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# New properties of the core–EP pre-order

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**Abstract.** The first aim of this paper is to present new characterizations of the core–EP pre-order $A \leq^{\textcircled{D}} B$ between two Hilbert space operators based on corresponding self-adjoint operators and the powers of the core–EP inverse. Under the relation $A \leq^{\textcircled{D}} B$, we further establish equivalent conditions for the forward order law $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$ to hold. We give conditions for the equivalence between the forward order law $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$ and the reverse order law $(AB)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$. Also, in the case that $A \leq^{\textcircled{D}} B$, necessary and sufficient conditions for $(B-A)^{\textcircled{D}} = B^{\textcircled{D}} - A^{\textcircled{D}}$ are studied. Applying our results, we obtain characterizations for the core partial order and the forward order law for the core inverse.

## 1. Introduction

Let $X$ and $Y$ be arbitrary Hilbert spaces and $\mathcal{B}(X,Y)$ be the set of all bounded linear operators from $X$ to $Y$. Set $\mathcal{B}(X) = \mathcal{B}(X,X)$. The adjoint, null space and range of $A \in \mathcal{B}(X,Y)$ are denoted by $A^*$, $N(A)$ and $R(A)$, respectively.

The concept of generalized inverses has significant applications in different fields of mathematics such as matrix theory, operator theory, differential equations, and in technics and engineering [2]. The definitions of well-known generalized inverses are given now. The Moore–Penrose inverse of $A \in \mathcal{B}(X,Y)$ presents the unique operator $B \in \mathcal{B}(Y,X)$ (denoted by $A^\dagger$) [2] such that $ABA = A$, $BAB = B$, $(AB)^* = AB$, $(BA)^* = BA$. Note that $A^\dagger$ exists if and only if $R(A)$ is closed in $Y$.

The Drazin inverse of $A \in \mathcal{B}(X)$ is the unique operator $B \in \mathcal{B}(X)$ (denoted by $A^D$) [2] which satisfies $BAB = B$, $AB = BA$ and $A^{k+1}B = A^k$, where $k = i(A)$ is the index of $A$, i.e. the smallest non-negative integer $k$ in this definition. When $i(A) \leq 1$, $A^D = A^\#$ is the group inverse of $A$. We use $\mathcal{B}(X)^D$ and $\mathcal{B}(X)^\#$ to mark the sets of all Drazin invertible and group invertible operators in $\mathcal{B}(X)$, respectively.

The notion of the core–EP inverse was presented for a square matrix [18] and extended to a Drazin invertible Hilbert space operator [14, 16]. For $A \in \mathcal{B}(X)^D$ with $i(A) = k$, there exists its unique core–EP

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inverse $B \in \mathcal{B}(X)$ (denoted by $A^{\textcircled{\dagger}}$) satisfying [14]

$$BAB = B \quad \text{and} \quad R(B) = R(B^*) = R(A^k),$$

and represented by [6]

$$A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger.$$

Especially, if $i(A) \leq 1$, $A^{\textcircled{\dagger}} = A^{\textcircled{\#}} = A^\# A A^\dagger$ is the core inverse of $A$ [1].

It is known that a binary relation on a non-empty set is a pre-order if it is reflexive and transitive. If it is also anti-symmetric, then it is called a partial order. Generalized inverses have an important role in defining and studying pre-orders and partial orders.

Using the core–EP inverse, the core–EP pre-order was defined in [14], for $A, B \in \mathcal{B}(X)^D$, as

$$A \leq^{\textcircled{\dagger}} B \quad \text{when} \quad AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}} \quad \text{and} \quad A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B.$$

For various characterizations of the core–EP pre-order see [21] for complex matrix case, [14] for operators, [15] for elements of $C^*$-algebra and [5, 7] for elements of a ring with involution. In particular, when $A, B \in \mathcal{B}(X)^\#$, the core partial order is given by [1, 19]

$$A \leq^{\textcircled{\#}} B \quad \text{when} \quad AA^{\textcircled{\#}} = BA^{\textcircled{\#}} \quad \text{and} \quad A^{\textcircled{\#}}A = A^{\textcircled{\#}}B.$$

Interesting properties of the core partial orders can be found in [9].

One of fundamental problems in the theory of generalized inverses is to find generalized inverses of products [3, 4, 10, 13]. For the core inverse, necessary and sufficient conditions for the reverse order law $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$ and the forward order law $(AB)^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$ to hold were investigated by many authors [8, 20, 22, 24]. Note that, by [23, Theorem 2.10], the core partial order $A \leq^{\textcircled{\#}} B$ implies $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}} = (BA)^{\textcircled{\#}}$. The assumptions under which the reverse order law $(AB)^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}A^{\textcircled{\dagger}}$ is satisfied for the core–EP inverse were presented in [6, 12].

Motivated by importance and different results about core-EP pre-order and core partial order, we continue to investigate this topic and get more properties applying these binary relations. Our first goal is to establish new characterizations for the core–EP pre-order $A \leq^{\textcircled{\dagger}} B$ using corresponding self-adjoint operators and the powers of the core–EP inverse. Further, in the case that $A \leq^{\textcircled{\dagger}} B$, we show characterizations of the forward order law $(AB)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}B^{\textcircled{\dagger}}$. Necessary and sufficient conditions for the equivalence between the forward order law $(AB)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}B^{\textcircled{\dagger}}$ and the reverse order law $(AB)^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}A^{\textcircled{\dagger}}$ are considered. Under the condition $A \leq^{\textcircled{\dagger}} B$, we give equivalent conditions for $(B - A)^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}$ to be satisfied. Applying previous mentioned results, we get characterizations for the core partial order and properties of the forward order law for the core inverse.

