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Ulam type stabilities for oscillatory Volterra integral equations

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Abstract. This paper investigates Hyers–Ulam stability and Hyers–Ulam–Rassias stability of classes of integral equations with kernels depending on sine and cosine functions. An example is given to show the applicability of our obtained results.

1. Introduction and Preliminaries

S. M. Ulam's [33] concept of stability, established in 1940, has gained popularity among researchers due to its various uses. The initial stability results for functional equations focused on determining when an approximation of an equation's solution is close enough to the exact solution and if a solution exists. D. H. Hyers was the first to provide a partial answer for Banach spaces, namely for the additive Cauchy equation $f(x+y) = f(x) + f(y)$, cf., [7] resulting in Hyers–Ulam stability. T. M. Rassias [36] contributed new ideas and developed the Hyers–Ulam–Rassias stability. Considering the possibility of using different norms involved, other types of equations, and different mathematical techniques for obtaining approximate solutions, different generalizations of stabilities were obtained by other researchers, in particular by Rassias [37], Gajda [39], and Aoki [38]. The interested reader can obtain a detailed description of these advances in other studies [8, 16, 27] and Brzdek et al [15].

In order to investigate stability in general, a variety of approaches have been utilized, such as Gronwall's lemma, the direct technique (see [7]), fixed–point theory (see [13, 14]), the invariant means method (see [22]), and many more (see, e.g., [40]). The Gronwall lemma is an extremely useful and appropriate tool for demonstrating the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of integral equations. In [10], Rus used the Gronwall lemma to prove the Hyers–Ulam–Rassias stability of integral equations.

Gronwall's lemma occupies a significant position in both pure and applied mathematics due to its numerous applications (see, e.g., [11, 26, 28]). It is an essential tool for analyzing functional, differential, and integral equations; in particular, it helps determine limits, uniqueness, and stability of solutions. Gronwall's lemma is often used in the study of ordinary differential equations to demonstrate the uniqueness of solutions to initial value problems, guaranteeing that solutions that begin with the same initial conditions

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continue to be consistent across time. Its significance extends to stability analysis, where it aids in proving that small changes to a system's initial parameters or conditions result in similarly tiny changes to the solution, thereby verifying the stability of the system. It can also be applied to integral equations, where it can be used to guarantee the uniqueness of solutions and set limitations on them in a variety of situations. Gronwall's lemma plays a crucial part in theoretical investigations as well as real–world problem–solving because of its capacity to create strict limitations and guarantee that solutions depend continuously on inputs.

The stability of integral equations has been studied by many authors. For instance, in 2007, using the fixed–point method, Jung [34] investigated the uniqueness of the solution and the Hyers–Ulam–Rassias stability of the Volterra integral equation. Indeed, when it comes to the literature on Ulam stabilities of integral equations, Jung's work [34] has served as a foundation. In 2011, M. Akkouchi (see [23]) established the stability of both the Hyers–Ulam and the Hyers–Ulam–Rassias for a number of integral equations using several different fixed–point theorems. Using the open mapping theorem, the authors in [3] investigated the Hyers–Ulam stability of linear impulsive Volterra integral equations. They looked into the existence and uniqueness of solutions to a particular class of nonlinear impulsive integral equations in their analysis. Additionally, they examined the Hyers–Ulam and Hyers–Ulam–Rassias stability of the same class of integral equations using Gronwall lemma.

The authors of [4] used the fixed–point theorem to study the Hyers–Ulam and Hyers–Ulam–Rassias stability of nonlinear impulsive systems on time scales. With the help of the Picard operator, the existence and uniqueness of the solution to nonlinear impulsive Volterra integral delay dynamic systems are demonstrated. Banach fixed–point theorem and abstract Gronwall lemma were the primary methods used to demonstrate the findings. The nonlinear Volterra system's Hyers–Ulam and Hyers–Ulam–Rassias stability results on time scales with fractionally integrable impulses were published by the authors in [35]. The existence and uniqueness of solutions are found using the Picard fixed–point theorem. They established Hyers–Ulam stability and Hyers–Ulam–Rassias stability results using abstract Gronwall lemma and Gronwall lemma on time scales. The authors in [41] investigated the Hyers–Ulam and Hyers–Ulam–Rassias stabilities of two forms of Volterra integral equations that were provided on time scales and generalized them. They also demonstrated stability results on time scales for the nonlinear, inhomogeneous Volterra integral equation and offered an example to verify the findings. Furthermore, they demonstrated the Hyers–Ulam–Rassias stability of the general Volterra type integral equation given on time scales.

Throughout this paper, let $(\mathbb{B}, |.|)$ denote a (real or complex) Banach space with the norm $|.|$.

In particular, $C([a,b],\mathbb{B})$ denotes the space of continuous operators from $[a,b]$ in $\mathbb{B}$.

