa^*) + m_\theta(\kappa^*, H\kappa_{n_0})\right], \\ &\leq \theta(\kappa_{n_0}, H\kappa_{n_0}) m_\theta(\kappa_{n_0}, \kappa^*) + \theta(\kappa_{n_0}, H\kappa_{n_0}) m_\theta(\kappa^*, H\kappa_{n_0}), \\ &\leq \frac{1}{2} m_\theta(\kappa_{n_0}, H\kappa_{n_0}) + \frac{1}{2} m_\theta(H\kappa_{n_0}, H^2\kappa_{n_0}), \\ &< \frac{1}{2} m_\theta(\kappa_{n_0}, H\kappa_{n_0}) + \frac{1}{2} m_\theta(\kappa_{n_0}, H\kappa_{n_0}), \\ &= m_\theta(\kappa_{n_0}, H\kappa_{n_0}). \end{aligned} \tag{16}$$

Which is a contradiction. Hence, inequality (13) holds. Then,
either

$$\begin{aligned} \tau + F(m_\theta(H\kappa_n, H\kappa^*)) \leq &\alpha F(m_\theta(\kappa_n, \kappa^*)) + \beta F(m_\theta(\kappa_n, H\kappa_n)) \\ &+ \gamma F(m_\theta(\kappa^*, H\kappa^*)), \end{aligned} \tag{17}$$

or

$$\begin{aligned} \tau + F(m_\theta(H^2\kappa_n, H\kappa^*)) \leq &\alpha F(m_\theta(H\kappa_n, \kappa^*)) + \beta F(m_\theta(H\kappa_n, H^2\kappa_n)) \\ &+ \gamma F(m_\theta(\kappa^*, H\kappa^*)), \end{aligned} \tag{18}$$

Letting $n \to \infty$ and applying equation (5) and (12), in (17), deduce

$$\lim_{n\to\infty} F(m_\theta(H\kappa_n, H\kappa^*)) = -\infty,$$
$$\Rightarrow \lim_{n\to\infty}(m_\theta(H\kappa_n, H\kappa^*)) = 0. \tag{19}$$

Considering the triangular inequality, observe that

$$\begin{aligned} m_\theta(\kappa^*, H\kappa^*) &\leq \theta(\kappa^*, H\kappa^*)\Big[m_\theta(\kappa^*, H\kappa_n) + m_\theta(H\kappa_n, H\kappa^*)\Big], \\ &\leq \theta(\kappa^*, H\kappa^*)\Big[m_\theta(\kappa^*, \kappa_{n+1}) + m_\theta(H\kappa_n, H\kappa^*)\Big]. \end{aligned}$$

Let $n \to \infty$ in the inequality above, as well as the constraints in (12) and (19), deduce that, $m_\theta(\kappa^*, H\kappa^*) = 0$. Thus κ^* is a fixed point of H.
By equation (5) and (12)

$$\lim_{n\to\infty} F(m_\theta(\kappa_{n+1}, \kappa^*)) = -\infty, \quad \lim_{n\to\infty} F(m_\theta(\kappa_{n+1}, H\kappa_{n+1})) = -\infty. \tag{20}$$

Letting $n \to \infty$ and applying equation (5) and (12), in (18), deduce

$$\lim_{n\to\infty} F(m_\theta(H^2\kappa_n, H\kappa^*)) = -\infty,$$
$$\Rightarrow \lim_{n\to\infty}(m_\theta(H^2\kappa_n, H\kappa^*)) = 0. \tag{21}$$

Considering the triangular inequality, we observe that

$$\begin{aligned} m_\theta(\kappa^*, H\kappa^*) &\leq \theta(\kappa^*, H\kappa^*)\Big[m_\theta(\kappa^*, H^2\kappa_n) + m_\theta(H^2\kappa_n, H\kappa^*)\Big], \\ &= \theta(\kappa^*, H\kappa^*)\Big[m_\theta(\kappa^*, \kappa_{n+2}) + m_\theta(H^2\kappa_n, H\kappa^*)\Big]. \end{aligned}$$

Let $n \to \infty$ in the inequality above, as well as the constraints in (12) and (21), deduce that $m_\theta(\kappa^*, H\kappa^*) = 0$. Thus κ^* is a fixed point of H. □