The content of this paper in details follows. Section 2 contains new characterizations of the core–EP pre-order and consequently characterizations of the core partial order. Several equivalent conditions for the forward order law $(AB)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}B^{\textcircled{\dagger}}$ and the difference $(B - A)^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}$ are part of Section 3.

## 2. Characterizations of the core–EP pre-order

This section is devoted to new characterizations of the core–EP pre-order. At the beginning, we state one auxiliary result which will be useful in the sequel.

**Lemma 2.1.** [14, Corollary 3.7] *For $A \in \mathcal{B}(X)^D$ with $k = i(A)$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger}} B$;

(ii) *there are the following matrix representations with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 \\ 0 & B_3 \end{bmatrix},$$

*where $A_1 \in \mathcal{B}(R(A^k))$ is invertible and $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent.*

In the first theorem, we develop equivalent conditions, which include $BA^{\textcircled{\dagger}}$ is self-adjoint, for the pre-order $A \leq^{\textcircled{\dagger}} B$ to hold.

**Theorem 2.2.** *For $A \in \mathcal{B}(X)^D$ with $k = i(A)$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger}} B$;

(ii) $BA^{\textcircled{\dagger}}$ *is self-adjoint and* $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$;

(iii) $BA^{\textcircled{\dagger}}$ *is self-adjoint and* $AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}B$;

(iv) $BA^{\textcircled{\dagger}}$ *is self-adjoint and* $A^*A^k = B^*A^k$;

(v) $BA^{\textcircled{\dagger}}$ *is self-adjoint and* $A^*A^{\textcircled{\dagger}} = B^*A^{\textcircled{\dagger}}$;

(vi) $BA^{\textcircled{\dagger}}$ *is self-adjoint and* $A^*A^D = B^*A^D$.

*Proof.* (i) ⇔ (ii): By [14, Corollary 2.2], $A$ and $A^{\textcircled{\dagger}}$ can be represented with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ as

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad A^{\textcircled{\dagger}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where $A_1 \in \mathcal{B}(R(A^k))$ is invertible and $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent. Set

$$B = \begin{bmatrix} B_1 & B_2 \\ B_4 & B_3 \end{bmatrix} : \begin{bmatrix} R(A^k) \\ N((A^k)^*) \end{bmatrix} \to \begin{bmatrix} R(A^k) \\ N((A^k)^*) \end{bmatrix}. \tag{2}$$

Firstly, notice that $BA^{\textcircled{\dagger}} = \begin{bmatrix} B_1A_1^{-1} & 0 \\ B_4A_1^{-1} & 0 \end{bmatrix}$ is self-adjoint if and only if $B_1A_1^{-1}$ is self-adjoint and $B_4 = 0$. Further, $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$ is equivalent to

$$\begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1}B_1 & A_1^{-1}B_2 \\ 0 & 0 \end{bmatrix},$$

that is, $B_1 = A_1$ and $B_2 = A_2$. Using Lemma 2.1, we deduce that statements (i) and (ii) are equivalent.

(ii) ⇔ (iii): It is evident by elementary calculations and $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.

(iii) ⇔ (iv): We have, by $A^{\textcircled{\dagger}} = A^DA^k(A^k)^\dagger$ and $(A^k)^* = (A^k)^*A^k(A^k)^\dagger$, that $AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}B$ if and only if $A^k(A^k)^\dagger A = A^k(A^k)^\dagger B$, which is equivalent to $(A^k)^*A = (A^k)^*B$, i.e. $A^*A^k = B^*A^k$.

(iv) ⇔ (v): This part is clear by $A^{\textcircled{\dagger}} = A^kA^D(A^k)^\dagger$ and $A^k = A^{\textcircled{\dagger}}A^{k+1}$.

(iv) ⇔ (vi): By properties of the Drazin inverse, it is clear. □

In the case that $i(A) = 1$ in Theorem 2.2, we obtain the following characterizations for the core partial order.

**Corollary 2.3.** *For $A \in \mathcal{B}(X)^\#$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\#}} B$;

(ii) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$;

(iii) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A = AA^{\textcircled{\#}}B$;

(iv) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A^*A = B^*A$;

(v) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A^*A^{\textcircled{\#}} = B^*A^{\textcircled{\#}}$;

(vi) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A^*A^\# = B^*A^\#$.

We study necessary and sufficient conditions, which involve $(AA^{\textcircled{\dagger}}A)^*B$ is self-adjoint, for the relation $A \leq^{\textcircled{\dagger}} B$ to be satisfied.

**Theorem 2.4.** *For $A \in \mathcal{B}(X)^D$ with $k = i(A)$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger}} B$;

(ii) $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$;

(iii) $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint and* $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}A$;

(iv) $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint and* $A^{k+1} = BA^k$;

(v) $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint and* $A^{\textcircled{\dagger}}A^* = A^{\textcircled{\dagger}}B^*$;

(vi) $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint and* $AA^D = BA^D$.

