Filomat 39:3 (2025), 983–988
https://doi.org/10.2298/FIL2503983O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Homotopy extendability and selections for acyclic type maps

Donal O'Regan[a]

[a]*School of Mathematical and Statistical Sciences, University of Galway, Galway, Ireland*

Abstract. In this paper for acyclic type maps we consider null-homoptic, extendabilty and essentiality and present some new Leray-Schauder results and some new fixed point theorems.

1. Introduction

In this paper we consider acyclic type maps and also maps which have upper semicontinuous selections. Using these classes we present a homotopy principle (Theorem 2.6) based on null–homotopic [7] and a Leray-Schauder principle (Theorem 2.9) based on extendability. A new fixed point result (Theorem 2.12) is also given for acyclic maps. Our homotopy principles motivate the notion of an essential map which we will then use to obtain an existence result (Theorem 2.20). Our theory here was motivated from some ideas in the literature [1, 2, 6, 7, 10, 11, 12].

We first present the following classes of maps from the literature. Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. First we consider the acyclic maps, namely $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ if $F : X \to Ac(Z)$ is upper semicontinuous. The following result was established in [6 pp 161].

Theorem 1.1. *Suppose $X \in AR$ and $G \in AC(X, X)$ is a compact map. Then G has a fixed point.*

Now we describe the class of maps in [1, 12]. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 and let F be a multifunction. We say $F \in W(X, Y)$ if $F : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) and there exists a $\theta : X \to 2^Y$ which is lower semicontinuous with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for each $x \in X$. It is of interest to note [9] that if $\theta : X \to 2^Y$ is lower semicontinuous then $\overline{co}\,\theta$ is lower semicontinuous so one can say $F \in W(X, Y)$ if $F : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with closed convex values and with $\theta(x) \subseteq F(x)$ for each $x \in X$. The following result was established in [1, 12].

2020 *Mathematics Subject Classification*. Primary 47H10, 54H25
Keywords. Fixed points, set-valued maps.
Received: 23 May 2024; Revised: 29 December 2024; Accepted: 04 January 2025
Communicated by Adrian Petrusel
Email address: donal.oregan@nuigalway.ie (Donal O'Regan)
ORCID iD: https://orcid.org/0000-0002-4096-1469 (Donal O'Regan)

Theorem 1.2. *Let X be a paracompact subset of a Hausdorff topological vector space E_1 and Y a metrizable, complete subset of a Hausdorff locally convex linear topological space E_2. If $F \in W(X,Y)$ then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(x) \subseteq F(x)$ for $x \in X$; here $CK(Y)$ denotes the family of nonempty convex compact subsetes of Y.*

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$.

The following result was established in [3].

Theorem 1.3. *Let X be a regular topological space, $F : X \to 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.4. *From Theorem 1.3 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [4, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [8, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [5]). Also note in Theorem 1.3 if F is compact valued, then the assumption that X is regular can be removed. In Section 2 when we apply this result our space X will be a metric space so in particular a uniform space.*

2. Fixed point results

To obtain our first main result (Theorem 2.9) we will use the notion of null–homotopic [7] to present our continuation principle (Theorem 2.6). In this section E will be a completely regular topological space (i.e. a Tychonoff space).

Definition 2.1. *We say $F \in ACW(E,E)$ if $F : E \to Ac(E)$ is a upper semicontinuous compact map (i.e. $F \in ACW(E,E)$ if $F \in AC(E,E)$ is a compact map).*

Definition 2.2. *We say $F \in AW(E,E)$ if $F : E \to 2^E$ is a compact map and there exists a upper semicontinuous selection $G : E \to Ac(E)$ of F (i.e. $F \in AW(E,E)$ if F is a compact map and there exists a selector $G \in AC(E,E)$ of F).*

Remark 2.3. *(i). Note G in Definition 2.2 is a compact map since $G(E) \subseteq F(E)$ and F is a compact map, so $G \in ACW(E,E)$ in Definition 2.2. Thus we say $F \in AW(E,E)$ if F is a compact map and there exists a selector $G \in ACW(E,E)$ of F.*

(ii). If E is a Fréchet space (so E is metrizable so paracompact) and $F \in W(E,E)$ is a compact map, then from Theorem 1.2 there exists an upper semicontinuous selection $G : X \to CK(Y)$ of F so G has convex (so acyclic) values (i.e. $G \in ACW(E,E)$).