Example 3.3. *Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:*

$$S_1 = 1, \quad S_2 = 1 + 2^2, \quad \cdots$$
$$S_n = 1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $E = \{S_n : n \in \mathbb{N}\}$ and $m_\theta(x, y) = |x-y|$. Then (E, m_θ) is complete extended b-metric space, (where $\theta(x, y) = 1$). Define the mapping $H : X \to X$ by $H(S_1) = S_1$ and $H(S_n) = S_{n-1}$ for every $n > 1$. Since,

$$\lim_{n\to\infty} \frac{m_\theta(H(S_n), H(S_1))}{m_\theta(S_n, S_1)} = 1.$$

H is not a Banach contraction and a Suzuki contraction. On the other hand taking $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$, we obtain the result that H is an generalized F-Suzuki type contraction with $\tau = 2$. To see this, let us consider the following calculation. First observe that,

$$\frac{1}{2} m_\theta(S_n, HS_n) < m_\theta(S_n, S_m) \quad \Leftrightarrow \quad [(1 = n < m) \vee (1 \leq m < n) \vee (1 < n < m)].$$

For $1 = n < m$, we have

$$|H(S_m) - H(S_1)| = |S_{m-1} - S_1| = 2^2 + 3^2 + \cdots + (m-1)^2.$$
$$|S_m - S_1| = 2^2 + 3^2 + \cdots + m^2.$$

Since $m > 1$ *and* $\frac{-1}{2^2+3^2+\cdots+(m-1)^2} < \frac{-1}{2^2+3^2+\cdots+m^2}$, *we have*

$$\begin{aligned}
&2 - \frac{1}{|H(S_m) - H(S_1)|} + |H(S_m) - H(S_1)| \\
&\quad = 2 - \frac{1}{2^2 + 3^2 + \cdots + (m-1)^2} + [2^2 + 3^2 + \cdots + (m-1)^2], \\
&\quad < 2 - \frac{1}{2^2 + 3^2 + \cdots + m^2} + [2^2 + 3^2 + \cdots + (m-1)^2)], \\
&\quad \leq -\frac{1}{2^2 + 3^2 + \cdots + m^2} + [2^2 + 3^2 + \cdots + (m-1)^2] + m^2, \\
&\quad = -\frac{1}{2^2 + 3^2 + \cdots + m^2} + [2^2 + 3^2 + \cdots + m^2], \\
&\quad = -\frac{1}{|S_m - S_1|} + |S_m - S_1|.
\end{aligned}$$

For $1 = m < n$, *similar to* $1 = n < m$, *we have*

$$2 - \frac{1}{|H(S_m) - H(S_1)|} + |H(S_m) - H(S_1)| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|.$$

For $1 < n < m$, *we have*

$$\begin{aligned}
|H(S_m) - H(S_n)| &= n^2 + (n+1)^2 + \cdots + (m-1)^2. \\
|S_m - S_n| &= (n+1)^2 + (n+2)^2 + \cdots + m^2.
\end{aligned}$$

We know that $\frac{-1}{n^2+(n+1)^2+\cdots+(m-1)^2} < \frac{-1}{(n+1)^2+(n+2)^2+\cdots+m^2}$. *Therefore,*

$$\begin{aligned}
&2 - \frac{1}{|H(S_m) - H(S_n)|} + |H(S_m) - H(S_n)|, \\
&\quad = 2 - \frac{1}{n^2 + (n+1)^2 + \cdots + (m-1)^2} + [n^2 + (n+1)^2 + \cdots + (m-1)^2], \\
&\quad < 2 - \frac{1}{(n+1)^2 + (n+2)^2 + \cdots + m^2} + [n^2 + (n+1)^2 + \cdots + (m-1)^2], \\
&\quad = -\frac{1}{(n+1)^2 + (n+2)^2 + \cdots + m^2} + [(n+1)^2 + (n+2)^2 + \cdots + (m-1)^2] + m^2, \\
&\quad \leq -\frac{1}{(n+1)^2 + (n+2)^2 + \cdots + m^2} + [(n+1)^2 + (n+2)^2 + \cdots + m^2], \\
&\quad < -\frac{1}{|S_m - S_n|} + |S_m - S_n|.
\end{aligned}$$

Therefore, $\tau + F(m_\theta(H(S_m), H(S_n))) \leq m_\theta(S_m, S_n)$, *for all* $m, n \in \mathbb{N}$. *Hence,* H *is a generalized F-Suzuki type contraction and* $H(S_1) = S_1$.