*Proof.* (i) ⇔ (ii): Suppose that $A$, $A^{\textcircled{\dagger}}$ and $B$ are expressed as in (1) and (2). Now, $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ if and only if

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1A_1^{-1} & 0 \\ B_4A_1^{-1} & 0 \end{bmatrix},$$

which is equivalent to $B_1 = A_1$ and $B_4 = 0$. Then

$$(AA^{\textcircled{\dagger}}A)^*B = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}\begin{bmatrix} A_1 & B_2 \\ 0 & B_3 \end{bmatrix} = \begin{bmatrix} A_1^*A_1 & A_1^*B_2 \\ A_2^*A_1 & A_2^*B_2 \end{bmatrix}$$

is self-adjoint if and only if $B_2 = A_2$. Lemma 2.1 implies that (i) is equivalent to (ii).

Similarly as Theorem 2.2, we complete this proof. □

Remark that the condition $(AA^{\textcircled{\dagger}}A)^*B$ is self-adjoint is equivalent to $A^*AA^{\textcircled{\dagger}}B$ (or $BAA^{\textcircled{\dagger}}A$) is self-adjoint. Theorem 2.4 implies the next consequence about the core partial order.

**Corollary 2.5.** *For $A \in \mathcal{B}(X)^\#$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\#}} B$;

(ii) $A^*B$ *is self-adjoint and* $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$;

(iii) $A^*B$ *is self-adjoint and* $A = BA^{\textcircled{\#}}A$;

(iv) $A^*B$ *is self-adjoint and* $A^2 = BA$;

(v) $A^*B$ *is self-adjoint and* $A^{\textcircled{\#}}A^* = A^{\textcircled{\#}}B^*$;

(vi) $A^*B$ *is self-adjoint and* $AA^\# = BA^\#$.

Based on the powers of the core–EP inverse $A^{\textcircled{\dagger}}$, we characterize $A \leq^{\textcircled{\dagger}} B$ as follows.

**Theorem 2.6.** *For $n \in \mathbb{N}$, $A \in \mathcal{B}(X)^D$ with $k = i(A)$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger}} B$;

(ii) $B(A^{\textcircled{\dagger}})^{n+1} = (A^{\textcircled{\dagger}})^n$ *and* $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$;

(iii) $B(A^D)^{n+1} = (A^D)^n$ *and* $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$;

(iv) $B(A^{\textcircled{\dagger}})^{n+1} = (A^{\textcircled{\dagger}})^n$ *and* $(AA^{\textcircled{\dagger}}A)^*B$ *is self-adjoint;*

(v) $B(A^{\textcircled{\dagger}})^{n+1} = (A^{\textcircled{\dagger}})^n$ *and* $(AA^{\textcircled{\dagger}}A)^*(A - B) = 0$;

(vi) $BA^{\textcircled{D}}$ *is self-adjoint and* $(AA^{\textcircled{D}}A)^*(A-B)=0$.

*Proof.* (i) ⇒ (ii): The hypothesis $A \leq^{\textcircled{D}} B$ implies $AA^{\textcircled{D}} = BA^{\textcircled{D}}$ and $A^{\textcircled{D}}A = A^{\textcircled{D}}B$. Hence, $(A^{\textcircled{D}})^n = A(A^{\textcircled{D}})^{n+1} = B(A^{\textcircled{D}})^{n+1}$.

(ii) ⇒ (iii): By [17, Lemma 2.1], $(A^{\textcircled{D}})^n = (A^D)^n A^k (A^k)^\dagger$, which gives

$$\begin{aligned} (A^{\textcircled{D}})^n AA^D &= (A^D)^n A^k (A^k)^\dagger AA^D = (A^D)^n (A^k (A^k)^\dagger A^k)(A^D)^k \\ &= (A^D)^n A^k (A^D)^k = (A^D)^n AA^D = (A^D)^n, \end{aligned} \tag{3}$$

for arbitrary $n \in \mathbb{N}$. Using (3) and $B(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n$, it follows

$$B(A^D)^{n+1} = (B(A^{\textcircled{D}})^{n+1})AA^D = (A^{\textcircled{D}})^n AA^D = (A^D)^n.$$

(iii) ⇒ (i): Applying $A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$ and $B(A^D)^{n+1} = (A^D)^n$, we obtain

$$\begin{aligned} BA^{\textcircled{D}} &= BA^D A^k (A^k)^\dagger = (B(A^D)^{n+1})A^{k+n}(A^k)^\dagger \\ &= (A^D)^n A^{k+n} (A^k)^\dagger = A(A^D A^k (A^k)^\dagger) \\ &= AA^{\textcircled{D}}. \end{aligned}$$

(i) ⇔ (iv): Assume that $A$, $A^{\textcircled{D}}$ and $B$ are given by (1) and (2). We observe that $B(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n$ if and only if

$$\begin{bmatrix} B_1 A_1^{-(n+1)} & 0 \\ B_4 A_1^{-(n+1)} & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-n} & 0 \\ 0 & 0 \end{bmatrix},$$

which is equivalent to $B_1 = A_1$ and $B_4 = 0$. Now, $(AA^{\textcircled{D}}A)^*B = \begin{bmatrix} A_1^*A_1 & A_1^*B_2 \\ A_2^*A_1 & A_2^*B_2 \end{bmatrix}$ is self-adjoint if and only if $B_2 = A_2$. By Lemma 2.1, (i) and (iv) are equivalent.