Definition 2.4. *(i). Suppose $F \in ACW(E,E)$ and $u_0 \in E$. We say $F \cong \{u_0\}$ in $ACW(E,E)$ if there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = F$ and $H_0 = \{u_0\}$; here $H_t(x) = H(x,t)$.*

(ii). Suppose $F \in AW(E,E)$ and $u_0 \in E$. We say $F \cong \{u_0\}$ in $AW(E,E)$ if for any selector $G \in ACW(E,E)$ of F there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = G$ and $H_0 = \{u_0\}$.

Remark 2.5. *If E is a Hausdorff topological vector space and $u_0 = 0$ then one could take $H(x,t) = t\,F(x)$ in Definition 2.4 (i). Note for a fixed $t \in [0,1]$ and a fixed $x \in E$ we have that $H_t(x)$ is acyclic valued (recall homeomorphic spaces have isomorphic homology groups).*

Theorem 2.6. *Let E be a completely regular topological space, U an open subset of E and $u_0 \in U$.*

(i). Let $F \in ACW(E,E)$ and suppose there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = F$ and $H_0 = \{u_0\}$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following condition is satisfied:

$$(2.1) \qquad \begin{cases} \text{for any } \theta \in ACW(E,E) \text{ with } \theta \cong \{u_0\} \text{ in } ACW(E,E) \\ \text{we have that } \theta \text{ has a fixed point in } E. \end{cases}$$

Then F has a fixed point in U.

(ii). Let $F \in AW(E,E)$ and for any selector $G \in ACW(E,E)$ of F suppose there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = G$ and $H_0 = \{u_0\}$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.1) holds. Then F has a fixed point in U.

Proof. We will only consider (ii) (since (i) is the same by replacing G with F). Let $G \in ACW(E,E)$ be a selector of F and let H be as in the statement of Theorem 2.6 (ii). Now let

$$B = \{x \in E \backslash U : \ x \in H_t(x) \text{ for some } t \in [0,1]\}.$$

We consider two cases, namely $B \neq \emptyset$ and $B = \emptyset$.

Case (i). $B = \emptyset$.

Then for every $t \in [0,1]$ we have $x \notin H_t(x)$ for $x \in E \backslash U$. Also from $H_1 \cong \{u_0\}$ in $ACW(E,E)$ then from (2.1) we have that there exists a $y \in E$ with $y \in H_1(y)$. Now since $x \notin H_1(x)$ for $x \in E\backslash U$ then $y \in U$. Thus $y \in U$ and $y \in H_1(y) = G(y) \subseteq F(y)$.

Case (ii). $B \neq \emptyset$.

Note B is closed and compact (recall $H : E \times [0,1] \to K(E)$ is a upper semicontinuous compact map) and $B \cap \overline{U} = \emptyset$ since $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$. Then (recall E is completely regular) there exists a continuous map $\mu : E \to [0,1]$ with $\mu(B) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to K(E)$ by $R(x) = H(x,\mu(x)) = H \circ w(x)$ where $w(x) = (x,\mu(x))$. If x is fixed, note $R(x)\,(= H_{\mu(x)}(x))$ is acyclic valued so $R : E \to Ac(E)$ and as a result $R \in ACW(E,E)$. We claim $R \cong \{u_0\}$ in $ACW(E,E)$. To see this let $\Omega : E \times [0,1] \to K(E)$ be given by $\Omega(x,t) = H(x,t\,\mu(x)) = H \circ \tau(x,t)$ where $\tau(x,t) = (x,t\,\mu(x))$. Note Ω is a upper semicontinuous compact map and $\Omega_t \in AC(E,E)$ for $t \in [0,1]$ (for fixed $x \in E$ note $\Omega_t(x) = H_{t\,\mu(x)}(x)$ is acyclic valued) and $\Omega_0 = \{u_0\}$ (note $\Omega(x,0) = H(x,0) = \{u_0\}$ for $x \in E$) and $\Omega_1 = R$. Thus $R \cong \{u_0\}$ in $ACW(E,E)$. Now (2.1) guarantees that there exists a $x \in E$ with $x \in R(x) = H_{\mu(x)}(x)$. If $x \in E\backslash U$ then since $x \in B$ we have $\mu(x) = 0$ and so $x \in H_0(x) = \{u_0\}$, a contradiction since $u_0 \in U$. Thus $x \in U$ so $\mu(x) = 1$ and so $x \in R(x) = H_{\mu(x)}(x) = H_1(x) = G(x) \subseteq F(x)$. □

In applications one is usually interested in maps $F : \overline{U} \to 2^E$ where U is an open subset of E. One can adjust the statement of Theorem 2.6 to this situation as the following result shows (this also motivates the definition of an essential map in this situation).

