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Approximation by Bézier type associated shifted knots of (λ, q)-Bernstein operators

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Abstract. In this article, we introduce the (λ, q)-Bernstein operators associated with Bézier basis functions using the features of shifted knots. We first construct the (λ, q)-Bernstein operators using shifted knot polynomials that are connected by a Bézier basis function. We then look into Korovkin's theorem, prove a local approximation theorem, and obtain the convergence theorem for Peetre's K-functional and Lipschitz continuous functions. The Voronovskaja type asymptotic formula is finally found in the concluding section of this paper.

1. Introduction and preliminaries

Among the most famous Weierstrass approximation theorems, the shortest and an elegant proof was given by one of the most famous mathematicians in the world, S. N. Bernstein, who first invented the sequence of positive linear operators implied by $\{B_s\}_{s\geq 1}$. It was shown in Bernstein's study that for all $g \in C[0,1]$, the class of all continuous functions on $[0,1]$ can be uniformly approximated by a function called the famous Bernstein polynomial, which is defined in [9]. Accordingly, the famous Bernstein polynomial can be defined as follows for all $y \in [0,1]$.

$$B_s(g; y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) b_{s,i}(y),$$

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where $s \in \mathbb{N}$ (positive integers) and $b_{s,i}(y)$ are the Bernstein polynomials of degree at most s defined by

$$b_{s,i}(y) = \binom{s}{i} y^i (1-y)^{s-i} \qquad (i = 0, 1, \cdots, s; y \in [0,1])$$

and

$$b_{s,i}(y) = 0 \qquad (i < 0 \text{ or } i > s).$$

It is very easy to verify the recursive relation for the Bernstein polynomials
The recursive relationship for Bernstein polynomials $b_{s,i}(y)$ is very simple to prove such that

$$b_{s,i}(y) = (1-y) b_{s-1,i}(y) + y b_{s-1,i-1}(y).$$

In 2010, Cai et al., defined the Bernstein-polynomials by introduce of new Bézier bases with shape parameter $\lambda \in [-1, 1]$, known as the λ-Bernstein operators as follows:

$$B_{s,\lambda}(g; y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) \tilde{b}_{s,i}(\lambda; y), \tag{1}$$

where the new Bernstein basis function $\tilde{b}_{s,i}(\lambda; y)$ in terms of the Bernstein polynomial $b_{s,i}(y)$ defined by Ye et al. [46] as follows:

$$\begin{aligned}
\tilde{b}_{s,0}(\lambda; y) &= b_{s,0}(y) - \frac{\lambda}{s+1} b_{s+1,1}(y), \\
\tilde{b}_{s,i}(\lambda; y) &= b_{s,i}(y) + \lambda\Big(\frac{s-2i+1}{s^2-1} b_{s+1,i}(y) - \frac{s-2i-1}{s^2-1} b_{s+1,i+1}(y)\Big), \text{ for } 1 \le i \le s-1 \\
\tilde{b}_{s,s}(\lambda; y) &= b_{s,s}(y) - \frac{\lambda}{s+1} b_{s+1,s}(y)
\end{aligned}$$

In 2010, Gadjiev et al., introduced the Recent Bernstein-type Stancu polynomials by means of shifted knots [18] such as:

$$S_{s,\mu,\beta}(g; y) = \left(\frac{s+\nu_2}{s}\right)^s \sum_{i=0}^{s} \binom{s}{i} \left(y - \frac{\mu_2}{s+\nu_2}\right)^i \left(\frac{s+\mu_2}{s+\nu_2} - y\right)^{s-i} g\left(\frac{i+\mu_1}{s+\nu_1}\right) \tag{2}$$

where $y \in [\frac{\mu_2}{m+\nu_2}, \frac{s+\mu_2}{s+\nu_2}]$ and $\mu_i,\ \nu_i,\ i = 1, 2$ are positive real numbers provided $0 \le \mu_2 \le \mu_1 \le \nu_1 \le \nu_2$.