Theorem 3.4. *Let* (E, m) *be a complete b-metric space and a mapping* $H : E \to E$ *be a self-map. Assume, there is a* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that* $\forall\ \kappa, \lambda \in E$ *with* $\kappa \neq \lambda$,

$$\begin{aligned}
&\frac{1}{2s} m(\kappa, H\kappa) < m(\kappa, \lambda) \\
&\Rightarrow \tau + F(m(H\kappa, H\lambda)) \leq \alpha F(m(\kappa, \lambda)) + \beta F(m(\kappa, H\kappa)) + \gamma F(m(\lambda, H\lambda)).
\end{aligned} \tag{22}$$

Where $0 \leq \alpha, \beta \leq 1$, $0 \leq \gamma < 1$ *with* $1 = \alpha + \beta + \gamma$

Then, H *has a fixed point* $\kappa^* \in E$.

Proof. By setting $\theta(\kappa,\lambda) = s$ in Theorem (3.2), the desired proof is obtained. □

Theorem 3.5. *Let (E, m) be a complete metric space and $H : E \to E$ be a mapping . Assume, there is one $F \in \mathcal{F}$ and $\tau > 0$ such that for all $\kappa, \lambda \in E$ with $\kappa \neq \lambda$,*

$$\frac{1}{2}m(\kappa, H\kappa) < m(\kappa,\lambda) \Rightarrow \tau + F(m(H\kappa, H\lambda)) \leq F(m(\kappa,\lambda)).$$

Then, H has a unique fixed point $\kappa^* \in E$.

Proof. By setting $\alpha = 1$, $\beta = \gamma = 0$ and $\theta(\kappa,\lambda) = 1$ in Theorem (3.2), then find H has a fixed point. Let fixed point is $\kappa^* \in E$. Furthermore, if $\exists\ \lambda^* \in E$ such that $H(\lambda^*) = \lambda^*$ and $\kappa^* \neq \lambda^*$ then,

$$\begin{aligned}\frac{1}{2}m(\kappa^*, H\kappa^*) < m(\kappa^*,\lambda^*) &\Rightarrow \tau + F(m(H\kappa^*, H\lambda^*)) \leq F(m(\kappa^*,\lambda^*)),\\ &\Rightarrow \tau + F(m(\kappa^*,\lambda^*)) \leq F(m(\kappa^*,\lambda^*)).\end{aligned}$$

Which is contradiction. So, H has a unique fixed point. □

Remark 3.6. *(I) When $\theta(\kappa,\lambda) = s \geq 1$. Then, Theorem (3.2) reduces to main result of Alsulami, karapnar and Piri [1]. Hence, Theorem (3.2) is a proper generalization of theorem (9) of [1].*

Definition 3.7. *Let (E, m_θ) be an extended b-metric space and $H : E \to E$ be a mapping. If $\exists\ F \in \mathfrak{F}$ and $\tau > 0$ such that $\forall\ \kappa, \lambda \in E$ with $\kappa \neq \lambda$,*

$$\begin{aligned}&0 < m_\theta(H\kappa, H\lambda),\\ &\Rightarrow \tau + F(m_\theta(H\kappa, H\lambda)) \leq \alpha F(m_\theta(\kappa,\lambda)) + \beta F(m_\theta(\kappa, H\kappa)) + \gamma F(m_\theta(\lambda, H\lambda)).\end{aligned} \tag{23}$$

Where, $0 \leq \alpha, \beta \leq 1$, $0 \leq \gamma < 1$ with $1 = \alpha + \beta + \gamma$. Then, H is called generalized F- contraction on extended b-metric space (E, m_θ) .