In this article, we consider the following oscillatory Volterra integral equation:

$$\vartheta(x) = \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)dy, \tag{1}$$

where $x \in [a,b]$; a and b are fixed real numbers; $\vartheta \in C([a,b],\mathbb{B})$; and $\alpha, \beta \in \mathbb{R}$.

In this paper, we prove the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability of the oscillatory Volterra integral Equation (1) by means of the Gronwall lemma (Lungu [26], Rus [10]).

Volterra and Fredholm integral equations are widely used in science and engineering, including electrostatics, mechanics, and fluid mechanics (see, e.g., earlier studies [1, 5, 9, 17, 20, 21, 31]). Integral equations with oscillatory convolution kernels are among the most well–studied. Many numerical methods have been developed to get approximate solutions to integral equations with highly oscillatory kernels, as exact analytical solutions are sometimes difficult or impossible to obtain. Thus, in this paper, we consider a Volterra type equation where the different coefficients in the cosine and sine functions are chosen to constitute the oscillatory kernel of the equation. Exponential functions are frequently used to define the oscillatory kernel. Oscillatory Volterra integral equations are useful in modeling and understanding systems or phenomena that exhibit oscillatory behavior. These equations help us analyze and predict how oscillations evolve over time. By studying these equations, we can gain insights into the dynamics and behavior of oscillatory systems. This work focuses on oscillatory effects caused by parameters interacting directly with sine and cosine functions, a topic that has received limited attention in the stability literature. For more details on stability, we recommend [2, 6, 12, 18, 19, 24, 25, 29, 30, 32].

The formal definition of the above mentioned stabilities is now introduced for the oscillatory Volterra integral Equation (1).

Definition 1.1. *Equation* (1) *is said to have the Hyers–Ulam stability with respect to* ϵ *if there exists a constant* $c > 0$ *such that for each solution* $\vartheta \in C([a,b],\mathbb{B})$ *of the inequality*

$$\left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy) + \beta\sin(xy)]\vartheta(y)dy\right| \leq \epsilon,$$

there exists a solution $\vartheta^* \in C([a,b],\mathbb{B})$ *of Equation* (1) *such that:*

$$|\vartheta(x) - \vartheta^*(x)| \leq c\times\epsilon, \quad \forall\, x \in [a,b].$$

Definition 1.2. *Equation* (1) *is said to have the Hyers–Ulam–Rassias stability with respect to* $\phi(x)$ *if there exists a constant* $c > 0$ *such that for each solution* $\vartheta \in C([a,b],\mathbb{B})$ *of the inequation*

$$\left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy) + \beta\sin(xy)]\vartheta(y)dy\right| \leq \phi(x),$$

there exists a solution $\vartheta^* \in C([a,b],\mathbb{B})$ *of Equation* (1) *such that:*

$$|\vartheta(x) - \vartheta^*(x)| \leq c\times\phi(x), \quad \forall\, x \in [a,b].$$

This paper is organized as follows: In Section 2, we discuss the Hyers–Ulam stability of oscillatory Volterra integral Equation (1). In Section 3, we discuss the Hyers–Ulam–Rassias stability of oscillatory Volterra integral Equation (1). Section 4 consists of the example of the results. Finally, the conclusion of the paper is presented in Section 5.

2. Hyers–Ulam stability of oscillatory Volterra integral equation

We now give the Hyers–Ulam stability result with regard to oscillatory Volterra integral Equation (1) in Theorem 2.1.

Theorem 2.1. *Suppose that we have*
($As1$) $\vartheta \in C([a,b],\mathbb{B})$.
($As2$) *There exist positive constant* $\sqrt{\frac{\alpha^2+\beta^2}{2\pi}}$ *such that*

$$\frac{1}{\sqrt{2\pi}}\Big|[\alpha\cos(xy) + \beta\sin(xy)]\vartheta_1 - [\alpha\cos(xy) + \beta\sin(xy)]\vartheta_2\Big| \leq \sqrt{\frac{\alpha^2+\beta^2}{2\pi}}\,\Big|\vartheta_1 - \vartheta_2\Big|$$

for all $x, y \in [a,b]$; $\alpha, \beta \in \mathbb{R}$ *and* $\vartheta_1, \vartheta_2 \in \mathbb{B}$.
Then,
(a) Equation (1) *has a unique solution* ϑ^* *in* $C([a,b],\mathbb{B})$*;*
(b) For each $\epsilon > 0$*, if* $\vartheta \in C([a,b],\mathbb{B})$ *is a solution of the inequality*

$$\left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy) + \beta\sin(xy)]\vartheta(y)\,dy\right| \leq \epsilon, \forall\, x \in [a,b], \tag{2}$$

