Filomat 39:3 (2025), 975–981
https://doi.org/10.2298/FIL2503975A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some spectral properties of complex unit gain graph

Vahid Adish[a], Maryam Khosravi[a,*]

[a] *Department of Pure Mathematics, Faculty of Mathematics and computer, Shahid Bahonar University of Kerman, Kerman, Iran*

Abstract. Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. In this note we show that Φ has one positive and negative eigenvalues if and only if it is switching equivalent to a balanced complete bipartite graph or complete tripartite with special weights. In addition, we found some lower bounds for the energy of a $\mathbb{T}$-gain graph.

1. Introduction and preliminaries

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A subgraph H of a graph G is called an induced subgraph if it is formed from a subset of the vertices of G and all of the edges of G connecting them. For a subgraph H of G, the *complement* of H in G, denoted by $G - H$, defined as the induced subgraph of G with vertex set $V(G) \backslash V(H)$. The subgraphs H and $G - H$ are called *complementary induced subgraphs* in G.

A *matching* in a graph G is a set of edges of G without common vertices. The *matching number* of G which is denoted by $\mu(G)$ is the cardinality of a matching with the maximum number of edges. A matching that meets all the vertices of G is called a *perfect matching* of G.

For each $\{v_i, v_j\} \in E$, by e_{ij}, we mean the directed edge from v_i to v_j and $\overrightarrow{E(G)} = \{e_{ij}, e_{ji} : \{v_i, v_j\} \in E\}$.

Definition 1.1. *A triple $\Phi = (G; \mathbb{T}; \varphi)$ (or $\Phi = (G, \varphi)$) in which*

(i) $G = (V; E)$ *is a simple finite graph,*

(ii) $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ *is the unit complex circle, and*

(iii) The gain function $\varphi : \overrightarrow{E(G)} \to \mathbb{T}$ *is a map such that* $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$,

is called a $\mathbb{T}$-gain graph (or complex unit gain graph). Briefly, a $\mathbb{T}$-gain graph is a simple graph in which a unit complex number is assigned to each directed edge, and its inverse is assigned to the reverse directed edge.

2020 *Mathematics Subject Classification*. Primary 05C50; Secondary 15A18.
Keywords. $\mathbb{T}$-gain graph, eigenvalue, matching number, energy of graph
Received: 24 April 2024; Revised: 28 October 2024; Accepted: 29 October 2024
Communicated by Paola Bonacini
* Corresponding author: Maryam Khosravi
Email addresses: vahidadish1@gmail.com (Vahid Adish), khosravi_m@uk.ac.ir (Maryam Khosravi)
ORCID iDs: https://orcid.org/0009-0000-8705-3882 (Vahid Adish), https://orcid.org/0000-0002-5974-5037 (Maryam Khosravi)

The adjacency matrix $A(\Phi) = [a_{ij}]$ for a $\mathbb{T}$-gain graph Φ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}), & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

If two vertices v_i and v_j are adjacent, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \overline{a_{ji}}$, where $\bar{z}$ denotes the conjugate of the complex number z . Thus the adjacency matrix of a $\mathbb{T}$-gain graph is hermitian and so its eigenvalues are real.

Definition 1.2. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A(\Phi)$. The energy of Φ is defined by $E(\Phi) = \sum_{j=1}^{n} |\lambda_j|$.*

Definition 1.3. *The gain of a cycle (with some orientation) $C = v_1v_2...v_kv_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is*

$$\varphi(C) = \varphi(e_{12})\varphi(e_{23})...\varphi(e_{(k-1)k})\varphi(e_{k1}).$$

A cycle C is said to be neutral if $\varphi(C) = 1$, and a gain graph is said to be balanced if all its cycles are neutral.