As a result of research conducted in the approximation process, Bernstein type operators have been obtained by researchers within the past few years, for example, q-Phillips operators [1], Bernstein-Kantorovich-Stancu Shifted Knots Operators [2], Stancu variant of Bernstein-Kantorovich operators [26], Genuine modified Bernstein-Durrmeyer operators [27], new family of Bernstein-Kantorovich operators [28], q-Bernstein shifted operators [29], $(\lambda; q)$-Bernstein-Stancu operators [30], Generalized q-Bernstein-Schurer operators [31], Bézier bases with Schurer polynomials [34], Generalized Bernstein–Schurer operators [35], Bernstein operators Based on Bézier bases [43] and q-Baskakov operators via wavelets [42], etc. For more details and recent published research we prefer to see [7, 32, 33, 39–41]. Moreover, for the approximation behaviour more fundamental results we prefer to see [3–5, 12–14, 23, 36, 44, 45].

Most recent M. Ayman-Mursaleen et al. constructed the λ-Bernstein shifted knots operators in terms of Bézier bases function (see [6]) by:

$$B_{s,\lambda}^{\varkappa_1,\varkappa_2}(g; y) = \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda; y) g\left(\frac{i}{s}\right), \tag{3}$$

where Bézier bases $\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}$ and Bernstein basis function $b_{s,j}^{\varkappa_1,\varkappa_2}$ are defined as:

$$\tilde{b}_{s,0}^{\varkappa_1,\varkappa_2}(\lambda;y) = b_{s,0}^{\varkappa_1,\varkappa_2}(y) - \frac{\lambda}{s+1} b_{s+1,1}^{\varkappa_1,\varkappa_2}(y),$$

$$\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(\lambda;y) = b_{s,j}^{\varkappa_1,\varkappa_2}(y) + \lambda\Big(\frac{s-2j+1}{s^2-1} b_{s+1,j}^{\varkappa_1,\varkappa_2}(y) - \frac{s-2j-1}{s^2-1} b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y)\Big), \text{ for } 1 \le j \le s-1$$

$$\tilde{b}_{s,s}^{\varkappa_1,\varkappa_2}(\lambda;y) = b_{s,s}^{\varkappa_1,\varkappa_2}(y) - \frac{\lambda}{s+1} b_{s+1,s}^{\varkappa_1,\varkappa_2}(y),$$

$$b_{s,j}^{\varkappa_1,\varkappa_2}(y) = \binom{s}{j}\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^j \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s-j}. \tag{4}$$

Lupaş [25] introduced the first Bernstein polynomials by using the q-analogue, in which the some approximation properties as well as shape-preserving properties were calculated. Phillips [38] in 1997, gives the classical-Bernstein polynomials in another form of q-analogue as follows:

$$\tilde{B}_{n,q}(g;y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k \prod_{k=0}^{n-k-1} (1-q^k y)\ g\left(\frac{[k]_q}{[n]_q}\right), \ \ y \in [0,1].$$

The q-analogue of Lupas Bernstein operators with shifted knots were introduced by M. Mursaleen et al. (see [29]) by:

$$B_{s,j}^{\varkappa_1,\varkappa_2}(f;y,q) = \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s} \sum_{j=0}^{s} \begin{bmatrix} s \\ j \end{bmatrix}_q \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^j \left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2} - y\right)^{s-j} f\left(\frac{[j]_q}{[s]_q}\right). \tag{5}$$

For q-integers and its properties we easily get some basic properties (see [21, 22]), for example, for any $q \in (0,1)$, and $0 \le n \le k$ the binomial coefficient for q-integer given by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$ and it's satisfy the recurrence relations as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

The q-binomial polynomial is given by

$$(1+y)_q^{n-k} = \begin{cases} (1+y)(1+qy)\cdots(1+q^{n-k-1}y) & (n,k \in \mathbb{N}) \\ 1 & (n=k=0). \end{cases}$$

2. Operators and basic estimation

We take the Bernstein basis function $b_{s,j}^{\varkappa_1,\varkappa_2}$ by means of shifted knots (see [18, 29]) as follows:

$$b_{s,j}^{\varkappa_1,\varkappa_2}(y;q) = \begin{bmatrix} s \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}} \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^j \left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2} - y\right)^{s-j}. \tag{6}$$

