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# Bi-periodic Fibonomial coefficients

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**Abstract.** In the present study, we introduce a new generalization of Fibonomial coefficients known as bi-periodic Fibonomial coefficients, which can be expressed in relation to bi-periodic Fibonacci numbers. We establish various properties of these coefficients, including recurrence relation and recurrence formulas for powers of the bi-periodic Fibonacci numbers. Moreover, we provide combinatorial interpretation through weighted tilings generated by lattice paths and offer combinatorial proofs for bi-periodic Fibonomial identities.

## 1. Introduction

The Fibonomial coefficients, also known as Fibonacci-binomial coefficients, are a sequence of integers similar to binomial coefficients. They are defined as follows:

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k},$$

where $n$ and $k$ are non-negative integers with $k \leq n$, and $F_n$ is the $n$-th Fibonacci number.

Many identities for Fibonomial coefficients have been proven algebraically [12, 17]. In 2010, Sagan and Savage proposed two combinatorial interpretations for Fibonomial coefficients. These interpretations are discussed in several papers, see [6, 13, 16, 17]. The interpretations involve statistics on integer partitions inside a rectangle and tilings of Young diagrams inside a rectangle.

The bi-periodic Fibonacci sequence is a generalization of the Fibonacci sequence, defined as follows, see [10],

$$t_n = \begin{cases} at_{n-1} + t_{n-2}, & \text{if } n \text{ is even,} \\ bt_{n-1} + t_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

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with the initial conditions $t_0 = 0$ and $t_1 = 1$. Note that for $a = b = 1$, we get the classical Fibonacci sequence. The Binet formula of the bi-periodic Fibonacci sequence is given as follows

$$t_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor n/2 \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right),$$

where $\xi(n) = n - 2\lfloor n/2 \rfloor$, i.e., $\xi(n) = 0$ if $n$ is even and $\xi(n) = 1$ if n is odd. The constants $\alpha$ and $\beta$ are roots of the quadratic equation $z^2 - abz - ab = 0$, and can be expressed as:

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}.$$

The bi-periodic Fibonacci sequence has been studied in several papers, and various generalizations of this sequence have been suggested by researchers [1–3, 14, 15].

Given a board of length $n$ ($n$-board) with cells labeled 1 to $n$ from left to right, the aim is to tile this board using squares and dominoes. The number of distinct tilings possible for the $n$-board can be represented by the $(n + 1)$-th Fibonacci number. This interpretation has been used to give a combinatorial interpretation of the bi-periodic Fibonacci sequence using weighted tilings. The squares are assigned weights based on their positions: a weight of $a$ if the square is in an odd position and a weight of $b$ if the square is in an even position.

Now, if we inverse the weight of such squares, i.e., we assign a weight of $b$ if the square is in an odd position and a weight of $a$ if the square is in an even position, we get the dual bi-periodic Fibonacci sequence

$$t'_n = \begin{cases} bt'_{n-1} + t'_{n-2}, & \text{if } n \text{ is even,} \\ at'_{n-1} + t'_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

with the initial condition $t'_0 = 0$ and $t'_1 = 1$. The connection between the bi-periodic Fibonacci sequence $(t_n)$ and the sequence $(t'_n)$ is established through the following relationship:

$$t'_n = \left( \frac{b}{a} \right)^{\xi(n+1)} t_n. \tag{1}$$

The focus of the present paper is to define and study bi-periodic Fibonomial coefficients. We establish recurrence relations and formulas for powers of the bi-periodic Fibonacci numbers, along with various other properties. Furthermore, we give a combinatorial interpretation based on weighted tilings created by lattice paths and provide combinatorial proofs for several identities involving bi-periodic Fibonomial coefficients.

## 2. Bi-periodic Fibonomial coefficient

Consider the sequence $(t_n)_{n \geq 0}$, known as the bi-periodic Fibonacci sequence. The $t$-factorial of a term $t_n$ is defined as the product of all preceding terms in the sequence up to $t_1$.
Specifically, the $t$-factorial is denoted as $t_n!$ and is computed as $t_n t_{n-1} \cdots t_1$. Additionally, by convention, we set $t_0! = 1$.