Theorem 1.4. *[11, Theorem 6.1] Let $\Phi = (C, \varphi)$ be a $\mathbb{T}$-gain graph on an oriented cycle C on n vertices with $\varphi(C) = e^{i\theta}$. Then*

$$A(\Phi) = \{2\cos(\frac{\theta + 2\pi j}{n}) : j = 0, 1, \ldots, n-1\}.$$

Definition 1.5. *Two gain graphs $\Phi_1 = (G; \varphi_1)$ and $\Phi_2 = (G; \varphi_2)$ are called switching equivalent, and written as $\Phi_1 \sim \Phi_2$, if there exists a switching function $\eta : V \longrightarrow \mathbb{T}$ such that*

$$\varphi_2(e_{ij}) = \eta(v_i)^{-1}\varphi_1(e_{ij})\eta(v_j).$$

or equivalently, there is a diagonal matrix D with diagonal entries from $\mathbb{T}$, such that

$$A(\Phi_2) = D^{-1}A(\Phi_1)D.$$

For any gain graph (G, φ) and a given vertex $v \in V(G)$, there exists a switching equivalent gain graph (G, ψ) such that $\psi(e) = 1$ for all edges e which are incident to v.

It is known that a $\mathbb{T}$-gain graph $\Phi = (G; \varphi)$ is balanced if and only if $\Phi \sim (G; 1)$ [11, Lemma 1.1].

Since two switching equivalent graphs have the same spectrum, for balanced graph (specially for trees), all known results about the energy of simple graphs are valid.

This paper, has two parts. In Section 2, we characterize all $\mathbb{T}$-gain graph with one positive and one negative eigenvalues. Some related results can be found in [7], [9, Theorem 1], [12, Theorem 4.2] and [9].

In Section 3, we find some lower bounds for the energy of a $\mathbb{T}$-gain graph. Our results are extensions of what obtained in [3] and [8, Theorem 5.2] for simple graphs. Also there are some related results in [14].

2. Graphs with one positive and negative eigenvalues

In the theory of simple graphs (without weight), it is well known that the only graphs with one negative eigenvalue and one positive eigenvalue are the complete bipartite graphs together with possibly some isolated vertices [4, p. 163]. This result fails for $\mathbb{T}$-gain graphs. For instance, the graph $\Phi = (C_3, \varphi)$, with $\varphi(C_3) = i$, has 3 eigenvalues $-\sqrt{3}, 0, \sqrt{3}$. In this section, we shall characterize all gain graphs with this property.

In [12], it is proved that if $\Phi = (G, \varphi)$ is a bipartite graph, then it has one positive eigenvalue if and only if it is a balanced complete bipartite. Along the lines of the same arguments, we generalize this result as follows.

Theorem 2.1. *Let $\Phi = (G, \varphi)$ be any triangle-free $\mathbb{T}$-gain graph. Then Φ has exactly one positive eigenvalue if and only if Φ consists of a balanced complete bipartite graph and possibly some isolated vertices.*

Proof. We assume that Φ is connected. First note that Φ contains no induced (P_4, φ). Since (P_4, φ) is a tree, it has the same spectrum as P_4 and so it has two positive eigenvalues. In this case, by interlacing theorem Φ has at least two positive eigenvalue, a contradiction. It follows that the diameter of G is at most 2.

Let v_1 be an arbitrary vertex of G, $u_1, \ldots, u_p$ be the neighbors of v_1. Since G is triangle-free, it follows that there is no edge between u_i's. Let $v_2, \ldots, v_q$ be non-adjacent vertices to v_1 (If there are not such vertices, all done). Every vertex v_k $(2 \le k \le q)$ must have a common neighbor with v_1, say u_1. We claim that v_k should be adjacent to all u_i $(2 \le i \le p)$, if there is any. Because the path $v_k u_1 v_1 u_i$ can not be an induced subgraph of G. Again, since G is triangle-free, there is no edge between v_k's. This means that $\Phi = (K_{p,q}, \varphi)$.