Theorem 3.8. *Let (E, m_θ) be a complete extended b-metric space, H be a continuous self-mapping on E and $\theta : E \times E \to [1,\infty)$ is a bounded function. If H is a generalized F-contraction on E, then H has a fixed point $\kappa^* \in E$.*

Proof. Let fix $\kappa_0 \in E$. Consider an iterative sequence $\{\kappa_n\}$ as follows:

$$\kappa_1 = H\kappa_0, \kappa_2 = H\kappa_1 = H^2\kappa_0, \ldots, \kappa_{n+1} = H\kappa_n = H^{n+1}\kappa_0,\ \forall\ n \in \mathbb{N} \cup \{0\}.$$

If $\kappa_n = \kappa_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, therefore result is obviously true.

Now, consider that $\kappa_n \neq \kappa_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, this yields

$$m_\theta(\kappa_n, H\kappa_n) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Thus, by the hypothesis of theorem, we have

$$\begin{aligned}&\tau + F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq \alpha F(m_\theta(\kappa_n, H\kappa_n)) + \beta F(m_\theta(\kappa_n, H\kappa_n)) + \gamma F(m_\theta(H\kappa_n, H^2\kappa_n)),\\ &\Rightarrow \tau + (1-\gamma)F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq (\alpha+\beta)F(m_\theta(\kappa_n, H\kappa_n)).\end{aligned}$$

Since $\alpha + \beta + \gamma = 1$.

$$F(m_\theta(H\kappa_n, H^2\kappa_n)) \leq F(m_\theta(\kappa_n, H\kappa_n)) - \frac{\tau}{\alpha+\beta} \leq F(m_\theta(\kappa_n, H\kappa_n)). \tag{24}$$

From $(\mathcal{F}_1)$, conclude that

$$m_\theta(\kappa_{n+1}, H\kappa_{n+1}) = m_\theta(H\kappa_n, H^2\kappa_n) < m_\theta(\kappa_n, H\kappa_n), \text{ for all } n \in \mathbb{N}. \tag{25}$$

Therefore, $\{m_\theta(\kappa_n, H\kappa_n)\}_{n=1}^{\infty}$ is a bounded below and monotonically decreasing sequence. Thus, $\{m_\theta(\kappa_n, H\kappa_n)\}_{n=1}^{\infty}$ converges and

$$\begin{aligned} F(m_\theta(\kappa_n, H\kappa_n)) &= F(m_\theta(H\kappa_{n-1}, H^2\kappa_n)), \\ &\leq F(m_\theta(\kappa_{n-1}, H\kappa_{n-1})) - \frac{\tau}{\alpha+\beta}, \\ &\leq F(m_\theta(\kappa_{n-2}, H\kappa_{n-2})) - 2\frac{\tau}{\alpha+\beta}, \\ &\vdots \\ &\leq F(m_\theta(\kappa_0, H\kappa_0)) - n\frac{\tau}{\alpha+\beta}. \end{aligned}$$

Hence we have

$$\lim_{n\to\infty} F(m_\theta(\kappa_n, H\kappa_n)) = -\infty \Rightarrow \lim_{n\to\infty} m_\theta(\kappa_n, H\kappa_n) = 0. \tag{26}$$

Our aims to demonstrate this in the subsequent stage,

$$\lim_{n,k\to\infty} m_\theta(\kappa_n, \kappa_k) = 0.$$

If not satisfy this equation. Then, consider that there exists $\epsilon > 0$ such that sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ satisfying,

$$r_n > s_n > n, \quad m_\theta(\kappa_{r_n}, \kappa_{s_n}) \geq \epsilon, \; m_\theta(\kappa_{r_{n-1}}, \kappa_{s_n}) < \epsilon, \text{ for all } n \in \mathbb{N}. \tag{27}$$

By triangle inequality,

$$\begin{aligned} m_\theta(\kappa_{r_n}, \kappa_{s_n}) &\leq \theta(\kappa_{r_n}, \kappa_{s_n})\Big[m_\theta(\kappa_{r_n}, \kappa_{r_{n-1}}) + m_\theta(\kappa_{r_{n-1}}, \kappa_{s_n})\Big], \\ &\leq \theta(\kappa_{r_n}, \kappa_{s_n}) m_\theta(\kappa_{r_n}, \kappa_{r_{n-1}}) + \theta(\kappa_{r_n}, \kappa_{s_n})\epsilon, \\ &\leq \theta(\kappa_{r_n}, \kappa_{s_n}) m_\theta(\kappa_{r_{n-1}}, H\kappa_{r_{n-1}}) + \theta(\kappa_{r_n}, \kappa_{s_n})\epsilon, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{28}$$