**Definition 2.1.** *We define the bi-periodic Fibonomial coefficient for* $0 \leq k \leq n$

$$\binom{n}{k}_t = \frac{t_n!}{t_k! t_{n-k}!} = \frac{t_n t_{n-1} \cdots t_{n-k+1}}{t_k t_{k-1} \cdots t_1},$$

*with* $\binom{n}{0}_t = 1$ *for all* $n \geq 0$.

The first terms of the bi-periodic Fibonomial coefficient are as follows:

| n\k | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | | | | | |
| 1 | 1 | 1 | | | | |
| 2 | 1 | a | 1 | | | |
| 3 | 1 | $ab+1$ | $ab+1$ | 1 | | |
| 4 | 1 | $a^2b+2a$ | $a^2b^2+3ab+2$ | $a^2b+2a$ | 1 | |
| 5 | 1 | $a^2b^2+3ab+1$ | $a^3b^3+5a^2b^2+7ab+2$ | $a^3b^3+5a^2b^2+7ab+2$ | $a^2b^2+3ab+1$ | 1 |

**Table 1:** Bi-periodic Fibonomial Triangle.

The following theorem provides the recurrence relation for the bi-periodic Fibonomial coefficients.

**Theorem 2.2.** *For any positive integers $0 \leq k \leq n$, the bi-periodic Fibonomial coefficient satisfies*

$$\binom{n}{k}_t = \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\binom{n-1}{k}_t + \left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_t.$$

*Proof.* Using identity [18, Theorem 3]

$$t_{n+m-1} = \left(\frac{b}{a}\right)^{\xi(mn+n-m-1)} t_m t_n + \left(\frac{b}{a}\right)^{\xi(mn)} t_{m-1}t_{n-1},$$

with $m = k$ and $n = n-k+1$, we get

$$\begin{aligned}
\binom{n}{k}_t &= \frac{t_n t_{n-1}\cdots t_{n-k+1}}{t_k t_{k-1}\cdots t_1} \\
&= \left(\left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}t_k + \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{n-k}t_{k-1}\right)\cdot \frac{t_{n-1}\cdots t_{n-k+1}}{t_k t_{k-1}\cdots t_1} \\
&= \left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\frac{t_{n-1}\cdots t_{n-k+1}}{t_{k-1}\cdots t_1} + \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\frac{t_{n-1}\cdots t_{n-k+1}t_{n-k}}{t_k t_{k-1}\cdots t_1} \\
&= \left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_t + \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\binom{n-1}{k}_t.
\end{aligned}$$

□

The connection between the bi-periodic Fibonomial and the classical Fibonomial coefficients is presented in the following result.

**Lemma 2.3.** *For positive integers $n, k, a, b$ such that $n \geq k \geq 0$, we have*

$$\binom{n}{k}_t = \left(\sqrt{\frac{a}{b}}\right)^{\xi(k)\xi(n+k)} \binom{n}{k}_{F\left(\sqrt{ab},1\right)}, \tag{2}$$

*where $F_n(x,y)$ is the n-th generalized Fibonacci polynomial, defined by a recurrence relation of the form*

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y),$$

*with the initial values $F_0(x,y) = 0$ and $F_1(x,y) = 1$.*

*Proof.* Let $A(n,k)$ be the right side of (2), the coefficients $A(n,k)$ and $\binom{n}{k}_t$ satisfy the same recurrence relation and the same initial conditions.
In fact, $t_n = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n+1)} F_n\left(\sqrt{ab},1\right)$, we get

$$\begin{aligned}
A(n,k) &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(k)\xi(n+k)} \binom{n}{k}_{F\left(\sqrt{ab},1\right)} \\
&= \left(\sqrt{\frac{a}{b}}\right)^{\xi(k)\xi(n+k)} \left(F_{k-1}\left(\sqrt{ab},1\right)\binom{n-1}{k}_{F\left(\sqrt{ab},1\right)} + F_{n-k+1}\left(\sqrt{ab},1\right)\binom{n-1}{k-1}_{F\left(\sqrt{ab},1\right)}\right) \\
&= \left(\sqrt{\frac{b}{a}}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\binom{n-1}{k}_{F\left(\sqrt{ab},1\right)} + \left(\sqrt{\frac{b}{a}}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_{F\left(\sqrt{ab},1\right)} \\
&= \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}A(n-1,k) + \left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}A(n-1,k-1),
\end{aligned}$$

which gives the desired result. □

The $q$-integer or $q$-analogue of the positive integer $n$ is defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q = 1. \end{cases}$$

The Gaussian $q$-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[n]_q!$ denotes the $q$-factorial or $q$-analogue of the number $n$, defined as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q \cdots [1]_q, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

The relation between the $q$-Gaussian coefficient and bi-periodic Fibonomial coefficients is given by the following lemma.