Let E be a Tychonoff space and U an open subset of E.

Definition 2.7. *We say $F \in ACW(\overline{U},E)$ if $F : \overline{U} \to Ac(E)$ is a upper semicontinuous compact map (i.e. $F \in ACW(\overline{U},E)$ if $F \in AC(\overline{U},E)$ is a compact map).*

Definition 2.8. *We say $F \in AW(\overline{U},E)$ if $F : \overline{U} \to 2^E$ is a compact map and there exists a upper semicontinuous selection $G : \overline{U} \to Ac(E)$ of F (i.e. $F \in AW(\overline{U},E)$ if F is a compact map and there exists a selector $G \in AC(\overline{U},E)$ of F).*

Theorem 2.9. *Let E be a completely regular topological space, U an open subset of E and $u_0 \in U$.*

(i). Let $F \in ACW(\overline{U},E)$ and suppose there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(E)$ with $\Lambda_t \in AC(\overline{U},E)$ for each $t \in [0,1]$, $\Lambda_1 = F$ and $\Lambda_0 = \{u_0\}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.1) holds. Then F has a fixed point in U.

(ii). Let $F \in AW(\overline{U}, E)$ and for any selector $G \in ACW(\overline{U}, E)$ of F suppose there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(E)$ with $\Lambda_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$, $\Lambda_1 = G$ and $\Lambda_0 = \{u_0\}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.1) *holds. Then F has a fixed point in U.*

Proof. We will only consider (ii) (since (i) is the same by replacing G with F). Let $G \in ACW(E, E)$ be a selector of F and let Λ be as in the statement of Theorem 2.9 (ii). Now let

$$D = \{x \in \overline{U} : \ x \in \Lambda_t(x) \ \text{ for some } \ t \in [0,1]\}.$$

Now $D \neq \emptyset$ (since $u_0 \in U$), D is closed and compact and $D \cap (E \backslash U) = \emptyset$. Then there exists a continuous map $\sigma : E \to [0,1]$ with $\sigma(D) = 1$ and $\sigma(E \backslash U) = 0$. Let $H : E \times [0,1] \to K(E)$ be given by

$$H(x,t) = \begin{cases} \Lambda(x, t\,\sigma(x)), & x \in \overline{U} \\ \{u_0\}, & x \in E \backslash U \end{cases}$$

(note for $x \in \partial U$ we have $H(x,t) = \Lambda(x,0) = \Lambda_0(x) = \{u_0\}$). Now $H : E \times [0,1] \to K(E)$ is a upper semicontinuous compact map and for each fixed $t \in [0,1]$ and fixed $x \in E$ note $H_t(x)$ is acyclic valued, so $H_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$. Note $H_0 = \{u_0\}$. Also $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$ since if there exists a $x \in \partial U$ and $t \in (0,1]$ with $x \in H_t(x)$, then $x \in \Lambda_{t\,\sigma(x)}(x)$, so $x \in D$, and as a result $\sigma(x) = 1$, so $x \in \Lambda_t(x)$, a contradiction. Thus H satisfies the conditions in Theorem 2.6 (i) with $F = H_1$ there (note $H_1 \in ACW(E, E)$). From Theorem 2.6 (i) there exists a $x \in U$ with $x \in H_1(x)$, so $x \in \Lambda_{\sigma(x)}(x)$. Thus $x \in D$ which implies $\sigma(x) = 1$. As a result we have $x \in \Lambda_1(x) = G(x) \subseteq F(x)$. □

For conditions in the literature to guarantee (2.1) we refer the reader to [1, 2, 10, 12]. We now present two general results. Let Ω be a class of topological spaces (in our results below Ω is a metric space). A space E is called a $ANR(\Omega)$ (written $E \in ANR(\Omega)$; see [6, 7]) if $E \in \Omega$ and given any closed embedding K of E in a space $X \in \Omega$ there exists an open set $U \subseteq X$ containing K and a continuous retraction $r : U \to K$. A more general concept is the following. A space E is called an Approximative $ANR(\Omega)$ (written $E \in$ Approximative $ANR(\Omega)$; see [7]) if $E \in \Omega$ and given any closed embedding K of E in a space $X \in \Omega$ there exists for any covering $\alpha \in Cov_X(K)$ an open set $U_\alpha \subseteq X$ containing K and a continuous map $r_\alpha : U_\alpha \to K$ such that $r_\alpha|_K$ and the identity $i|_K : K \to K$ are α–close.