(i) ⇔ (v): If $A$, $A^{\textcircled{D}}$ and $B$ are presented by (1) and (2), as in the part (i) ⇔ (iv), note that $B(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n$ is equivalent to $B_1 = A_1$ and $B_4 = 0$. Then, from

$$(AA^{\textcircled{D}}A)^*(A-B) = \begin{bmatrix} 0 & A_1^*(A_2 - B_2) \\ 0 & A_2^*(A_2 - B_2) \end{bmatrix},$$

$(AA^{\textcircled{D}}A)^*(A-B) = 0$ if and only if $B_2 = A_2$.

(i) ⇔ (vi): Let $A$, $A^{\textcircled{D}}$ and $B$ have the forms as in (1) and (2). Because

$$(AA^{\textcircled{D}}A)^*(A-B) = \begin{bmatrix} A_1^*(A_1 - B_1) & A_1^*(A_2 - B_2) \\ A_2^*(A_1 - B_1) & A_2^*(A_2 - B_2) \end{bmatrix},$$

we conclude that $(AA^{\textcircled{D}}A)^*(A-B) = 0$ is equivalent to $B_1 = A_1$ and $B_2 = A_2$. Also, $BA^{\textcircled{D}} = \begin{bmatrix} I & 0 \\ B_4 A_1^{-1} & 0 \end{bmatrix}$ is self-adjoint if and only if $B_4 = 0$. □

By Theorem 2.6, we verify the next result. Note that part (iv) of Corollary 2.7 recovers part (iv) of [25, Theorem 3.10].

**Corollary 2.7.** *For* $n \in \mathbb{N}$, $A \in \mathcal{B}(X)^{\#}$ *and* $B \in \mathcal{B}(X)$, *the following statements are equivalent:*

(i) $A \leq^{\textcircled{\#}} B$;

(ii) $B(A^{\textcircled{\#}})^{n+1} = (A^{\textcircled{\#}})^n$ *and* $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$;

(iii) $B(A^{\#})^{n+1} = (A^{\#})^n$ *and* $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$;

(iv) $B(A^{\textcircled{\#}})^{n+1} = (A^{\textcircled{\#}})^n$ *and* $A^*B$ *is self-adjoint;*

(v) $B(A^{\textcircled{\#}})^{n+1} = (A^{\textcircled{\#}})^n$ *and* $A^*A = A^*B$;

(vi) $BA^{\textcircled{\#}}$ *is self-adjoint and* $A^*A = A^*B$.

We also prove the following characterizations for $A \leq^{\textcircled{D}} B$.

**Theorem 2.8.** *For* $A \in \mathcal{B}(X)^D$ *with* $k = i(A)$ *and* $B \in \mathcal{B}(X)$, *the following statements are equivalent:*

(i) $A \leq^{\textcircled{D}} B$;

(ii) $BA^{\textcircled{D}} = AA^{\textcircled{D}}BA^{\textcircled{D}}$ *and* $A^{\textcircled{D}}B = A^{\textcircled{D}}A$;

(iii) $BA^{\textcircled{D}}A = AA^{\textcircled{D}}BA^{\textcircled{D}}A$ *and* $AA^{\textcircled{D}}B = AA^{\textcircled{D}}A$;

(iv) $BA^{\textcircled{D}} = AA^{\textcircled{D}}BA^{\textcircled{D}}$ *and* $B^*A^k = A^*A^k$;

(v) $BA^{\textcircled{D}} = AA^{\textcircled{D}}BA^{\textcircled{D}}$, $A^{\textcircled{D}}BA^{\textcircled{D}}A = A^{\textcircled{D}}A$ *and* $A^{\textcircled{D}}B(I - AA^{\textcircled{D}}) = A^{\textcircled{D}}A(I - AA^{\textcircled{D}})$;

(vi) $BA^{\textcircled{D}}A = AA^{\textcircled{D}}BA^{\textcircled{D}}A$, $A^{\textcircled{D}}BA^{\textcircled{D}} = A^{\textcircled{D}}$ *and* $AA^{\textcircled{D}}B(I - AA^{\textcircled{D}}) = AA^{\textcircled{D}}A(I - AA^{\textcircled{D}})$.

*Proof.* (i) ⇔ (ii): Let $A$, $A^{\textcircled{D}}$ and $B$ be given by (1) and (2). Then $BA^{\textcircled{D}} = AA^{\textcircled{D}}BA^{\textcircled{D}}$ if and only if $(I - AA^{\textcircled{D}})BA^{\textcircled{D}} = 0$ which is equivalent to

$$\begin{bmatrix} 0 & 0 \\ B_4A_1^{-1} & 0 \end{bmatrix} = 0,$$

that is, $B_4 = 0$. As in the proof of Theorem 2.2, $A^{\textcircled{D}}B = A^{\textcircled{D}}A$ if and only if $B_1 = A_1$ and $B_2 = A_2$. The rest is clear by Lemma 2.1.

(ii) ⇔ (iii) ⇔ (iv): These equivalences can be verified as in Theorem 2.2.

(i) ⇔ (v): Suppose that $A$, $A^{\textcircled{D}}$ and $B$ have the forms as in (1) and (2). As in part (i) ⇔ (ii), we show that $BA^{\textcircled{D}} = AA^{\textcircled{D}}BA^{\textcircled{D}}$ is equivalent to $B_4 = 0$. The equality $A^{\textcircled{D}}BA^{\textcircled{D}}A = A^{\textcircled{D}}A$ holds if and only if

$$\begin{bmatrix} A_1^{-1}B_1 & A_1^{-1}B_1A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

if and only if $B_1 = A_1$. Also, $A^{\textcircled{D}}B(I - AA^{\textcircled{D}}) = A^{\textcircled{D}}A(I - AA^{\textcircled{D}})$ is equivalent to

$$\begin{bmatrix} 0 & A_1^{-1}B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix},$$

i.e. $B_2 = A_2$.