The equality (6) can be also written as:

$$b_{s,j}^{\varkappa_1,\varkappa_2}(y;q) = \begin{bmatrix} s \\ j \end{bmatrix}_q \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^j \prod_{w=0}^{s-j-1} \left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2} - q^w y\right). \tag{7}$$

We take the Bézier bases function $\tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)$ by means of Bernstein basis function $b^{\varkappa_1,\varkappa_2}_{s,j}(y;q)$ (see [10, 46]) such as follows:

$$\tilde{b}^{\varkappa_1,\varkappa_2}_{s,0}(y;q,\lambda) = b^{\varkappa_1,\varkappa_2}_{s,0}(y;q) - \frac{\lambda}{[s]_q+1} b^{\varkappa_1,\varkappa_2}_{s+1,1}(y;q),$$

$$\tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda) = b^{\varkappa_1,\varkappa_2}_{s,j}(y;q) + \lambda\Big(\frac{[s]_q-2[j]_q+1}{[s]_q^2-1} b^{\varkappa_1,\varkappa_2}_{s+1,j}(y;q) - \frac{[s]_q-2q[j]_q-1}{[s]_q^2-1} b^{\varkappa_1,\varkappa_2}_{s+1,j+1}(y;q)\Big), \text{ for } 1 \le j \le s-1$$

$$\tilde{b}^{\varkappa_1,\varkappa_2}_{s,s}(y;q,\lambda) = b^{\varkappa_1,\varkappa_2}_{s,s}(y;q) - \frac{\lambda}{[s]_q+1} b^{\varkappa_1,\varkappa_2}_{s+1,s}(y;q).$$

Thus for all $\frac{\varkappa_1}{[s]_q+\varkappa_2} \le y \le \frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}$ and the real number $0 \le \varkappa_1 \le \varkappa_2$, we define the Lupas q-Bernstein shifted knots operators in terms of Bézier bases function $\tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)$ as follows:

$$\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(f;y,q) = \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda) f\left(\frac{[j]_q}{[s]_q}\right), \tag{8}$$

where $C[0,1]$ be the set of all continuous functions defined on $[0,1]$ and $s \in \mathbb{N}$ (the set of positive integers).

Remark 2.1. *We have the following observations:*

1. *If we put $q = 1$ in the equality* (8)*, then our operators $\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(f;y,q)$ reduced to equality* (3) *by [6].*
2. *If we put $q = 1$ and $\varkappa_1 = \varkappa_2 = 0$ in the equality* (8)*, then our operators $\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(f;y,q)$ reduced to equality* (7) *by [11].*
3. *It is obvious that, in contrast to [6, 11, 25, 29], our new operators are the newest and most generalized operators.*

In general, this paper is organized as follows: For our new operators, (8), we examine their moments and central moments. We study an asymptotic formula for Voronovskaja, prove a local approximation theorem, present a convergence theorem for Lipschitz continuous functions, and analyze a Korovkin approximation theorem.

Lemma 2.2. *For $f(t) = 1, t$, we have*

$$\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(1;y,q) = 1$$

$$\begin{aligned}
\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(t;y,q) &= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q}\left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right) + \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q-1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q - \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\right\} \\
&+ \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \frac{-2\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q^2-1}\Big[\frac{1}{[s]_q}\Big\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q - \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s_q\Big\} \\
&+ q\left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\Big\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^{s-1}_q - \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\Big\}\Big] \\
&+ \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \frac{\lambda}{q[s]_q([s]_q+1)}\Big[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^{s+1}_q - \prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2} - q^w y\right) \\
&- \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s+1}_q - [s+1]_q\left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\Big\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q - \left(y - \frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\Big\}\Big]
\end{aligned}$$

$$+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda}{[s]_q([s]_q^2-1)}\Big[2[s]_q[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}$$
$$-\frac{2}{q}[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s}\right\}$$
$$+\frac{2}{q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right\}\Big].$$