Because of (26), there one exists $N_2 \in \mathbb{N}$ satisfying,

$$m_\theta(\kappa_{r_n}, H\kappa_{r_n}) < \epsilon, \text{ for all } n > N_2, \tag{29}$$

Put in (28) find that

$$\begin{aligned} &m_\theta(\kappa_{r_n}, \kappa_{s_n}) < 2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon, \text{ for all } n > N_2, \\ &\Rightarrow F(m_\theta(\kappa_{r_n}, \kappa_{s_n})) < F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon), \text{ for all } n > N_2. \end{aligned}$$

On the other hand from (26),

$$\epsilon \leq m_\theta(\kappa_{r_n+1}, \kappa_{s_n+1}) = m_\theta(H\kappa_{r_n}, H\kappa_{s_n}).$$

Then,

$$\begin{aligned} \tau + F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) \leq &\alpha F(m_\theta(\kappa_{r_n}, \kappa_{s_n})) + \beta F(m_\theta(\kappa_{r_n}, H\kappa_{r_n})) \\ &+ \gamma F(m(\kappa_{s_n}, H\kappa_{s_n})), \end{aligned}$$

$$\begin{aligned} \Rightarrow \tau + F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) \leq &\alpha F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon) + \beta F(m_\theta(\kappa_{r_n}, H\kappa_{r_n})) \\ &+ \gamma F(m_\theta(\kappa_{s_n}, H\kappa_{s_n})). \end{aligned} \tag{30}$$

There exist $M \in \mathbb{R}$ such that $F(2\theta(\kappa_{r_n}, \kappa_{s_n})\epsilon) \leq M$, because θ is bounded function. So from (26) and $(\mathcal{F}_2)$, obtain that

$$\lim_{n\to\infty} F(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) = -\infty,$$
$$\Rightarrow \lim_{n\to\infty}(m_\theta(H\kappa_{r_n}, H\kappa_{s_n})) = 0.$$

This is a contradiction with the relation in (27). Hence, $\lim_{n,k\to\infty} m_\theta(\kappa_n, \kappa_k) = 0$; that is $\{\kappa_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Because of completion of (E, m_θ), there exists $\kappa^* \in E$ as well as.

$$\lim_{n\to\infty} m_\theta(\kappa_n, \kappa^*) = 0. \tag{31}$$

due to continuity of H, observe that

$$\lim_{n\to\infty} m_\theta(H\kappa_n, H\kappa^*) = 0.$$

As $m(\kappa^*, H\kappa^*) \leq \theta(\kappa^*, H\kappa^*)\Big[m_\theta(\kappa^*, \kappa_n) + m_\theta(\kappa_n, H\kappa^*)\Big]$, thus, $m_\theta(\kappa^*, H\kappa^*) = 0$ and so κ^* is a fixed point of H. □

Example 3.9. *Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:*

$$S_1 = 1, \quad S_2 = 1 + 2^3, \quad \cdots$$
$$S_n = 1 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Let $E = \{S_n : n \in \mathbb{N}\}$ and $m_\theta(x, y) = |x - y|$. Then (E, m_θ) is complete extended b-metric space, (where $\theta(x, y) = 1$). Define the mapping $H : X \to X$ by $H(S_1) = S_1$ and $H(S_n) = S_{n-1}$ for every $n > 1$. Since,

$$\lim_{n\to\infty} \frac{m_\theta(H(S_n), H(S_1))}{m_\theta(S_n, S_1)} = 1.$$

H is not a Banach contraction and a Suzuki contraction. On the other hand taking $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$, we obtain the result that H is an generalized F-contraction with $\tau = 1.5$. To see this, let us consider the following calculation. First observe that,

$$0 < m_\theta(HS_n, HS_m).$$

Then,

For $1 = n < m$, we have

$$|H(S_m) - H(S_1)| = |S_{m-1} - S_1| = 2^3 + 3^3 + \cdots + (m-1)^3,$$
$$|S_m - S_1| = 2^3 + 3^3 + \cdots + m^3.$$