(vi) ⇔ (v): It is clear. □

Consequently, we characterize $A \leq^{\textcircled{\#}} B$ in the next manner.

**Corollary 2.9.** *For* $A \in \mathcal{B}(X)^\#$ *and* $B \in \mathcal{B}(X)$, *the following statements are equivalent:*

(i) $A \leq^{\textcircled{\#}} B$;

(ii) $BA^{\textcircled{\#}} = AA^{\textcircled{\#}}BA^{\textcircled{\#}}$ *and* $A^{\textcircled{\#}}B = A^{\textcircled{\#}}A$;

(iii) $BA^{\textcircled{\#}}A = AA^{\textcircled{\#}}BA^{\textcircled{\#}}A$ *and* $AA^{\textcircled{\#}}B = A$;

(iv) $BA^{\textcircled{\#}} = AA^{\textcircled{\#}}BA^{\textcircled{\#}}$ *and* $B^*A = A^*A$;

(v) $BA^{\textcircled{\#}} = AA^{\textcircled{\#}}BA^{\textcircled{\#}}$, $A^{\textcircled{\#}}BA^{\textcircled{\#}}A = A^{\textcircled{\#}}A$ *and* $A^{\textcircled{\#}}B(I - AA^{\textcircled{\#}}) = A^{\textcircled{\#}}A - AA^{\textcircled{\#}}$;

(vi) $BA^{\textcircled{\#}}A = AA^{\textcircled{\#}}BA^{\textcircled{\#}}A$, $A^{\textcircled{\#}}BA^{\textcircled{\#}} = A^{\textcircled{\#}}$ *and* $AA^{\textcircled{\#}}B(I - AA^{\textcircled{\#}}) = A(I - AA^{\textcircled{\#}})$.

Several characterizations of $A \leq^{\textcircled{D}} B$ are developed in terms of the product $BA^{\textcircled{D}}B$.

**Theorem 2.10.** *For $A \in \mathcal{B}(X)^D$ with $k = i(A)$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger}} B$;

(ii) $AA^{\textcircled{\dagger}}(BA^{\textcircled{\dagger}}B - A) = 0$ *and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$;

(iii) $A^{\textcircled{\dagger}}(BA^{\textcircled{\dagger}}B - A) = 0$ *and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$;

(iv) $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}B$ *and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

*Proof.* (i) ⇔ (ii): Assume that $A$, $A^{\textcircled{\dagger}}$ and $B$ have representations as in (1) and (2). As in the proof of Theorem 2.4, we deduce that $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ is equivalent to $B_1 = A_1$ and $B_4 = 0$. Now, by

$$BA^{\textcircled{\dagger}}B = \begin{bmatrix} A_1 & B_2 \\ 0 & 0 \end{bmatrix}$$

and

$$AA^{\textcircled{\dagger}}(BA^{\textcircled{\dagger}}B - A) = \begin{bmatrix} 0 & B_2 - A_2 \\ 0 & 0 \end{bmatrix},$$

$AA^{\textcircled{\dagger}}(BA^{\textcircled{\dagger}}B - A) = 0$ if and only if $B_2 = A_2$.

(ii) ⇔ (iii): This is obvious.

(i) ⇔ (ii): Again, using the representations of $A$, $A^{\textcircled{\dagger}}$ and $B$ as in (1) and (2), $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ if and only if $B_1 = A_1$ and $B_4 = 0$. Furthermore, $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}B$ is equivalent to

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & B_2 \\ 0 & 0 \end{bmatrix},$$

that is, $B_2 = A_2$. □

As a consequence, we characterize the relation $A \leq^{\textcircled{\#}} B$.

**Corollary 2.11.** *For $A \in \mathcal{B}(X)^{\#}$ and $B \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) $A \leq^{\textcircled{\#}} B$;

(ii) $AA^{\textcircled{\#}}(BA^{\textcircled{\#}}B - A) = 0$ *and* $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$;

(iii) $A^{\textcircled{\#}}(BA^{\textcircled{\#}}B - A) = 0$ *and* $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$;

(iv) $A = BA^{\textcircled{\#}}B$ *and* $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$.

## 3. Core–EP inverse of product and difference of two operators

Under the assumption $A \leq^{\textcircled{\dagger}} B$, we consider the core–EP inverse of the product and difference of $A$ and $B$ in this section.

We firstly study necessary and sufficient conditions for $A^{\textcircled{\dagger}} \leq^{\textcircled{\dagger}} B^{\textcircled{\dagger}}$ to hold when $A \leq^{\textcircled{\dagger}} B$.

**Theorem 3.1.** *For $A, B \in \mathcal{B}(X)^D$ such that $A \leq^{\textcircled{\dagger}} B$, the following statements are equivalent:*

(i) $A^{\textcircled{\dagger}} \leq^{\textcircled{\dagger}} B^{\textcircled{\dagger}}$;

(ii) $A^{\textcircled{\dagger}}B^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}A^{\textcircled{\dagger}}$;

(iii) $(A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}B^{\textcircled{\dagger}}$.