Proof. For prove the our equality we taking into account the equality $[j+1]_q = 1+q[j]_q$, then easy to get

$$\begin{aligned}
\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y,q) &= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Big\{\sum_{j=0}^{s} b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)-\frac{\lambda}{[s]_q+1}b_{s+1,1}^{\varkappa_1,\varkappa_2}(y;q)+\lambda\frac{[s]_q-2[1]_q+1}{[s]_q^2-1}b_{s+1,1}^{\varkappa_1,\varkappa_2}(y;q)\\
&-\lambda\frac{[s]_q-2q[1]_q-1}{[s]_q^2-1}b_{s+1,2}^{\varkappa_1,\varkappa_2}(y;q)+\lambda\frac{[s]_q-2q[2]_q+1}{[s]_q^2-1}b_{s+1,2}^{\varkappa_1,\varkappa_2}(y;q)\\
&-\lambda\frac{[s]_q-2q[2]_q-1}{[s]_q^2-1}b_{s+1,3}^{\varkappa_1,\varkappa_2}(y;q)+\cdots+\lambda\frac{[s]_q-2[s-1]_q+1}{s^2-1}b_{s+1,s-1}^{\varkappa_1,\varkappa_2}(y;q)\\
&-\lambda\frac{[s]_q-2q[s-1]_q-1}{[s]_q^2-1}b_{s+1,s}^{\varkappa_1,\varkappa_2}(y;q)-\frac{\lambda}{[s]_q+1}b_{s+1,s}^{\varkappa_1,\varkappa_2}(y;q)\Big\}\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s} b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s}\begin{bmatrix} s\\ j\end{bmatrix}_q q^{\frac{j(j-1)}{2}}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^j\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-y\right)^{s-j}\\
&= 1
\end{aligned}$$

$$\begin{aligned}
\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) &= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\frac{[j]_q}{[s]_q}\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Big\{\sum_{j=1}^{s-1}\frac{[j]_q}{[s]_q}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)+\tilde{b}_{s,s}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\Big\}\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=1}^{s-1}\frac{[j]_q}{[s]_q}\Big[b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)+\lambda\Big(\frac{[s]_q-2[j]_q+1}{[s]_q^2-1}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q)\\
&-\frac{[s]_q-2q[j]_q-1}{[s]_q^2-1}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q)\Big)\Big]+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Big\{b_{s,s}^{\varkappa_1,\varkappa_2}(y;q)-\frac{\lambda}{1+[s]_q}b_{s+1,s}^{\varkappa_1,\varkappa_2}(y;q)\Big\},
\end{aligned}$$

$$\begin{aligned}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) &= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg\{\sum_{j=0}^{s}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\frac{[j]_q}{[s]_q}+\lambda\sum_{j=0}^{s}\frac{[j]_q}{[s]_q}\frac{[s]_q-2[j]_q+1}{[s]_q^2-1}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q)\\
&-\lambda\sum_{j=1}^{s-1}\frac{[j]_q}{[s]_q}\frac{[s]_q-2[j]_q-1}{[s]_q^2-1}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q)\Bigg\}\\
&= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg\{y+\frac{\lambda[s+1]_q}{[s]_q([s]_q-1)}\sum_{j=1}^{s}\frac{[j]_q}{[s+1]_q}\frac{[s+1]_q!}{[1+s-j]_q![j]_q!}y^j\prod_{w=0}^{s-j}(1-q^w y)\\
&-\frac{2\lambda[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=1}^{s}\frac{[j]_q^2}{[s+1]_q}\frac{[s+1]_q!}{[1+s-j]_q![j]_q!}y^j\prod_{w=0}^{s-j}(1-q^w y)\\
&-\frac{\lambda[s+1]_q}{[s]_q(1+[s]_q)}\sum_{j=1}^{s-1}\frac{[j]_q}{[s+1]_q}\frac{[s+1]_q!}{[s-j]_q![j+1]_q!}y^{j+1}\prod_{w=0}^{s-j-1}(1-q^w y)\\
&+\frac{2q\lambda[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=1}^{s-1}\frac{[j]_q^2}{[s+1]_q}\frac{[s+1]_q!}{[s-j]_q![j+1]_q!}y^{j+1}\prod_{w=0}^{s-j-1}(1-q^w y)\Bigg\}\\
&:= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Big\{y+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\Big\}\text{ (suppose).}\end{aligned}$$