Since $m > 1$ and $\frac{-1}{2^3+3^3+\cdots+(m-1)^3} < \frac{-1}{2^3+3^3+\cdots+m^3}$, we have

$$\begin{aligned}
1.5 - \frac{1}{|H(S_m) - H(S_1)|} &+ |H(S_m) - H(S_1)| \\
&= 1.5 - \frac{1}{2^3 + 3^3 + \cdots + (m-1)^3} + [2^3 + 3^3 + \cdots + (m-1)^3], \\
&< 1.5 - \frac{1}{2^3 + 3^3 + \cdots + m^3} + [2^3 + 3^3 + \cdots + (m-1)^3], \\
&\leq -\frac{1}{2^3 + 3^3 + \cdots + m^3} + [2^3 + 3^3 + \cdots + (m-1)^3] + m^3, \\
&= -\frac{1}{2^3 + 3^3 + \cdots + m^3} + [2^3 + 3^3 + \cdots + m^3], \\
&= -\frac{1}{|S_m - S_1|} + |S_m - S_1|.
\end{aligned}$$

For $1 = m < n$*, similar to* $1 = n < m$*, we have*

$$1.5 - \frac{1}{|H(S_m) - H(S_1)|} + |H(S_m) - H(S_1)| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|.$$

For $1 < n < m$*, we have*

$$|H(S_m) - H(S_n)| = n^3 + (n+1)^3 + \cdots + (m-1)^3,$$
$$|S_m - S_n| = (n+1)^3 + (n+2)^3 + \cdots + m^3.$$

We know that $\frac{-1}{n^3+(n+1)^3+\cdots+(m-1)^3} < \frac{-1}{(n+1)^3+(n+2)^3+\cdots+m^3}$*. Therefore,*

$$\begin{aligned}
1.5-&\frac{1}{|H(S_m) - H(S_n)|} + |H(S_m) - H(S_n)|,\\
&= 1.5 - \frac{1}{n^3 + (n+1)^3 + \cdots + (m-1)^3} + [n^3 + (n+1)^3 + \cdots + (m-1)^3],\\
&< 2 - \frac{1}{(n+1)^3 + (n+2)^3 + \cdots + m^3} + [n^3 + (n+1)^3 + \cdots + (m-1)^3],\\
&= -\frac{1}{(n+1)^3 + (n+2)^3 + \cdots + m^3} + [(n+1)^3 + (n+2)^3 + \cdots + (m-1)^3] + m^3,\\
&\leq -\frac{1}{(n+1)^3 + (n+2)^3 + \cdots + m^3} + [(n+1)^3 + (n+2)^3 + \cdots + m^3],\\
&< -\frac{1}{|S_m - S_n|} + |S_m - S_n|.
\end{aligned}$$

Therefore, $\tau + F(m_\theta(H(S_m), H(S_n))) \leq m_\theta(S_m, S_n)$*, for all* $m, n \in \mathbb{N}$*. Hence, H is a F-Suzuki contraction and* $H(S_1) = S_1$*.*

Theorem 3.10. *Let* (E, m) *be a complete b-metric space and a mapping* $H : E \to E$ *be a continuous map. Assume, there is one* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that* $\forall\ \kappa, \lambda \in E$ *with* $\kappa \neq \lambda$*,*

$$0 < m(H\kappa, H\lambda) \Rightarrow \tau + F(m(H\kappa, H\lambda)) \leq \alpha F(m(\kappa, \lambda)) + \beta F(m(\kappa, H\kappa)) + \gamma F(m(\lambda, H\lambda)).$$

Where, $0 \leq \alpha, \beta \leq 1$*,* $0 \leq \gamma < 1$ *with* $1 = \alpha + \beta + \gamma$*. Then, H has a fixed point* $\kappa^* \in E$*.*

Proof. By setting $\theta(\kappa, \lambda) = s$ in Theorem (3.8), the desired proof is obtained. □

Theorem 3.11. *Let* (E, m) *be a complete metric space and a mapping* $H : E \to E$ *be a continuous map. Assume, there is one* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that* $\forall\ \kappa, \lambda \in E$ *with* $\kappa \neq \lambda$*,*

$$0 < m(H\kappa, H\lambda) \Rightarrow \tau + F(m(H\kappa, H\lambda)) \leq F(m(\kappa, \lambda)).$$