Remark 2.10. *In our case Ω is a metric space so E is a metric space so from the Arens-Eells theorem [6 pg 4, 7 pg 597] (recall the Arens–Eells theorem states that any metric space can be isometrically embedded as a closed subset in a normed linear space) E can be regarded as a closed subset of a normed space X, so when we apply our situation we can replace K above by E.*

Theorem 2.11. *Let $E \in ANR(metric)$. Then* (2.1) *holds.*

Proof. Let $\theta \in ACW(E, E)$ with $\theta \cong \{u_0\}$ in $ACW(E, E)$; here in fact we can take $u_0 \in E$. Then there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E, E)$ for each $t \in [0,1]$, $H_1 = \theta$ and $H_0 = \{u_0\}$. Note E can be regarded as a closed subset of a normed space X. Since $E \in ANR$(metric) there exists an open neighborhood V of E in X and a continuous retraction $r : \overline{V} \to E$ (note there exists an open set U of E in X and a continuous retraction $r : U \to E$ so since X is a normal topological space there exists an open set V with $E \subset V \subset \overline{V} \subset U$). Let $\eta : X \to [0,1]$ be a continuous map with $\eta(X \backslash V) = 0$ and $\eta(E)$=1. Let

$$Q(x) = \begin{cases} H(r(x), \eta(x)), & x \in \overline{V} \\ \{u_0\}, & x \in X \backslash V \end{cases}$$

(note for $x \in \partial V$ we have $Q(x) = H(r(x), \eta(x)) = H(r(x), 0) = \{u_0\}$). Now $Q : X \to K(X)$ is a upper semicontinuous compact map and $Q \in AC(X, X)$ (for a fixed $x \in X$ note $Q(x)$ is acyclic valued). Now Theorem 1.1 guarantees that there exists a $x \in X$ with $x \in Q(x)$. If $x \in X \backslash V$ then $x \in \{u_0\}$, a contradiction since $u_0 \in E$ and $E \subseteq V$. Thus $x \in \overline{V}$. If $x \in \overline{V} \backslash E$ then since $Q : X \to K(E)$ (note $H : E \times [0,1] \to K(E)$) we have since $x \in Q(x)$ that $x \in E$, a contradiction. Thus $x \in E$ so as a result $r(x) = x$ (note $E \subseteq V$) and $\eta(x) = 1$. Consequently $x \in Q(x) = H(x, 1) = \theta(x)$. Thus (2.1) holds. □

Theorem 2.12. *Let $E \in$ Approximative ANR(metric). Then* (2.1) *holds.*

Proof. Let $\theta \in ACW(E,E)$ with $\theta \cong \{u_0\}$ in $ACW(E,E)$; here in fact we can take $u_0 \in E$. Then there exists a upper semicontinuous compact map $H : E \times [0,1] \to K(E)$ with $H_t \in AC(E,E)$ for each $t \in [0,1]$, $H_1 = \theta$ and $H_0 = \{u_0\}$. Note E can be regarded as a closed subset of a normed space X. Let $\beta \in Cov_X(\overline{\theta(E)})$. Then $\alpha = \beta \cup \left\{X \backslash \ \overline{\theta(E)}\right\}$ is an open covering of X (and of E) so $\alpha \in Cov_X(E)$. Now since $E \in$ Approximative ANR(metric) there exists an open set $V_\alpha \subseteq X$ containing E and a continuous map $r_\alpha : \overline{V_\alpha} \to E$ (note X is a normal topological space) such that $r_\alpha|_E$ and the identity $i|_E : E \to E$ are α–close. Let $\eta_\alpha : X \to [0,1]$ be a continuous map with $\eta_\alpha(X \backslash V_\alpha) = 0$ and $\eta_\alpha(E)$=1. Let

$$Q_\alpha(x) = \begin{cases} H(r_\alpha(x), \eta_\alpha(x)), & x \in \overline{V_\alpha} \\ \{u_0\}, & x \in X \backslash V_\alpha. \end{cases}$$