Here we easily conclude the Σ_1, Σ_2, Σ_3 and Σ_4 as follows:

$$\begin{aligned}\Sigma_1 &= \frac{\lambda[s+1]_q}{[s]_q([s]_q-1)}\sum_{j=1}^{s}\frac{[j]_q}{[s+1]_q}\frac{[s+1]_q!}{[s+1-j]_q![j]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^j\prod_{w=0}^{s-j}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\\
&= \frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q-1)}\sum_{j=0}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)\\
&= \frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q-1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\right\}\end{aligned}$$

$$\begin{aligned}\Sigma_2 &= -\frac{2\lambda[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=1}^{s}\frac{[j]_q^2}{[s+1]_q}\frac{[s+1]_q!}{[s+1-j]_q![j]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^j\prod_{w=0}^{s-j}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\\
&= -\frac{2\lambda[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=1}^{s}\frac{[j]_q}{[s+1]_q}\frac{[s+1]_q!}{[s+1-j]_q![j]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^j\prod_{w=0}^{s-j}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\\
&+\frac{-2q\lambda[s+1]_q}{[s]_q^2-1}\sum_{j=1}^{s}\frac{[j]_q[j-1]_q}{[s+1]_q[s]_q}\frac{[s+1]_q!}{[j]_q![s+1-j]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^j\prod_{w=0}^{s-j}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\\
&= \frac{-2\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=0}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)+\frac{-2q\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2[s]_q[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=0}^{s-2}b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q)\end{aligned}$$

$$= \frac{-2\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q^2-1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s - q^{\frac{s(s-1)}{2}}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+\frac{-2q\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2[s]_q[s+1]_q}{[s]_q([s]_q^2-1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s - q^{\frac{s(s-1)}{2}}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\right.$$

$$\left.-q^{\frac{(s-1)(s-2)}{2}}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-y\right)\right\}.$$

$$= \frac{-2\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q^2-1)}\Big[\frac{1}{[s]_q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\Big].$$

To obtain Σ_3, we use $[j]_q = \frac{[j+1]_q}{q} - \frac{1}{q}$, thus

$$\Sigma_3 = -\frac{\lambda[s+1]_q}{[s]_q([s]_q+1)}\sum_{j=1}^{s-1}\frac{[j]_q}{[s+1]_q}\frac{[s+1]_q!}{[j+1]_q![s-j]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{j+1}\prod_{w=0}^{s-j-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)$$

$$= -\frac{\lambda[s+1]_q}{q[s]_q([s]_q+1)}\sum_{j=1}^{s-1}\frac{[j+1]_q}{[s+1]_q}\frac{[s+1]_q!}{[s-j]_q![j+1]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{j+1}\prod_{w=0}^{s-j-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)$$

$$+\frac{\lambda}{[s]_q([s]_q+1)}\sum_{j=1}^{s-1}\frac{b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q)}{q}$$

$$= -\frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q+1)}\sum_{j=1}^{s-1}\frac{b_{s,j}^{\varkappa_1}(y;q)}{q}+\frac{\lambda}{[s]_q([s]_q+1)}\sum_{j=1}^{s-1}\frac{b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q)}{q}$$

$$= -\frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{q[s]_q([s]_q+1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\prod_{w=0}^{s-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+\frac{\lambda}{q[s]_q([s]_q+1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\right.$$

$$\left.-[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\prod_{w=0}^{s-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right\}$$

$$= \frac{\lambda}{q[s]_q([s]_q+1)}\Big[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)$$

$$-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}-[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}\Big]$$

Similarly, we use $q[j]_q^2 = [j]_q\left([j+1]_q-1\right)$, then easy to get

$$\Sigma_4 = \frac{2q\lambda[s+1]_q}{[s]_q([s]_q^2-1)}\sum_{j=1}^{s-1}\frac{[j]_q^2}{[s+1]_q}\frac{[s+1]_q!}{[s-j]_q![j+1]_q!}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{j+1}\prod_{w=0}^{s-j-1}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)$$

$$
\begin{aligned}
&= \frac{2\lambda[s]_q[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2}{[s]_q([s]_q^2-1)}\sum_{j=0}^{s-2} b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q) - \frac{2\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{q([s]_q^2-1)[s]_q}\sum_{j=1}^{s-1} b_{s,j}^{\varkappa_1,\varkappa_2}(y;q)\\
&+\frac{2\lambda}{([s]_q^2-1)[s]_q}\sum_{j=1}^{s-1}\frac{b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q)}{q}\\
&= \frac{\lambda}{[s]_q^2-1}\Bigg[2[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\\
&-\frac{2}{q[s]_q}[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s}\right\}\\
&+\frac{2}{q[s]_q}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\Bigg\}\Bigg].
\end{aligned}
$$