**Lemma 2.4.** *For $n \geq k \geq 0$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\alpha^{k^2-nk}}{a^{\xi(k)\xi(n+1)}(ab)^{\lfloor (k^2-nk)/2 \rfloor}} \binom{n}{k}_t. \tag{3}$$

*Proof.* Taking $q = \frac{\beta}{\alpha}$ and using Binet's formula for the bi-periodic Fibonacci sequence, we arrive at the result. □

In the following theorem, we give the recurrence formula for the power of the bi-periodic Fibonacci numbers.

**Theorem 2.5.** *For $n > k \geq 0$, we have*

$$\sum_{j=0}^{k+1} (-1)^{\binom{j+1}{2}} \left(\sqrt{\frac{b}{a}}\right)^{k\xi(k+j+1)+\xi(k)\xi(j)} \binom{k+1}{j}_t t_{n-j}^k = 0.$$

*Proof.* From the Binet formula, we have

$$t_n^k = \frac{a^{k\xi(n+1)}}{(ab)^{k\lfloor n/2\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^k = \frac{a^{k\xi(n+1)}}{(ab)^{k\lfloor n/2\rfloor}(\alpha-\beta)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\alpha^{n(k-j)}\beta^{nj}.$$

Let $U$ be the shift operator $Uf_n = f_{n-1}$. The sequences $\left(\frac{\alpha^{n(k-j)}\beta^{nj}}{\left(\sqrt{ab}\right)^{(k-1)n}}\right)_{n\geq 0}$ satisfy the following recurrence relation

$$\left(1-\frac{\alpha^{k-j}\beta^j}{\left(\sqrt{ab}\right)^k}U\right)\left(\frac{\alpha^{n(k-j)}\beta^{nj}}{\left(\sqrt{ab}\right)^{(k-1)n}}\right)=0. \tag{4}$$

Using (4), we get

$$\prod_{j=0}^{k}\left(1-\frac{\alpha^{k-j}\beta^j}{\left(\sqrt{ab}\right)^k}U\right)\left(\left(\sqrt{\frac{b}{a}}\right)^{k\xi(n+1)}t_n^k\right)=0.$$

The $q$-binomial theorem (see [8, 9]) states that

$$\prod_{j=0}^{n-1}\left(1-q^jx\right)=\sum_{k=0}^{n}(-1)^kq^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k.$$

By taking $q=\frac{\beta}{\alpha}$ and applying Relation (3), we obtain

$$\begin{aligned}\prod_{j=0}^{k}\left(1-\frac{\alpha^{k-j}\beta^j}{\left(\sqrt{ab}\right)^k}U\right) &= \prod_{j=0}^{k}\left(1-\left(\frac{\beta}{\alpha}\right)^j\frac{\alpha^kU}{\left(\sqrt{ab}\right)^k}\right)\\ &= \sum_{j=0}^{k}(-1)^j\left(\frac{\beta}{\alpha}\right)^{\binom{j}{2}}\frac{\alpha^{j^2-(k+1)j}}{a^{\xi(j)\xi(k)}(ab)^{\lfloor (j^2-(k+1)j)/2\rfloor}}\binom{k+1}{j}_t\frac{\alpha^{kj}}{\left(\sqrt{ab}\right)^{kj}}U^j\\ &= \sum_{j=0}^{k}(-1)^j\frac{(\alpha\beta)^{\binom{j}{2}}}{a^{\xi(j)\xi(k)}\left(\sqrt{ab}\right)^{kj}(ab)^{\lfloor (j^2-(k+1)j)/2\rfloor}}\binom{k+1}{j}_tU^j.\end{aligned}$$

Since $\alpha\beta=-ab$, we get

$$\prod_{j=0}^{k}\left(1-\frac{\alpha^{k-j}\beta^j}{\left(\sqrt{ab}\right)^k}U\right)=\sum_{j=0}^{k}(-1)^{\binom{j+1}{2}}\left(\sqrt{\frac{b}{a}}\right)^{\xi(k)\xi(j)}\binom{k+1}{j}_tU^j.$$