If $p = 1$ or $q = 1$, the graph is tree and so is balanced. Let $p, q \le 2$. Note that the every cycle $v_i u_k v_j u_\ell v_i$ for some $1 \le i, j \le q$ and $1 \le k, \ell \le p$ is an induced subgraph of Φ and by the interlacing theorem it has at most one positive eigenvalue. In view of Theorem 1.4, this happens only when such a cycle is neutral. Since the gain of every even cycle can be written as a product of the gains of some C_4, it follows that Φ is balanced. □

Similarly, we have the following result.

Corollary 2.2. *Let $\Phi = (G, \varphi)$ be any triangle-free $\mathbb{T}$-gain graph. Then Φ has exactly one negative eigenvalue if and only if Φ consists of a balanced complete bipartite graph and possibly some isolated vertices.*

Example 2.3. *Let $\Phi = (G, \varphi)$ in which G is a complete 3-partite graph with parts V_1, V_2 and V_3 and*

$$\varphi(uv) = \begin{cases} i & \text{if } u \in V_2,\ v \in V_3, \\ -i & \text{if } u \in V_3,\ v \in V_2, \\ 1 & \text{otherwise.} \end{cases}$$

If V_1, V_2 and V_3 contain n, m and k vertices, then the adjacency matrix of Φ is the following block matrix

$$A(\Phi) = \begin{bmatrix} 0_{n\times n} & 1_{n\times m} & 1_{n\times k} \\ 1_{m\times n} & 0_{m\times m} & i_{m\times k} \\ 1_{k\times n} & -i_{k\times m} & 0_{k\times k} \end{bmatrix}$$

The $A(\Phi)$ has evidently rank 2 and thus Φ has one positive and one negative eigenvalues.

The next theorem states that the above graphs are all $\mathbb{T}$-gain graph with desired property. There is a different proof of this theorem in [9, 13].

Theorem 2.4. *Let $\Phi = (G, \varphi)$ be any connected $\mathbb{T}$-gain graph which have C_3. Then Φ has exactly one positive eigenvalue and one negative eigenvalues if and only if Φ is switching equivalent to a complete 3-partite graph with weighted described in Example 2.3.*

Proof. First of all, note that every C_3 subgraph of G should satisfy $\varphi(C_3) = i$ or $-i$. Suppose that there is a cycle $u_1 v_1 w_1 u_1$ and assume that all edges incident to v_1 have weight 1 and $\varphi(u_1 w_1) = i$. Consider three vertex sets $W = N(v_1) \cap N(u_1)$, $U_1 = N(v_1) \setminus N(u_1)$ and $U_2 = V \setminus N(v_1)$. If $w_2 \in W$, each probable edge from w_2 to w_1 or u_1 needs to have weight $\pm i$. Checking all possibilities, the only case for which the induced subgraph by v_1, u_1, w_1, w_2 has one positive and one negative eigenvalues is the case that $\varphi(u_1 w_2) = i$ and w_1 and w_2 are not adjacent. Similar discussion shows that there is no edges between two vertices in W or two vertices in U_1 and $\varphi(uw) = i$ for each $w \in W$ and $u \in U_1$.

As in the proof of Theorem 2.1, the diameter of G is at most 1. So each $v \in U_2$ should be adjacent to at least one of the vertices in $U_1 \cup W$. If there are one adjacent vertex and one non-adjacent vertex to v in this set, the induced subgraph with these two vertices and v_1 and v doesn't pass the property. This means that every $v \in U_2$ should be adjacent to all vertices in $U_1 \cap W$. By Theorem 2.1, $U_1 \cup U_2$ and $U_2 \cup W$ form balance complete bipartite graphs. Without loss of generality, assume that $\varphi(u_1 v) = 1$. Since for $u \in U_1$ the cycle $v_1 u_1 v u v_1$ is neutral, $\varphi(uv) = 1$. Similar discussion on induced graph on v_1, v, u_1, w for some $v \in U_2$ and $w \in W$ shows that $\varphi(vw) = 1$. □

3. Lower bounds for the energy

In this section, we present some lower bounds for the energy of $\mathbb{T}$-gain graphs based on the number of their edges and their matching number.