*Proof.* According to Lemma 2.1, $A \leq^{\textcircled{D}} B$ implies that it can be written, with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$, as

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 \\ 0 & B_3 \end{bmatrix},$$

where $k = i(A)$, $A_1 \in \mathcal{B}(R(A^k))$ is invertible and $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent. Notice that Drazin invertibility of $B$ yields Drazin invertibility of $B_3$. By [11, Lemma 2.3],

$$A^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (A^{\textcircled{D}})^{\textcircled{D}} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\textcircled{D}} \\ 0 & B_3^{\textcircled{D}} \end{bmatrix}.$$

(i) ⇔ (ii): Firstly, note that the equality

$$A^{\textcircled{D}}(A^{\textcircled{D}})^{\textcircled{D}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = B^{\textcircled{D}}(A^{\textcircled{D}})^{\textcircled{D}}$$

holds. Because

$$(A^{\textcircled{D}})^{\textcircled{D}}A^{\textcircled{D}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad (A^{\textcircled{D}})^{\textcircled{D}}B^{\textcircled{D}} = \begin{bmatrix} I & -A_2B_3^{\textcircled{D}} \\ 0 & 0 \end{bmatrix},$$

$(A^{\textcircled{D}})^{\textcircled{D}}A^{\textcircled{D}} = (A^{\textcircled{D}})^{\textcircled{D}}B^{\textcircled{D}}$ if and only if $A_2B_3^{\textcircled{D}} = 0$. Hence, (i) is equivalent to $A_2B_3^{\textcircled{D}} = 0$.

From

$$A^{\textcircled{D}}B^{\textcircled{D}} = \begin{bmatrix} A_1^{-2} & -A_1^{-2}A_2B_3^{\textcircled{D}} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B^{\textcircled{D}}A^{\textcircled{D}} = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix},$$

we have that $A^{\textcircled{D}}B^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ is equivalent to $A_2B_3^{\textcircled{D}} = 0$, that is, to (i).

(i) ⇔ (iii): Since

$$(A^{\textcircled{D}})^2 = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix},$$

$(A^{\textcircled{D}})^2 = A^{\textcircled{D}}B^{\textcircled{D}}$ if and only if $A_2B_3^{\textcircled{D}} = 0$, i.e. (i). □

In the case that $A \leq^{\textcircled{D}} B$, we present equivalent conditions for the forward order law $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$ to be satisfied. Notice that $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$ is related with the reverse order law $(AB)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$.

**Theorem 3.2.** *For $A, B \in \mathcal{B}(X)^D$ such that $A \leq^{\textcircled{D}} B$, the following statements are equivalent:*

(i) *$AB \in \mathcal{B}(X)^D$ and $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$;*

(ii) *$AB \in \mathcal{B}(X)^D$, $(AB)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ and $AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$;*

(iii) *$AB \in \mathcal{B}(X)^D$, $(AB)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ and $A^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$;*

(iv) *$AB \in \mathcal{B}(X)^D$, $(AB)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ and $AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$;*

(v) *$AB \in \mathcal{B}(X)^D$, $(AB)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ and $A^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$.*

*Proof.* Applying Lemma 2.1 and the assumption $A \leq^{\textcircled{D}} B$, with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$, we have the representations:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 \\ 0 & B_3 \end{bmatrix},$$

where $k = i(A)$, $A_1 \in \mathcal{B}(R(A^k))$ is invertible, $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent and $B_3 \in \mathcal{B}[N((A^k)^*)]^D$. Observe that

$$AB = \begin{bmatrix} A_1^2 & A_1A_2 + A_2B_3 \\ 0 & A_3B_3 \end{bmatrix}.$$

The Drazin invertibility of $AB$ implies the Drazin invertibility of $A_3B_3$. Utilizing [11, Lemma 2.3], we get

$$A^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\textcircled{D}} \\ 0 & B_3^{\textcircled{D}} \end{bmatrix}$$

and

$$(AB)^{\textcircled{D}} = \begin{bmatrix} A_1^{-2} & -A_1^{-2}(A_1A_2 + A_2B_3)(A_3B_3)^{\textcircled{D}} \\ 0 & (A_3B_3)^{\textcircled{D}} \end{bmatrix}.$$

(i) $\Leftrightarrow$ (ii): Evidently,

$$A^{\textcircled{D}}B^{\textcircled{D}} = \begin{bmatrix} A_1^{-2} & -A_1^{-2}A_2B_3^{\textcircled{D}} \\ 0 & 0 \end{bmatrix}.$$

Thus, $(AB)^{\textcircled{D}} = A^{\textcircled{D}}B^{\textcircled{D}}$ if and only if $(A_3B_3)^{\textcircled{D}} = 0$ and $A_2B_3^{\textcircled{D}} = 0$.

Since

$$(AB)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = \begin{bmatrix} 0 & -A_1^{-2}(A_1A_2 + A_2B_3)(A_3B_3)^{\textcircled{D}} \\ 0 & (A_3B_3)^{\textcircled{D}} \end{bmatrix},$$

$(AB)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ is equivalent to $(A_3B_3)^{\textcircled{D}} = 0$. The equality

$$AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = \begin{bmatrix} 0 & -A_1^{-1}A_2B_3^{\textcircled{D}} \\ 0 & 0 \end{bmatrix}$$

implies $AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ if and only if $A_2B_3^{\textcircled{D}} = 0$. So, the statements (i) and (ii) are equivalent.