then

$$|\vartheta(x) - \vartheta^*(x)| \leq c\times\epsilon, \forall\, x \in [a,b],$$

where

$$c = \exp\left(\sqrt{\frac{\alpha^2+\beta^2}{2\pi}}(b-a)\right)$$

with

$$\sqrt{\frac{\alpha^2+\beta^2}{2\pi}} > 0.$$

Hence, Equation (1) *is Hyers–Ulam stable.*

Proof. (a) The proof can be easily done as in Rus [11]. We will not give the proof of (a).
(b) According to ($As1$), ($As2$) and (2) we derive

$$\begin{aligned}
|\vartheta(x)-\vartheta^*(x)| &= \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta^*(y)\,dy\right| \\
&= \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta(y)\,dy\right. \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta(y)\,dy \\
&\quad \left. - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta^*(y)\,dy\right| \\
&\leq \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta(y)\,dy\right| \\
&\quad + \left|\frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta(y)\,dy\right. \\
&\quad \left. - \frac{1}{\sqrt{2\pi}}\int_a^x [\alpha\cos(xy)+\beta\sin(xy)]\vartheta^*(y)\,dy\right| \\
&\leq \epsilon + \left|\frac{1}{\sqrt{2\pi}}\int_a^x |\alpha\cos(xy)+\beta\sin(xy)||\vartheta(y)-\vartheta^*(y)|\,dy\right|.
\end{aligned}$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\epsilon + \left|\frac{1}{\sqrt{2\pi}}\int_a^x |\alpha\cos(xy)+\beta\sin(xy)||\vartheta(y)-\vartheta^*(y)|\,dy\right| \\
\leq\ &\epsilon + \left|\frac{1}{\sqrt{2\pi}}\int_a^x \sqrt{[\alpha^2+\beta^2][\cos^2(xy)+\sin^2(xy)]}\,|\vartheta(y)-\vartheta^*(y)|\,dy\right|.
\end{aligned}$$

Thus, we derive

$$|\vartheta(x)-\vartheta^*(x)| \leq \epsilon + \sqrt{\frac{\alpha^2+\beta^2}{2\pi}}\int_a^x |\vartheta(y)-\vartheta^*(y)|\,dy. \tag{3}$$

Using the Gronwall lemma (Lungu [26], Rus [10]), from (3), we obtain

$$|\vartheta(x)-\vartheta^*(x)| \leq \epsilon\exp\left(\sqrt{\frac{\alpha^2+\beta^2}{2\pi}}(b-a)\right). \tag{4}$$

Let

$$c = \exp\left(\sqrt{\frac{\alpha^2+\beta^2}{2\pi}}(b-a)\right)$$

and

$$0 < \sqrt{\frac{\alpha^2 + \beta^2}{2\pi}} < 1.$$

Hence, according to (4), we have

$$|\vartheta(x) - \vartheta^*(x)| \leq c \times \epsilon, \forall x \in [a, b].$$

As a result of the above inequality, according to the conditions of Theorem 2.1, we conclude that Equation (1) is Hyers–Ulam stable. □

3. Hyers–Ulam–Rassias stability of oscillatory Volterra integral equation

In this section, we give the stability result in the sense of Hyers–Ulam–Rassias for oscillatory Volterra integral Equation (1).

Theorem 3.1. *We assume that* $(As1)$ *and* $(As2)$ *of Theorem 2.1 hold and let* $\phi \in C([a, b], \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$ *and* ϕ *be an increasing function. Then, we have the following results:*
(a) Equation (1) *has a unique solution* ϑ^* *in* $C([a, b], \mathbb{B})$*;*
(b) If $\vartheta \in C([a, b], \mathbb{B})$ *is a solution of the inequality*

$$\left|\vartheta(x) - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy\right| \leq \phi(x), \forall\, x \in [a, b], \tag{5}$$

then

$$|\vartheta(x) - \vartheta^*(x)| \leq c \times \phi(x), \quad \forall\, x \in [a, b],$$

where

$$c = \exp\left(\sqrt{\frac{\alpha^2 + \beta^2}{2\pi}}(b - a)\right)$$

and

$$0 < \sqrt{\frac{\alpha^2 + \beta^2}{2\pi}} < 1.$$

Hence, Equation (1) *is Hyers–Ulam–Rassias stable.*

Proof. (a) The proof of this theorem can be easily done (see, Lungu [26]). Therefore, we will not give the proof of this theorem.