By applying this operator to $t_n^k$, we get

$$\begin{aligned}\prod_{j=0}^{k}\left(1-\frac{\alpha^{k-j}\beta^j}{\left(\sqrt{ab}\right)^k}U\right)\left(\left(\sqrt{\frac{b}{a}}\right)^{k\xi(n+1)}t_n^k\right) &= \sum_{j=0}^{k}(-1)^{\binom{j+1}{2}}\left(\sqrt{\frac{b}{a}}\right)^{\xi(k)\xi(j)}\binom{k+1}{j}_tU^j\left(\left(\sqrt{\frac{b}{a}}\right)^{k\xi(n+1)}t_n^k\right)\\ &= \sum_{j=0}^{k}(-1)^{\binom{j+1}{2}}\binom{k+1}{j}_t\left(\sqrt{\frac{b}{a}}\right)^{k\xi(n+1-j)+\xi(k)\xi(j)}t_{n-j}^k,\end{aligned}$$

which gives the desired result. □

**Theorem 2.6.** *For $n \geq 0$ and $k \geq 1$, we have the following alternative convolution relation*

$$\sum_{j=0}^{k} (-1)^{\binom{j+1}{2}} \left(\sqrt{\frac{b}{a}}\right)^{\xi(n+1)\xi(k+1)+2\xi(j)\xi(k+1)\xi(n)} \binom{k}{j}_t \binom{n-j}{k-1}_t = 0.$$

*Proof.* Lind [11] provided that

$$\sum_{j=0}^{k} (-1)^{\binom{j+1}{2}} \binom{k}{j}_F \binom{n-j}{k-1}_F = 0.$$

We use the above identity and Lemma 2.3 to obtain the result. □

## 3. Combinatorial interpretations

Partitioning an integer $n$ into at most $k$ parts, each not exceeding $n-k$, corresponds directly to ascending lattice paths within a $k \times (n-k)$ rectangle. These paths, allowing only rightward (→) and upward (↑) movements from $(0,0)$ to $(k, n-k)$ over $n$ steps, generate partitions $\lambda = (n_1, n_2, \ldots, n_k)$ (numbered from the bottom) and their complement $\lambda' = (m_1, m_2, \ldots, m_{n-k})$ (numbered from the left). The binomial coefficient $\binom{n}{k}$ represents the count of ascending lattice paths fitting into a $k \times (n-k)$ rectangle.
we provide the interpretation of the bi-periodic Fibonomial coefficients based on the interpretation introduced in [16].

**Theorem 3.1.** *For $1 \leq k \leq n$, $\binom{n}{k}_t$ counts the number of ways to draw a lattice path inside a $k \times (n-k)$ grid from $(0,0)$ to $(k, n-k)$, then tile each row above the lattice path and each column below it according to their position in the grid with dominoes and weighted squares, such that:*

- *Column bands tiling are not allowed to start with a square.*
- *If the tiling bands occur at odd positions, then squares take the weight $a$ in odd positions and the weight $b$ in even positions.*
- *If the tiling bands occur at even positions, squares take the weight $b$ in odd positions and the weight $a$ in even positions.*

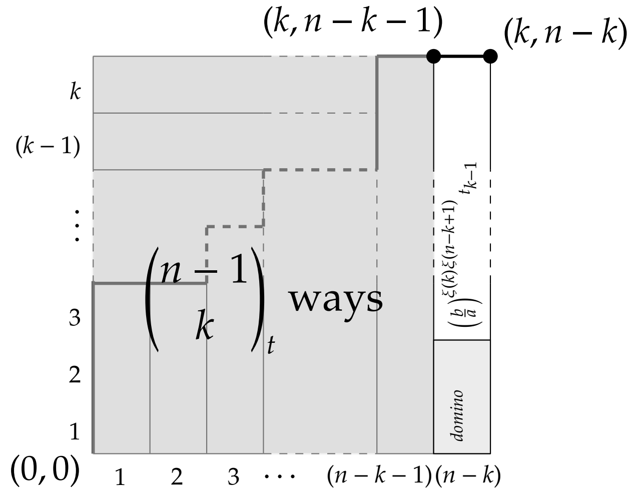
*Proof.* The proof is based on the recurrence relation of the theorem 2.2 and relies on considering the last step of the path and the parity of $k$ and $n$. The path moves through a grid of $k$ rows and $n-k$ columns, using only horizontal and vertical steps.