The following theorem based on the results of Section 2. Its analogue for simple graphs can be found in [8, Theorem 5.2].

Theorem 3.1. *For a $\mathbb{T}$-gain graph Φ with m edges,*

$$2\sqrt{m} \leq E(\Phi).$$

The equality holds if and only if Φ has a component switching equivalent to a complete bipartite graph $K_{a,b}$ or a complete 3-partite graph described in Example 2.3 and possibly some isolated vertices.

Proof. Let $\lambda_1, \ldots, \lambda_n$, be the eigenvalues of Φ. We know that $\sum_{i=1}^{n} \lambda_i = trace(A) = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = trace(A^2) = 2m$. From

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j,$$

it follows that $\sum_{i \neq j} \lambda_i \lambda_j = -2m$. In addition,

$$E(\Phi)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j} |\lambda_i \lambda_j| \geq \sum_{i=1}^{n} \lambda_i^2 + \left|\sum_{i \neq j} \lambda_i \lambda_j\right| = 4m.$$

The equality holds if and only if all $\lambda_i \lambda_j$ for $i \neq j$ have the same sign. Thus Φ has exactly one positive and one negative eigenvalues. Now the result follows from Theorems 2.4 and 2.1. □

In the next lemmas, we state how edge-delation decreases the energy of a graph.

Lemma 3.2. *[12, Lemma 4.2] Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph and F be a cut set of Φ. Then $E(\Phi - F) \leq E(\Phi)$.*

The next lemma is ststed in [5, Theorem 3.6] for simple graphs. We demonstrate that it is also true for $\mathbb{T}$-gain graphs.

Lemma 3.3. *Let L and M be two complementary induced subgraphs of a graph G and F be the cut set in between them. If F is not empty and all edges in F are incident to one and only one vertex in M, then $E(G - F) < E(G)$.*

Proof. Consider that G is connected. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$ be the positive eigenvalues of G. since $\Sigma_{j=1}^{n} \lambda_j = 0$, it follows that $E(G) = 2(\lambda_1 + \ldots + \lambda_r)$ Let $\mu_1, \mu_2, \ldots$, be the eigenvalues of $G - v$ by interlacing

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_r \geq \mu_s, \quad \text{and} \quad 0 \geq \lambda_{r+1} \geq \mu_{r+1}.$$

So the positive eigenvalues of $G - v$ are $\mu_1, \ldots, \mu_q$ where $s = r$ or $r - 1$. Also, from the Perron–Frobenius Theorem [6, p.31] $\lambda_1 > \mu_1$. Therefore,

$$E(G) = 2(\lambda_1 + \ldots + \lambda_r) > 2(\mu_1 + \ldots + \mu_s) = E(G - v).$$

□

Lemma 3.4. *[2, Theorem 1] Let A be a Hermitian matrix with the block form*

$$A = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix}.$$

Then, $E(A) \geq 2E(D)$.

Lemma 3.5. *Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph and F be a cut of Φ. Then, $E(F) \leq E(\Phi)$.*

Proof. By reordering of vertices, the adjacency matrix of Φ can be written as $A(\Phi) = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix}$, where $A(F) = \begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix}$ is the adjacency matrix of F. Note that the eigenvalues of $A(F)$ are the singular values of D and their negatives, so $E(F) = 2E(D)$. Then, Lemma 3.4 implies $E(\Phi) \geq 2E(D)$ and the result stands. □

Now, as a consequence of Lemma 3.5, we state the following result.