(b) According to the conditions ($As1$), ($As2$) and (5), we derive

$$
\begin{aligned}
|\vartheta(x) - \vartheta^*(x)| &= \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta^*(y)\, dy\right| \\
&= \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy\right. \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy \\
&\quad \left. - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta^*(y)\, dy\right| \\
&\leq \left|\vartheta(x) - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy\right| \\
&\quad + \left|\frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy\right. \\
&\quad \left. - \frac{1}{\sqrt{2\pi}} \int_a^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta^*(y)\, dy\right| \\
&\leq \phi(x) + \left|\frac{1}{\sqrt{2\pi}} \int_a^x |\alpha \cos(xy) + \beta \sin(xy)||\vartheta(y) - \vartheta^*(y)|\, dy\right|.
\end{aligned}
$$

By the Cauchy–Schwartz inequality, we have

$$
\begin{aligned}
&\phi(x) + \left|\frac{1}{\sqrt{2\pi}} \int_a^x |\alpha \cos(xy) + \beta \sin(xy)||\vartheta(y) - \vartheta^*(y)|\, dy\right| \\
\leq\ &\phi(x) + \left|\frac{1}{\sqrt{2\pi}} \int_a^x \sqrt{[\alpha^2 + \beta^2][\cos^2(xy) + \sin^2(xy)]}\, |\vartheta(y) - \vartheta^*(y)|\, dy\right|.
\end{aligned}
$$

Thus, we derive

$$
|\vartheta(x) - \vartheta^*(x)| \leq \phi(x) + \sqrt{\frac{\alpha^2 + \beta^2}{2\pi}} \int_a^x |\vartheta(y) - \vartheta^*(y)|\, dy. \tag{6}
$$

Using the Gronwall lemma (Lungu [26], Rus [10]), from (6), we obtain that

$$
|\vartheta(x) - \vartheta^*(x)| \leq \phi(x) \exp\left(\sqrt{\frac{\alpha^2 + \beta^2}{2\pi}}(b - a)\right). \tag{7}
$$

Let

$$
c = \exp\left(\sqrt{\frac{\alpha^2 + \beta^2}{2\pi}}(b - a)\right).
$$

Hence, it follows from (7) that

$$
|\vartheta(x) - \vartheta^*(x)| \leq c \times \phi(x). \tag{8}
$$

The above outcomes and (8) imply that Equation (1) is Hyers–Ulam–Rassias stable. Thus, the proof of Theorem 3.1 is completed. □

4. Illustrative example

To demonstrate that the conditions described above are attainable, we will provide an example. First, we will make some considerations about the solution of our equation.

If we consider $\vartheta \in C([0,1], \mathbb{B})$ with $\vartheta(x) \neq 0$ for $x \in [0,1]$ and $\vartheta(0) = 1$, the solution of

$$\vartheta(x) = \frac{1}{\sqrt{2\pi}} \int_0^x [\alpha \cos(xy) + \beta \sin(xy)]\vartheta(y)\, dy,$$

is

$$\vartheta^*(x) = \exp\left[\frac{\alpha}{2} C\left(\sqrt{\frac{2}{\pi}}\right) + \frac{\beta}{2} S\left(\sqrt{\frac{2}{\pi}}\right)\right]$$

with $C(.)$ and $S(.)$ the Fresnel C integral and the Fresnel S integral, respectively.

Example 4.1. *To verify that* $(As1)$ *and* $(As2)$ *of Theorem 2.1 hold, we let* $\alpha = 0.2$, $\beta = 0$ *and calculate:*

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\Big|[0.2\cos(xy)]\vartheta_1 - [0.2\cos(xy)]\vartheta_2\Big| &= \frac{0.2}{\sqrt{2\pi}}\Big|[\cos(xy)]\vartheta_1 - [\cos(xy)]\vartheta_2\Big| \\ &\leq \frac{0.2}{\sqrt{2\pi}}|\vartheta_1 - \vartheta_2|, \forall\, \vartheta_1, \vartheta_2 \in \mathbb{B}. \end{aligned}$$

Let the perturbation of the solution $\vartheta(x) = \exp(0.3x)$. *So we have*

$$|\vartheta(x) - \vartheta^*(x)| = \left|\exp(0.3x) - \exp\left[\frac{1}{10} C\left(\sqrt{\frac{2}{\pi}}\right)\right]\right| \leq \exp\frac{0.2}{\sqrt{2\pi}}\epsilon,$$

where

$$0 < \sqrt{\frac{\alpha^2 + \beta^2}{2\pi}} = \frac{0.2}{\sqrt{2\pi}} < 1,$$

$$c = \exp\left(\sqrt{\frac{\alpha^2 + \beta^2}{2\pi}}(b - a)\right) = \exp\frac{0.2}{\sqrt{2\pi}}.$$

So, all conditions of Theorem 2.1 are satisfied and we obtain the stability of the oscillatory Volterra integral Equation (1) *in the Hyers–Ulam sense.*

In addition, with choosing $\phi(x) = \exp(2x)$, *all the hypotheses of Theorem 3.1 are satisfied. Thus, Equation* (1) *is stable in the Hyers–Ulam–Rassias sense.*

5. Conclusion

In this manuscript, we investigated the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of oscillatory Volterra integral equations with kernels depending on sine and cosine functions by using a Gronwall lemma.

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