For lattice paths ending with a horizontal step (Figure 1a): In all such paths, the last column (counting from the right) has a length of $k$ and must start with a domino, can be tiled in $t_{k-1}$ ways when $n-k$ is odd and $t'_{k-1}$ when $n-k$ is even. The remaining lattice paths from $(0,0)$ to $(k, n-k-1)$ yield $\binom{n-1}{k}_t$ tiled lattice paths. Consequently, the total number of tiled lattice paths ending with a horizontal step is $t_{k-1}\binom{n-1}{k}_t$ when $n-k$ is odd and $(\frac{b}{a})^{\xi(k)} t_{k-1}\binom{n-1}{k}_t$ when $n-k$ is *even*.

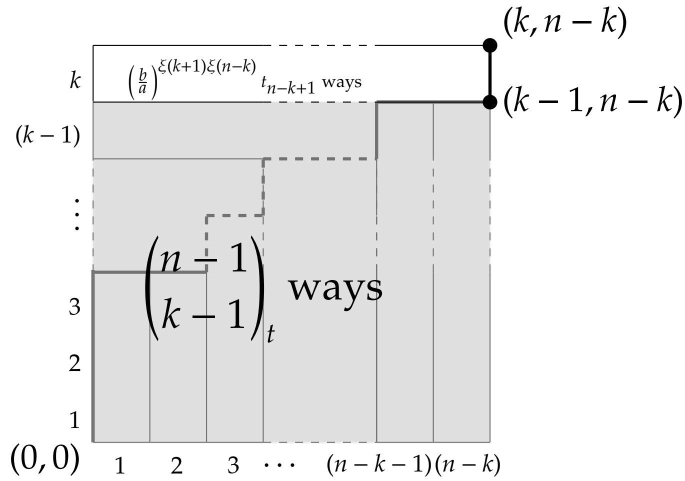
For lattice paths ending with a vertical step (Figure 1b): The last row, row $k$ (counting from the bottom), has a length of $n-k$ and can be tiled in $t_{n-k+1}$ ways when $k$ is odd and $t'_{n-k+1}$ ways when $k$ is *even*. The remaining part of the lattice path is determined by the lattice path from $(0,0)$ to $(k-1, n-k)$, resulting in $\binom{n-1}{k-1}_t$ tiled lattice paths. Therefore, the number of tiled lattice paths ending in a vertical step is $t_{n-k+1}\binom{n-1}{k-1}_t$ when $k$ is *odd* and $(\frac{b}{a})^{\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_t$ when $k$ is *even*.The total count of tiled lattice paths from point $(0,0)$ to $(k, n-k)$ is

$$\binom{n}{k}_t = \left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\binom{n-1}{k}_t + \left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_t.$$

□




**(a)** Tiled Lattice Paths Ending with a Horizontal Step $\left(\frac{b}{a}\right)^{\xi(k)\xi(n-k+1)} t_{k-1}\binom{n-1}{k}_t$.




**(b)** Tiled Lattice Paths Concluding with a Vertical Step $\left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k)} t_{n-k+1}\binom{n-1}{k-1}_t$.

**Figure 1:** Illustration of a Non-circular Partition Tiling in a $k \times (n-k)$ Grid.

Figure 2 shows a practical example to explain the earlier concept. This example demonstrates how the theorem works using $\binom{6}{4}_t$. Additionally, in Figure 3, we highlight the lattices that can't be tiled as the theorem predicts. This is due to columns with a height of 1, as explained by the theorem.

Considering the nature of the last $i$ steps within the ascending path of a grid with dimensions $(k, n-k)$, we present two distinct formulations for our coefficient. These formulations are given in Theorems 3.2 and 3.3.