Proposition 3.6. *Adding any number of edges to each part of a bipartite $\mathbb{T}$-gain graph, does not decrease its energy.*

Theorem 3.7. *Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph whose cycles are vertex-disjoint and have odd lengths. If T is an spanning tree of G, then $E(\Phi) \geq E(T)$.*

Proof. The tree T can be considered as a bipartite graph and all other edges of G (which are not in T) have two vertices in one of the parts. So by Proposition 3.6, we are done. □

Lemma 3.8. *[10, Corollary 3.1] Let $\Phi = (G, \varphi)$ be any $\mathbb{T}$-gain graph and $P_\Phi(x) = x^n + a_1 x^n ... + a_n$ be the characteristics polynomial of Φ. Suppose that $\mathcal{H}_i(G)$ is the set of elementary subgraphs of G with i vertices and $p(H)$ and $c(H)$ are the number of components and cycles of H, respectively. In this case*

$$a_i = \textstyle\sum_{H \in \mathcal{H}_i(G)} (-1)^{p(H)} 2^{c(H)} \prod_{C \in C(H)} \Re(C)$$

where $C(H)$ is the collection of cycles in H.

In proof of the Coulson integral formula [8, P. 23], we can see that the graph weights have no effect on the formula and we have

$$E(\Phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left[\left(\sum_{k \geq 0} (-1)^k a_{2k} x^{2k} \right)^2 + \left(\sum_{k \geq 0} (-1)^k a_{2k+1} x^{2k+1} \right)^2 \right] dx \tag{1}$$

Lemma 3.9. *Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph whose cycles are vertex disjoint and the length of every cycle is 2 (mod 4). If the tree T is obtained by removing one edge from each cycle of G, then $E(\Phi) \geq E(T)$.*

Proof. Suppose that $C_{2r_1}, \ldots, C_{2r_s}$ are all cycles of G with $r_i = 2l_i + 1$, $1 \leq i \leq s$. Since G and T does not have odd cycles, each elementary subgraph of them has even vertices. So from (1), we have

$$E(\Phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left(\sum_{k \geq 0} (-1)^k a_{2k} x^{2k} \right)^2 dx$$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left(\sum_{k \geq 0} (-1)^k a'_{2k} x^{2k} \right)^2 dx$$

Note that for each elementary subgraph H with $2k$ vertices, the number of connected components is k modulo 2. Because every cycle C_{2r_i} decreases the number of K_2 in $p(H)$ by r_i wich is an odd number and add 1 component to $p(H)$. Thus $b_{2k} = (-1)^k a_{2k} \geq 0$ and $b'_{2k} = (-1)^k a'_{2k} \geq 0$.

On the other hand, if $m_k(G)$ and $m_k(T)$ are the number of k-matchings in G and T, we have

$$b_{2k} \geq m_k(G) \geq m_k(T) = b'_{2k}.$$

Therefore, it follows that $E(\Phi) \geq E(T)$. □

The following lemma discusses about the underlaying graph and so are also valid in $\mathbb{T}$-gain graphs.

Lemma 3.10. *[3, lemma 3] Let G be any connected graph on n vertices and* $\mu(G) = k$*. Then G has a maximum matching M and an edge* $uv \in M$ *with* $\mu(G - \{u, v\}) = k - 1$ *in such a way that:*
• *if G has perfect matching, then* $G - \{u, v\}$ *is connected;*
• *if G does not have perfect matching, then* $G - \{u, v\}$ *has a connected component with at least* $2k - 1$ *vertices and possibly some isolated points.*

In the next result, we establish a connection between the energy and the matching number of a gain graph. In [12], it was established that $2\mu(\Phi)$ is a lower bound for energy of a $\mathbb{T}$-gain graph and the authors characterized the class of $\mathbb{T}$-gain graphs for which the equality holds. In what follows, we get a simpler proof for the equality condition of $E(\Phi) \geq 2\mu(\Phi)$. Our method is similar to the one presented in [3].