Now $Q_\alpha : X \to K(X)$ is a upper semicontinuous compact map and $Q_\alpha \in AC(X,X)$. Theorem 1.1 guarantees that there exists a $x \in X$ with $x \in Q_\alpha(x)$. If $x \in X \backslash V_\alpha$ then $x \in \{u_0\}$, a contradiction since $u_0 \in E$ and $E \subseteq V_\alpha$. If $x \in \overline{V_\alpha} \backslash E$ then since $Q_\alpha : X \to K(E)$ (note $H : E \times [0,1] \to K(E)$) we have since $x \in Q_\alpha(x)$ that $x \in E$, a contradiction. Thus $x \in E$ so $\eta_\alpha(x) = 1$. As a result $x \in H(r_\alpha(x), 1) = \theta(r_\alpha(x))$ i.e. $x \in E$ and $x \in \theta(r_\alpha(x))$. Now since $r_\alpha|_E$ and the identity $i|_E : E \to E$ are α–close then there exists a $U_x \in \alpha$ with $x \in U_x$ and $r_\alpha(x) \in U_x$. Note $x \in \theta(r_\alpha(x))$ implies $x \in \overline{\theta(E)}$ so $U_x \neq X \backslash \ \overline{\theta(E)}$, so as a result $U_x \in \beta$. Rewriting, we have shown that there exists a $U_x \in \beta$ with $x \in U_x$ and $r_\alpha(x) \in U_x$. Let $y = r_\alpha(x)$. Then $y \in U_x$ and $\theta(y) \cap U_x \neq \emptyset$ since $x \in U_x$ and $x \in \theta(r_\alpha(x)) = \theta(y)$ i.e. $U_x \in \beta$, $y \in U_x$ and $\theta(y) \cap U_x \neq \emptyset$. Thus θ has a β–fixed point (i.e. y) for every $\beta \in Cov_X(\overline{\theta(E)})$. Now Remark 1.4 guarantees that θ has a fixed point, so (2.1) holds. □

Our next result is based on the notion of an essential map. Let E be a a completely regular topological space and U an open subset of E.

Definition 2.13. *We say $F \in AW_{\partial U}(\overline{U}, E)$ if $F \in AW(\overline{U}, E)$ (see Definition 2.8) and $x \notin F(x)$ for $x \in \partial U$.*

Definition 2.14. *We say $G \in ACW_{\partial U}(\overline{U}, E)$ if $G \in ACW(\overline{U}, E)$ (see Definition 2.9) and $x \notin G(x)$ for $x \in \partial U$.*

Definition 2.15. *We say $F \in AW_{\partial U}(\overline{U}, E)$ is essential in $AW_{\partial U}(\overline{U}, E)$ if for any selection $G \in ACW(\overline{U}, E)$ of F and any map $\theta \in ACW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = G|_{\partial U}$ there exists a $x \in U$ with $x \in \theta(x)$.*

Remark 2.16. *(i). In Definition 2.15 note $G \in ACW_{\partial U}(\overline{U}, E)$ since $x \notin F(x)$ for $x \in \partial U$ and G is a selection of F.*
(ii). If $F \in AW_{\partial U}(\overline{U}, E)$ is essential in $AW_{\partial U}(\overline{U}, E)$ and if $G \in ACW(\overline{U}, E)$ is a selection of F, then there exists a $x \in U$ with $x \in G(x)$ (take $\theta = G$ in Definition 2.15), so in particular $x \in F(x)$.

Definition 2.17. *We say $F \in ACW_{\partial U}(\overline{U}, E)$ is essential in $ACW_{\partial U}(\overline{U}, E)$ if for any map $\theta \in ACW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $x \in \theta(x)$.*

Definition 2.18. *Let $F, G \in AW_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $AW_{\partial U}(\overline{U}, E)$ if for any selection $\Psi \in ACW_{\partial U}(\overline{U}, E)$ of G and any selection $\Phi \in ACW_{\partial U}(\overline{U}, E)$ of F there exists a upper semicontinuous compact map $H : \overline{U} \times [0,1] \to K(E)$ with $H_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$, $H_1 = \Phi$ and $H_0 = \Psi$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$.*

Definition 2.19. *Let $F, G \in ACW_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $ACW_{\partial U}(\overline{U}, E)$ if there exists a upper semicontinuous compact map $H : \overline{U} \times [0,1] \to K(E)$ with $H_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$, $H_1 = F$ and $H_0 = G$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$.*

We state a result for $AW_{\partial U}(\overline{U}, E)$ maps (there is an obvious analogue for $ACW_{\partial U}(\overline{U}, E)$ maps).