**Theorem 3.2 (Horizontal steps).** *For $n \geq k \geq 0$, we have*

$$\binom{n}{k}_t = \sum_{j=0}^{n-k} \left(\frac{b}{a}\right)^{\xi(k)\lfloor (j+1)/2 \rfloor + \xi(j+1)\xi(k)\xi(n-k+1) + \xi(k+1)\xi(n-k-j)} t_{k-1}^{j} t_{n-k-j+1} \binom{n-1-j}{k-1}_t.$$

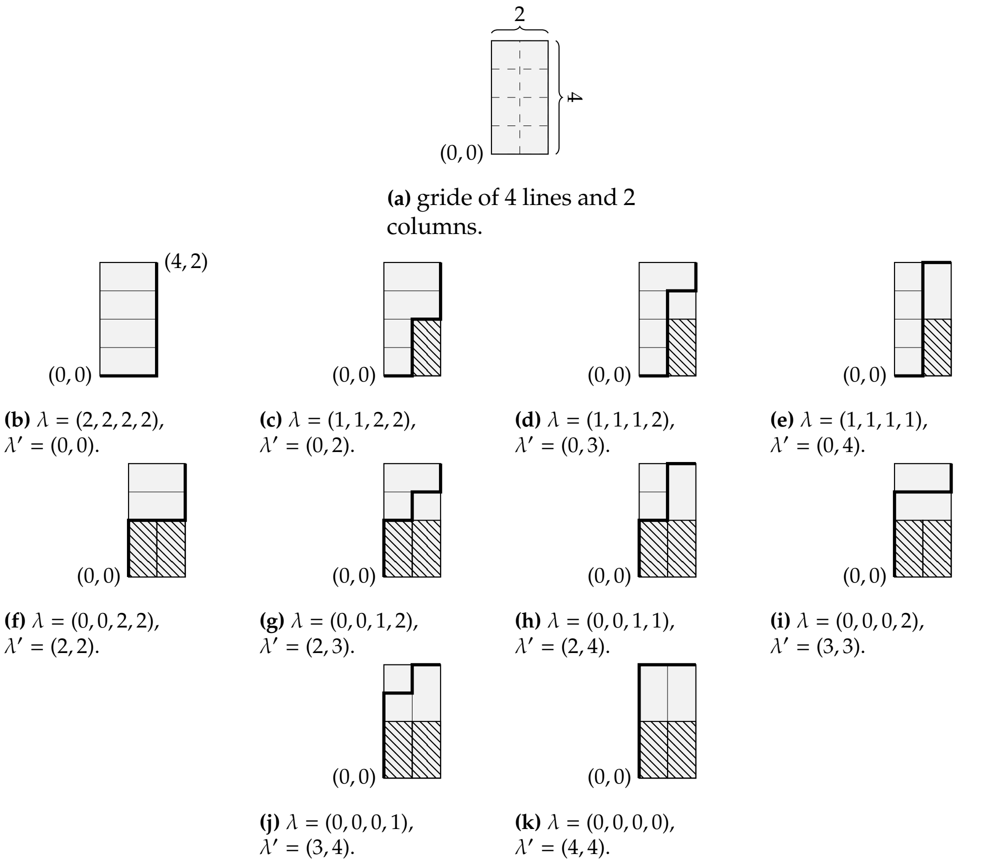



**Figure 2:** Suitable lattices allowing columns to begin with dominoes, illustrated here using the example $\binom{6}{4}_t$.

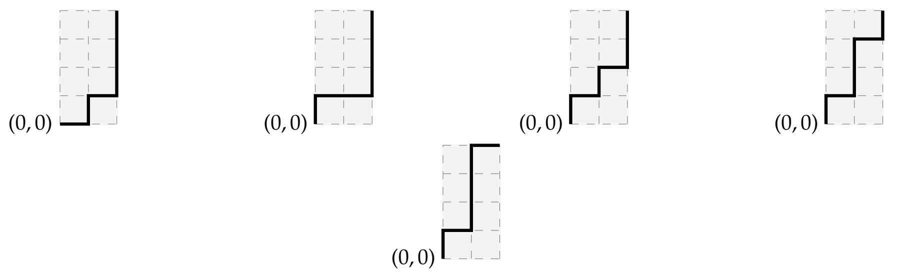



**Figure 3:** Ascendant lattice paths within a $4 \times 2$ grid, that cannot be tiled with dominoes due to columns of height 1.

*Proof.* Consider the number of horizontal steps at the end of the path. Suppose there are $j\,(0 \leq j \leq n-k)$ horizontal steps, then we have $j$ vertical tilings of length $k-2$ (starting with a domino), which can be tiled in $\left(\frac{b}{a}\right)^{\xi(k)\left(\xi(n-k+1)+\xi(n-k)+\cdots+\xi(n-k-j+1)\right)} t_{k-1}^{j}$ ways. Due to the horizontal step in the lattice path from $(k-1, n-k-j)$ to $(k, n-k-j)$, the last row with a length of $n-k-j$ can be tiled in $\left(\frac{b}{a}\right)^{\xi(k+1)\xi(n-k-j)} t_{n-k-j+1}$ ways.