Theorem 3.11. *Let* $\Phi = (G, \varphi)$ *be any* $\mathbb{T}$*-gain graph. Then* $E(\Phi) \geq 2\mu(\Phi)$ *and* $E(\Phi) = 2\mu(\Phi)$ *if and only if* Φ *is a union of some balanced complete bipartite graphs with parts of equal sizes that is switching equivalent to* $(K_{r,r}, 1)$ *and possibly some isolated vertices.*

Proof. We proceed by induction on $\mu(\Phi)$. With no loss of generality, we assume that G is connected. Let M be a maximum matching of G, $e_1 = uv \in M$, H_1 and H_2 be the induced subgraphs on $\{u, v\}$ and $V(G) \setminus \{u, v\}$, respectively. Then $\mu(H_1) = 1$ and $\mu(H_2) = k - 1$. If E is the cut between $\{u, v\}$ and $V(G) \setminus \{u, v\}$, then $G - E$ consists of two graph H_1 and H_2. By Lemma 3.2 and the induction hypothesis,

$$E(\Phi) \geqslant E(\Phi - E) = E((H_1, \varphi)) + E((H_2, \varphi)) \geqslant 2 + 2(k - 1) = 2\mu(\Phi).$$

For the equality case, if G is a balanced complete bipartite graphs with parts of equal sizes, then $\Phi \sim (K_{n,n}, 1)$ and $E(\Phi) = 2n = 2\mu(\Phi)$. For the converse, if $E(\Phi) = 2\mu(\Phi) = 2k$ and the maximum matching of G misses a vertex v, then by Lemma 3.3, $E(\Phi) > E(\Phi - v) \geq 2k$. So G has $2k$ vertices. Let $u_1v_1, ..., u_kv_k$ be a perfect matching of G. By induction on k, we show that $\Phi \sim (K_{k,k}, 1)$. This is obvious for $k = 1$ as $\Phi \sim (K_{1,1}, 1)$ and $E(G) = 2$. For $k = 2$, there are (up to switching equivalency) only 5 possible graphs, on 4 vertices with perfect matching as shown in Figure 1.

If Φ is as Figure 1 (i), then $E(\Phi) = 4.4721 > 4$ and for $(ii) - (v)$, by Theorem 3.1, the only desired case is $\Phi \sim (K_{2,2}, 1)$.

Let $k \geq 3$. As G is connected by Lemma 3.10, we may assume that the induced subgraph H on $u_2, v_2, ..., u_k, v_k$ is connected and $E((H, \varphi)) = 2\mu(H) = 2(k - 1)$. So by induction, $H \sim (K_{k-1,k-1}, 1)$. Thus let $\varphi(u_iv_j) = 1$ for all $2 \leq i, j \leq k$. Since G is connected, u_1 or v_1 is adjacent with some u_t or v_t. Without losing the generality, we can assume that u_1 is adjacent to v_2 and passing through a switching function η with appropriate value of u_1 and v_1, let $\varphi(u_1v_2) = \varphi(u_1v_1) = 1$. Since $G - \{u_k, v_k\}$ is connected, from induction hypothesis, it is balanced complete bipartite. For each $2 < i < k$, the gain of any cycle $u_1v_2u_iv_iu_1$ is 1. So $\varphi(u_1v_i) = 1$. Similarly, considering any cycle $u_1v_1u_jv_ju_1$ for $2 \leq j < k$, it follows that $\varphi(u_jv_1) = 1$. Again, by induction hypothesis $G - \{u_2, v_2\}$ is balanced complete bipartite. Thus $\varphi(u_1v_k) = \varphi(v_1u_k) = 1$ and the result follows.

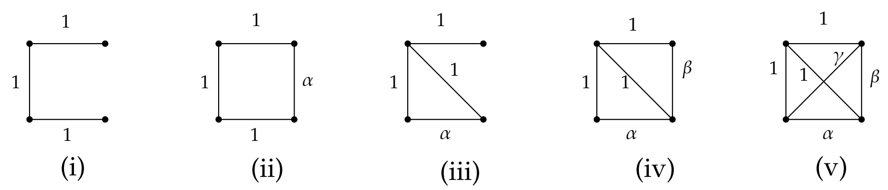

Figure 1: connected graphs with 4 vertices and matching number 2

□

Theorem 3.12. *Let $\Phi = (G, \varphi)$ be any connected $\mathbb{T}$-gain graph with no perfect matching and $\mu(\Phi) = k$. Then $E(\Phi) \geq 2k + 1$ except for $G \sim (K_{k,k+1}, 1)$.*