(ii) $\Leftrightarrow$ (iii): By direct calculations and the equality $A^{\textcircled{D}} = A^{\textcircled{D}}AA^{\textcircled{D}}$, we show this equivalence.

(i) $\Leftrightarrow$ (iv): Utilizing

$$B^{\textcircled{D}}A^{\textcircled{D}} = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix},$$

we have that $(AB)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ is satisfied if and only if $(A_3B_3)^{\textcircled{D}} = 0$. As in the part (i) $\Leftrightarrow$ (ii), one observes that $AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ if and only if $A_2B_3^{\textcircled{D}} = 0$.

(iv) $\Leftrightarrow$ (v): It follows as (ii) $\Leftrightarrow$ (iii). □

By [23, Theorem 2.10], $A \leq^{\textcircled{\#}} B$ yields $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}} = (BA)^{\textcircled{\#}}$. Based on this fact, Theorem 3.2 gives the next characterizations of the forward order law $(AB)^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$.

**Corollary 3.3.** *For $A, B \in \mathcal{B}(X)^{\#}$ such that $A \leq^{\textcircled{\#}} B$, the following statements are equivalent:*

(i) *$AB \in \mathcal{B}(X)^{\#}$ and $(AB)^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$;*

(ii) *$AB \in \mathcal{B}(X)^{\#}$ and $AA^{\textcircled{\#}}B^{\textcircled{\#}}(I - AA^{\textcircled{\#}}) = 0$;*

(iii) *$AB \in \mathcal{B}(X)^{\#}$ and $A^{\textcircled{\#}}B^{\textcircled{\#}}(I - AA^{\textcircled{\#}}) = 0$;*

(iv) *$AB \in \mathcal{B}(X)^{\#}$ and $B^{\textcircled{\#}}A^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$;*

(v) *$BA \in \mathcal{B}(X)^{\#}$ and $(BA)^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$.*

It is interesting that the similar result to Theorem 3.2 holds for $(BA)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ although the same assumption $A \leq^{\textcircled{D}} B$ is satisfied.

**Theorem 3.4.** *For $A, B \in \mathcal{B}(X)^D$ such that $A \leq^{\textcircled{D}} B$, the following statements are equivalent:*

(i) *$BA \in \mathcal{B}(X)^D$ and $(BA)^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$;*

(ii) *$BA \in \mathcal{B}(X)^D$, $(BA)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ and $AA^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$;*

(iii) *$BA \in \mathcal{B}(X)^D$, $(BA)^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$ and $A^{\textcircled{D}}B^{\textcircled{D}}(I - AA^{\textcircled{D}}) = 0$;*

(iv) $BA \in \mathcal{B}(X)^D$, $(BA)^{\oplus} = A^{\oplus}B^{\oplus}$ *and* $AA^{\oplus}B^{\oplus}(I - AA^{\oplus}) = 0$;

(v) $BA \in \mathcal{B}(X)^D$, $(BA)^{\oplus} = A^{\oplus}B^{\oplus}$ *and* $A^{\oplus}B^{\oplus}(I - AA^{\oplus}) = 0$.

*Proof.* Using the matrix expressions of $A$ and $B$ as in Lemma 2.1, we get

$$BA = \begin{bmatrix} A_1^2 & A_1A_2 + A_2A_3 \\ 0 & B_3A_3 \end{bmatrix},$$

and

$$(BA)^{\oplus} = \begin{bmatrix} A_1^{-2} & -A_1^{-2}(A_1A_2 + A_2A_3)(B_3A_3)^{\oplus} \\ 0 & (B_3A_3)^{\oplus} \end{bmatrix}.$$

We finish this proof as Theorem 3.2. □

Using Theorem 3.1 and Theorem 3.2, we obtain the equivalence between the forward order law $(AB)^{\oplus} = A^{\oplus}B^{\oplus}$ and the reverse order law $(AB)^{\oplus} = B^{\oplus}A^{\oplus}$.

**Corollary 3.5.** *For* $A, B \in \mathcal{B}(X)^D$ *such that* $A \leq^{\oplus} B$ *and* $A^{\oplus} \leq^{\oplus} B^{\oplus}$, *the following statements are equivalent:*

(i) $AB \in \mathcal{B}(X)^D$ *and* $(AB)^{\oplus} = A^{\oplus}B^{\oplus}$;

(ii) $AB \in \mathcal{B}(X)^D$ *and* $(AB)^{\oplus}(I - AA^{\oplus}) = 0$;

(iii) $AB \in \mathcal{B}(X)^D$ *and* $(AB)^{\oplus} = B^{\oplus}A^{\oplus}$.

*Proof.* Theorem 3.1 and the hypothesis $A^{\oplus} \leq^{\oplus} B^{\oplus}$ give $A^{\oplus}B^{\oplus} = B^{\oplus}A^{\oplus}$. The rest is clear by Theorem 3.2. □

Similarly, by Theorem 3.1 and Theorem 3.4, we get the next result for the core–EP inverse of the product $BA$.

**Corollary 3.6.** *For* $A, B \in \mathcal{B}(X)^D$ *such that* $A \leq^{\oplus} B$ *and* $A^{\oplus} \leq^{\oplus} B^{\oplus}$, *the following statements are equivalent:*

(i) $BA \in \mathcal{B}(X)^D$ *and* $(BA)^{\oplus} = B^{\oplus}A^{\oplus}$;

(ii) $BA \in \mathcal{B}(X)^D$ *and* $(BA)^{\oplus}(I - AA^{\oplus}) = 0$;

(iii) $BA \in \mathcal{B}(X)^D$ *and* $(BA)^{\oplus} = A^{\oplus}B^{\oplus}$.