The remaining tiling is composed of a lattice path from $(0, 0)$ to $(k-1, n-k-j)$, which is tiled in $\binom{n-j+1}{k-1}_t$ different ways. Summing over all possibilities for $j$ and some simplification, we get the result. □

Note that for $a = b = 1$, we get the following result (see [4])

$$\binom{n}{k}_F = \sum_{j=0}^{n-k} F_{k-1}^{j} F_{n-k-j+1} \binom{n-1-j}{k-1}_F.$$

**Theorem 3.3 (Vertical steps).** *For $n \geq k \geq 0$, we have*

$$\binom{n}{k}_t = \sum_{j=0}^{k} \left(\frac{b}{a}\right)^{\xi(n-k)\lfloor (j+1)/2 \rfloor + \xi(j+1)\xi(k+1)\xi(n-k) + \xi(k-j)\xi(n-k+1)} t_{n-k+1}^{j} t_{k-j-1} \binom{n-1-k}{k-j}_t.$$

*Proof.* Consider the number of vertical steps at the end of the path. Suppose there are $j\,(0 \leq j \leq k)$ vertical steps, then we have $j$ horizontal tilings of length $n-k$ which can be tiled in $\left(\frac{b}{a}\right)^{\xi(n-k)\left(\xi(k+1)+\xi(k)+\cdots+\xi(k-j+1)\right)} t_{n-k+1}^{j}$ ways. Due to the horizontal step in the lattice path from $(k-j, n-k-1)$ to $(k-j, n-k)$, the last column with a length of $k-j$ can be tiled in $\left(\frac{b}{a}\right)^{\xi(k-j)\xi(n-k+1)} t_{k-j-1}$ ways (the starting domino).

The remaining tiling is composed of a lattice path from $(0, 0)$ to $(k-j, n-k-1)$, which is tiled in $\binom{n-j-1}{k-j}_t$ different ways. We get the result by summing over all possibilities for $j$ and some simplification. □

Note that for $a = b = 1$, we obtain the following result (see [4])

$$\binom{n}{k}_F = \sum_{j=0}^{k} F_{n-k+1}^{j} F_{k-j-1} \binom{n-1-k}{k-j}_F.$$

**Theorem 3.4.** *For $n \geq k \geq 0$, we have*

$$\binom{n}{k}_t = \sum_{\sum_i^k n_i + \sum_j^{n-k} m_j = k\cdot(n-k)} \prod_{i=1}^{k} \left(\frac{b}{a}\right)^{\xi(i+1)\xi(n_i)} t_{n_i+1} \cdot \prod_{j=1}^{n-k} \left(\frac{b}{a}\right)^{\xi(j+1)\xi(m_j)} t_{m_j-1},$$

*with*

$$0 \leq m_1 \leq \cdots \leq m_{n-k} \leq n,\ 0 \leq n_1 \leq \cdots \leq n_k \leq n-k \text{ and } t_{-1} = 1$$

*the $n_i$ and $m_j$ are generated from ascendant partitions of a $(k \times n-k)$ grid.*

*Proof.* Based on the combinatorial interpretation, we consider all ascending lattice paths inside a $k \times (n-k)$ grid.
In each ascending lattice path, the parts $\lambda = (n_1, n_2, ..., n_k)$ represent the lengths of the lines that are above the partition when counted from the bottom, while the parts of the complementary $\lambda' = (m_1, m_2, ..., m_{n-k})$ denote the lengths of columns that are below the partition when counted from the left.
Counting the number of ways to tile the resulting bands using $(t_n)_n$ if the band is in an even position (resp $(t'_n)_n$ if the band is in an odd position), counting from the bottom for the lines band and from the left for the columns bands such that the columns band must start with a domino, and by considering all the cases according to the parity of $n$ and $k$ we'll have:

- If $k$ is even:
  - When $n$ is even (Figure 4a):

$$t_{n_1+1} t'_{n_2+1} \cdots t_{n_{k-1}+1} t'_{n_k+1} \cdot t_{m_1-1} t'_{m_2-1} \cdots t_{m_{n-k-1}-1} t'_{m_{n-k}-1},$$