Proof. We show that if G is connected with no perfect matching and $E(\Phi) < 2\mu(\Phi) + 1 = 2k + 1$, then $G \sim (K_{k,k+1}, 1)$. We prove this by induction on k. All connected graphs G with $n \leq 4$ vertices and with no perfect matching satisfy $E(\Phi) \geq 2\mu(\Phi) + 1$ ($\mu(\Phi) = 1$) except for $G \sim K_{1,2}$. Let $n \geq 5$. If $\mu(\Phi) = 1$, then $G \sim (K_{1,n-1}, 1)$ for which $E(\Phi) = 2\sqrt{n-1} > 3$. So assume that $\mu(\Phi) \geq 2$. Let $u_1v_1, ..., u_kv_k$ be a maximum matching of G. By Lemma 3.10, we have that $G - \{u_1, v_1\} = H \cup sK_1$, where H is connected with at least $2k-1$ vertices, $\mu(H) = k - 1$, and $s \geq 0$. We have

$$2k + 1 > E(\Phi) \geq 2 + E(\Phi - \{u_1, v_1\}) \geq 2 + E(H).$$

This means that $E(H) < 2k - 1$ and from the induction hypothesis it follows that $H \sim (K_{k-1,k}, 1)$. On the other hand, the possible isolated vertices of $G - \{u_1, v_1\}$ must be connected to u_1 or v_1 in G. Since G is connected, u_1 or v_1 is adjacent with some u_t or v_t. Arguing as in the proof of Theorem 3.11, we can deduce that $G \sim K_{k,k+1}$. □

Competing interest

The authors declare that they have no competing financial interests in this paper.

Acknowledgement

The authors would like to express their sincere gratitude to Dr. Ebrahim Ghorbani for his valuable guidance and support throughout this research and for his comments on the last version that greatly improved the manuscript.

References

[1] S. Akbari, K.C. Das, M. Ghahremani, F. Koorepazan Moftakhar, E. Raoufi, *Energy of graphs containing disjoint cycles*, Match Commun. Math. Comput. Chem. **86** (2021), 543–547.
[2] S. Akbari, A. Ghodrat, M. Hosseinzadeh, *Some Lower Bounds for the Energy of Graphs*, Linear Algebra Appl. **591** (2020), 205–214.
[3] F. Ashraf, *Energy, matching number and odd cycles of graphs*, Linear Algebra Appl. **577** (2019), 159–167.
[4] D. Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic, New York, 1980.
[5] J. Day, W. So, *Graph energy change due to edge deletion*, Linear Algebra Appl. **428** (2008), 2070–2078.
[6] C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993.
[7] X.C. He, L.H. Feng, L. Lu, *On $\mathbb{T}_4$-gain graphs with few positive eigenvalues*, Linear multilinear Algebra **71** (2023), 2780–2795.
[8] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
[9] L. Lu, J. Wang, Q. Huang, *Complex unit gain graphs with exactly one positive eigenvalue*, Linear Algebra Appl. **608** (2021), 270–281.
[10] R. Mehataria, M. R Kannan, A. Samanta, *On the adjacency matrix of a complex unit gain graph*, Linear Algebra Appl. **436** (2019), 1–14.
[11] N. Reff, *Spectral properties of complex unit gain graphs*, Linear Algebra Appl. **436** (2012), 3165–3176.
[12] A. Samanta, M.R. Kannan, *Bounds for the energy of a complex unit gain graph*, Linear Algebra Appl. **612** (2021), 1–29.
[13] Q. Xu, D. Zhou, D. Wong, F. Tian, *Complex unit gain graphs of rank 2*, Linear Algebra Appl. **597** (2020), 155-160.
[14] S. Zaman, X.C. He, *Relation between the inertia indices of a complex unit gain graph and those of its underlying graph*, Linear multilinear Algebra **70** (2022), 843–877.