Theorem 2.20. *Let E be a completely regular topological space, U an open subset of E, $G \in AW_{\partial U}(\overline{U}, E)$ is essential in $AW_{\partial U}(\overline{U}, E)$, $F \in AW_{\partial U}(\overline{U}, E)$ and $F \cong G$ in $AW_{\partial U}(\overline{U}, E)$. Then there exists a $x \in U$ with $x \in F(x)$.*

Proof. Let $\Psi \in ACW_{\partial U}(\overline{U}, E)$ be a selection of G and $\Phi \in ACW_{\partial U}(\overline{U}, E)$ be a selection of F. Since $F \cong G$ in $AW_{\partial U}(\overline{U}, E)$ there exists a a upper semicontinuous compact map $H : \overline{U} \times [0,1] \to K(E)$ with $H_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$, $H_1 = \Phi$ and $H_0 = \Psi$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$. Let

$$B = \left\{ x \in \overline{U} : \ x \in H_t(x) \ \text{ for some } \ t \in [0,1] \right\}.$$

Notice $B \neq \emptyset$ (since G is essential in $AW_{\partial U}(\overline{U}, E)$ and see Remark 2.16), B is closed and compact and $B \cap \partial U = \emptyset$ (note $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(B) = 1$ and $\mu(\partial U) = 0$. Define a map $R : \overline{U} \to K(E)$ by $H(x) = H(x, \mu(x)) = H \circ w$ where $w(x) = (x, \mu(x))$. Now R is a upper semicontinuous compact map and for each $x \in \overline{U}$ note $R(x) = H_{\mu(x)}(x)$ has acyclic values. Thus $R \in ACW(\overline{U}, E)$. In fact $R \in ACW_{\partial U}(\overline{U}, E)$ since $R|_{\partial U} = H_0|_{\partial U} = \Psi|_{\partial U}$. Since $G \in AW_{\partial U}(\overline{U}, E)$ is essential in $AW_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $x \in R(x)$ i.e. $x \in H_{\mu(x)}(x)$. Then $x \in B$ so $\mu(x) = 1$ and as a result $x \in H_1(x) = \Phi(x) \subseteq F(x)$. □

Declaration.
Ethical Approval: Not Applicable.
Competing Interests: The author declares no conflict of interest.
Authors Contribution: Not Applicable.
Funding: Not Applicable.
Availability of Data and Matherials: Not Applicable.

References

[1] R.P. Agarwal, J.H. Dshalalow and D.O'Regan, Generalized Leray–Schauder principles in Hausdorff topological spaces for acyclic maps, Mathematical Sciences Research Journal 8(2004), no. 4, 114–117
[2] R.P. Agarwal and D.O'Regan, Fixed point theory for maps with lower semicontinuous selections and equilibrium theory for abstract economies, Jour. Nonlinear Convex Anal. 2(2001), 31–46.
[3] H. Ben-El-Mechaiekh, The coincidence problem for compositions of set valued maps, Bull. Austral. Math. Soc.41(1990), 421–434.
[4] H. Ben-El-Mechaiekh, Spaces and maps approximation and fixed points, J. Comput. Appl. Math.113(2000), 283–308.
[5] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[6] L. Gorniewicz, Topological fixed point theory of multivalued mappings, Kluwer Acad. Publishers, Dordrecht, 1991.
[7] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.
[8] J.L. Kelley, General Topology, D.Van Nostrand, New York, 1955.
[9] E. Michael, Continuous selections I, Ann. of Math. 63(1956), 361–382.
[10] D.O'Regan, Multivalued *H*–essential maps of acyclic type on Hausdorff topological spaces, Journal of Applied Mathematics and Stochastic Analysis Vol2006(2006), Article ID 36461, 1–4.
[11] D.O'Regan, A note on the topological transversity theorem for maps with lower semicontinuous type selections, Linear Nonlinear Anal. 8(2022), 89–94.
[12] X. Wu, A new fixed point theorem and its applications, Proc. Amer. Math. Soc. 125(1997), 1779–1783.