– When $n$ is odd (Figure 4b):

$$t_{n_1+1}t'_{n_2+1}\cdots t_{n_{k-1}+1}t'_{n_k+1}\cdot t_{m_1-1}t'_{m_2-1}\cdots t'_{m_{n-k-1}-1}t_{m_{n-k}-1},$$

- If $k$ is odd:
  - When $n$ is even (Figure 4c):

$$t_{n_1+1}t'_{n_2+1}\cdots t'_{n_{k-1}+1}t_{n_k+1}\cdot t_{m_1-1}t'_{m_2-1}\cdots t'_{m_{n-k-1}-1}t_{m_{n-k}-1},$$

  - When $n$ is odd (Figure 4d):

$$t_{n_1+1}t'_{n_2+1}\cdots t'_{n_{k-1}+1}t_{n_k+1}\cdot t_{m_1-1}t'_{m_2-1}\cdots t_{m_{n-k-1}-1}t'_{m_{n-k}-1}.$$

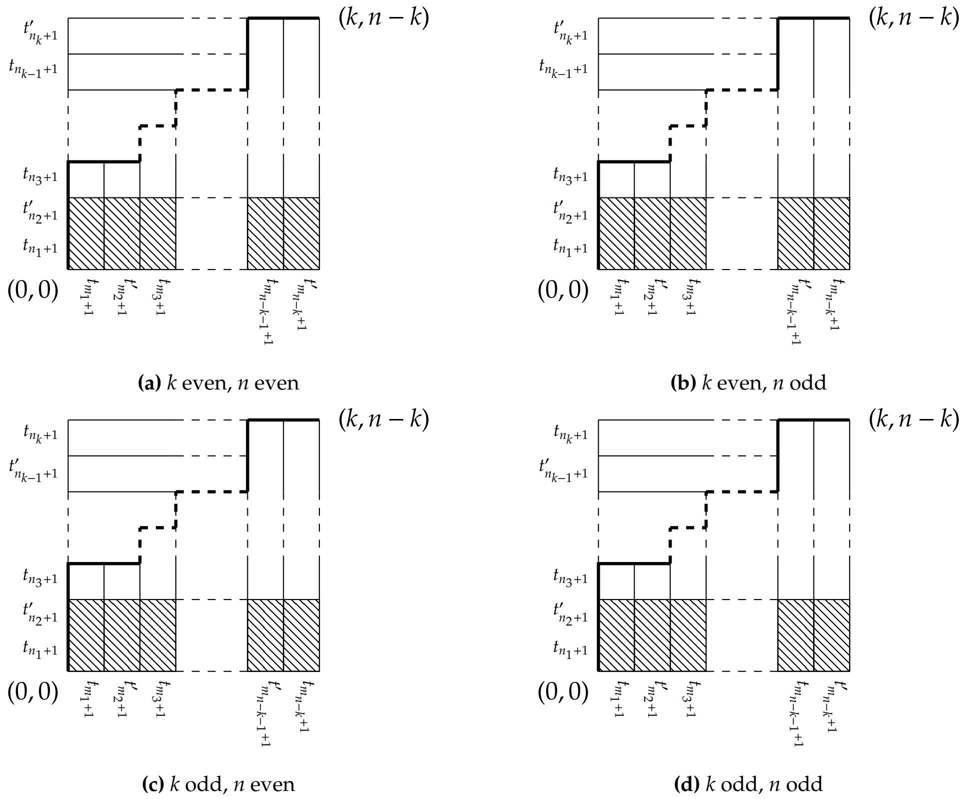



**Figure 4:** Illustration of Tiling of the resulting bands generated by one ascending lattice path according to the parity of $k$ and $n$, with consideration for the dominoes at the beginning of each column tiling.

Using 1, the four cases can be reduced to

$$\prod_{i=1}^{k}\left(\frac{b}{a}\right)^{\xi(i+1)\xi(n_i)} t_{n_i+1}\cdot\prod_{j=1}^{n-k}\left(\frac{b}{a}\right)^{\xi(j+1)\xi(m_j)} t_{m_j-1}.$$

By summing over all resulting partitions, we find the result. □

## Conclusion

The bi-periodic Fibonomial coefficients introduced in this study extend the classical Fibonomial coefficients. We established recurrence relations and combinatorial interpretations via weighted tilings. These findings offer new insights and potential avenues for further exploration in combinatorics.

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