In the case that $A \leq^{\oplus} B$, we investigate equivalent conditions for $(B - A)^{\oplus} = B^{\oplus} - A^{\oplus}$ to be satisfied.

**Theorem 3.7.** *For* $A, B \in \mathcal{B}(X)^D$ *such that* $A \leq^{\oplus} B$, *the following statements are equivalent:*

(i) $B - A \in \mathcal{B}(X)^D$ *and* $(B - A)^{\oplus} = B^{\oplus} - A^{\oplus}$;

(ii) $B - A \in \mathcal{B}(X)^D$ *and* $(B - A)^{\oplus} = B^{\oplus}(I - AA^{\oplus})$;

(iii) $B - A \in \mathcal{B}(X)^D$, $AA^{\oplus}B^{\oplus} = A^{\oplus}$ *and* $(B - A)^{\oplus} = (I - AA^{\oplus})B^{\oplus}$;

(iv) $B - A \in \mathcal{B}(X)^D$, $A^{\oplus}B^{\oplus} = (A^{\oplus})^2$ *and* $(B - A)^{\oplus} = (I - AA^{\oplus})B^{\oplus}$;

(v) $B - A \in \mathcal{B}(X)^D$, $AA^{\oplus}B^{\oplus}(I - AA^{\oplus}) = 0$ *and* $(B - A)^{\oplus} = (I - AA^{\oplus})B^{\oplus}$;

(vi) $B - A \in \mathcal{B}(X)^D$, $A^{\oplus}B^{\oplus}(I - AA^{\oplus}) = 0$ *and* $(B - A)^{\oplus} = (I - AA^{\oplus})B^{\oplus}$.

*Proof.* For $k = i(A)$, Lemma 2.1 and $A \leq^{\circledD} B$ give that, with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$,

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 \\ 0 & B_3 \end{bmatrix},$$

where $A_1 \in \mathcal{B}(R(A^k))$ is invertible, $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent and $B_3 \in \mathcal{B}[N((A^k)^*)]^D$. Using [11, Lemma 2.3], we have

$$A^{\circledD} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B^{\circledD} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\circledD} \\ 0 & B_3^{\circledD} \end{bmatrix}.$$

The fact $B - A = \begin{bmatrix} 0 & 0 \\ 0 & B_3 - A_3 \end{bmatrix} \in \mathcal{B}(X)^D$ yields $B_3 - A_3 \in \mathcal{B}[N((A^k)^*)]^D$ and thus

$$(B - A)^{\circledD} = \begin{bmatrix} 0 & 0 \\ 0 & (B_3 - A_3)^{\circledD} \end{bmatrix}.$$

Thus, the statement (i), i.e., $(B - A)^{\circledD} = B^{\circledD} - A^{\circledD}$ is equivalent to $A_2B_3^{\circledD} = 0$ and $(B_3 - A_3)^{\circledD} = B_3^{\circledD}$.

(i) ⇔ (ii): From

$$B^{\circledD}(I - AA^{\circledD}) = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\circledD} \\ 0 & B_3^{\circledD} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & -A_1^{-1}A_2B_3^{\circledD} \\ 0 & B_3^{\circledD} \end{bmatrix},$$

$(B - A)^{\circledD} = B^{\circledD}(I - AA^{\circledD})$ if and only if $A_2B_3^{\circledD} = 0$ and $(B_3 - A_3)^{\circledD} = B_3^{\circledD}$, which is equivalent to (i).

(i) ⇔ (iii): Note that, by

$$AA^{\circledD}B^{\circledD} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\circledD} \\ 0 & B_3^{\circledD} \end{bmatrix} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2B_3^{\circledD} \\ 0 & 0 \end{bmatrix},$$

$AA^{\circledD}B^{\circledD} = A^{\circledD}$ if and only if $A_2B_3^{\circledD} = 0$. Since

$$(I - AA^{\circledD})B^{\circledD} = \begin{bmatrix} 0 & 0 \\ 0 & B_3^{\circledD} \end{bmatrix},$$

$(B - A)^{\circledD} = (I - AA^{\circledD})B^{\circledD}$ if and only if $(B_3 - A_3)^{\circledD} = B_3^{\circledD}$. Therefore, statements (i) and (iii) are equivalent.

(iii) ⇔ (iv): It follows by elementary computations and $A^{\circledD} = A(A^{\circledD})^2$.

(i) ⇔ (v): We observe that

$$AA^{\circledD}B^{\circledD}(I - AA^{\circledD}) = \begin{bmatrix} 0 & A_1^{-1}A_2B_3^{\circledD} \\ 0 & 0 \end{bmatrix}$$

gives $AA^{\circledD}B^{\circledD}(I - AA^{\circledD}) = 0$ if and only if $A_2B_3^{\circledD} = 0$. As in part (i) ⇔ (iii), one can see that $(B - A)^{\circledD} = (I - AA^{\circledD})B^{\circledD}$ is equivalent to $(B_3 - A_3)^{\circledD} = B_3^{\circledD}$.

(v) ⇔ (vi): This equivalence is obvious. □

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