Then, H has a unique fixed point $\kappa^* \in E$*.*

Proof. By setting $\alpha = 1$, $\beta = \gamma = 0$ and $\theta(\kappa, \lambda) = 1$ in Theorem (3.8), then one can find that H has a fixed point. Furthermore, if there exists $\lambda^*, \kappa^* \in E$ such that $H(\lambda^*) = \lambda^*$ and $H(\kappa^*) = \kappa^*$ with $\kappa^* \neq \lambda^*$ then,

$$\begin{aligned}
0 < m(H\kappa^*, H\lambda^*) &\Rightarrow \tau + F(m(H\kappa^*, H\lambda^*)) \leq F(m(\kappa^*, \lambda^*)),\\
&\Rightarrow \tau + F(m(\kappa^*, \lambda^*)) \leq F(m(\kappa^*, \lambda^*)).
\end{aligned}$$

This is contraction. So, H has a unique fixed point. □

Remark 3.12. *(I) When,* $\theta(\kappa, \lambda) = s \geq 1$*.Then, Theorem (3.8) reduces to main result of Alsulami, karapnar and Piri [1]. Hence, Theorem (3.8) is a proper generalization of theorem (15) of [1].*

4. Applications

In this section, we find the solution of Fredholm integral equation by using Theorem (3.2). Let $E = C([0,1], \mathbb{R})$ be the set of all continuous real valued functions defined on [0,1]. Note that E is complete extended b-metric space by considering,

$$m_\theta(\lambda, \kappa) = \sup_{x\in[0,1]} |\lambda(x) - \kappa(x)|^2,$$

with $\theta(\lambda, \kappa) = |\lambda(x)| + |\kappa(x)| + 2$, where $\theta : E \times E \to [1, \infty)$.

Consider

$$\lambda(x) = \int_0^1 M(x, y, \lambda(y))dy + g(x), \quad x, s \in [a, b]. \tag{32}$$

be the Fredholm integral equation, where $g : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous function.

Let $H : E \to E$ be an operator is given by,

$$H\lambda(x) = \int_0^1 M(x, y, \lambda(y))dy + g(x), \quad x, y \in [0,1]. \tag{33}$$

Further, assume that

$$|M(x, y, \lambda(y)) - M(x, y, \kappa(y))| \leq e^{-\frac{\tau}{2}}|\lambda(y) - \kappa(y)|.$$

Then, the integral equation 32 has a solution.

Proof. For any $\lambda, \kappa \in E$ we have,

$$\begin{aligned}
|H(\lambda(x)) - H(\kappa(x))| &= \left|\int_0^1 M(x, y, \lambda(y))dy + g(x) - \int_0^1 M(x, y, \kappa(y))dy - g(x)\right| \\
&= \left|\int_0^1 M(x, y, \lambda(y))dy - \int_0^1 M(x, y, \kappa(y))dy\right| \\
&= \left|\int_0^1 (M(x, y, \lambda(y)) - M(x, y, \kappa(y)))\, dy\right| \\
&\leq \int_0^1 e^{-\frac{\tau}{2}}|\lambda(y) - \kappa(y)|dy \\
&\leq \int_0^1 e^{-\frac{\tau}{2}} \sqrt{m_\theta(\lambda, \kappa)}dy \\
&= e^{-\frac{\tau}{2}} \sqrt{m_\theta(\lambda, \kappa)}, \\
\Rightarrow \sqrt{m_\theta(H(\lambda(x)) - H(\kappa(x)))} &\leq e^{-\frac{\tau}{2}} \sqrt{m_\theta(\lambda, \kappa)}, \\
\Rightarrow m_\theta(H(\lambda(x)) - H(\kappa(x))) &\leq e^{-\tau} m_\theta(\lambda, \kappa), \\
\Rightarrow \tau + \log m_\theta(H(\lambda(x)) - H(\kappa(x))) &\leq \log m_\theta(\lambda, \kappa).
\end{aligned}$$

Hence, H satisfying all the conditions of Theorem (3.2). Therefore, the operator H has a fixed point that is, the fredholm integral equation has a solution. □

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5. Conflict of Interest

The authors declared no conflict of interests related to this publication.

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