These explanations gives $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)$. □

Lemma 2.3. *For the basic test functions $g(t)=t^2$, the operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}$ have the moments:*

$$
\begin{aligned}
\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y,q) &= \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q-1)}\\
&\times\Bigg[q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\Bigg]\\
&+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{-2\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q^2-1}\Bigg[\frac{1}{[s]_q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s}\right\}\\
&+q(2+q)\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\\
&+q^3[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-2}\right\}\Bigg]\\
&+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{-\lambda}{q[s]_q([s]_q+1)}\Bigg[[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\\
&-\frac{[s+1]_q}{q[s]_q}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s}\right\}\\
&+\frac{1}{q[s]_q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right]\\
&+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{2\lambda}{[s]_q([s]_q^2-1)}\Bigg[q[s-1]_q[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^3\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-2}\right\}\\
&-\frac{(1-q)[s+1]_q}{q}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}
\end{aligned}
$$

$$+\frac{[s+1]_q}{q^2[s]_q}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$
$$-\frac{1}{q^2[s]_q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right\}\bigg].$$

Proof. For $f(t)=t^2$, we have $f\left(\frac{[j]_q}{[s]_q}\right)=\left(\frac{[j]_q}{[s]_q}\right)^2$, then we easily conclude that

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y,q)$$
$$=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\frac{[j]_q^2}{[s]_q^2}$$
$$=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg\{\sum_{j=0}^{s}\frac{[j]_q^2}{[s]_q^2}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)+\lambda\sum_{j=0}^{s}\frac{[j]_q^2}{[s]_q^2}\frac{[s]_q-2[j]_q+1}{[s]_q^2-1}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)$$
$$-\lambda\sum_{j=1}^{s-1}\frac{[j]_q^2}{[s]_q^2}\frac{[s]_q-2q[j]_q-1}{[s]_q^2-1}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\Bigg\}$$
$$=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg\{\frac{1}{[s]_q}\left([s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2+\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-y\right)\right)$$
$$+\frac{\lambda}{[s]_q-1}\sum_{j=0}^{s}\frac{[j]_q^2}{[s]_q^2}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)-\frac{2\lambda}{[s]_q^2-1}\sum_{j=0}^{s}\frac{[j]_q^3}{[s]_q^2}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)$$
$$-\frac{\lambda}{[s]_q+1}\sum_{j=1}^{s-1}\frac{[j]_q^2}{[s]_q^2}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)+\frac{2q\lambda}{[s]_q^2-1}\sum_{j=1}^{s-1}\frac{[j]_q^3}{[s]_q^2}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\Bigg\}$$
$$=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\left[\frac{1}{[s]_q}\left\{[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2+\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-y\right)\right\}+\Sigma_5+\Sigma_6+\Sigma_7+\Sigma_8\right]\text{ (suppose).}$$

To obtain Σ_5, we use $[j]_q^2=[j]_q(1+q[j-1]_q)$, thus

$$\Sigma_5=\frac{\lambda}{[s]_q-1}\sum_{j=0}^{s}\frac{[j]_q^2}{[s]_q^2}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)$$
$$=\frac{\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{[s]_q([s]_q-1)}\sum_{j=0}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)+\frac{q\lambda[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2}{[s]_q^2([s]_q-1)}\sum_{j=0}^{s-2}b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)$$
$$=\frac{\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{[s]_q^2([s]_q-1)}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s$$
$$+q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\Bigg].$$

To obtain Σ_6, we use $[j]_q^3 = [j]_q + q(q+2)[j]_q[j-1]_q + q^3[j]_q[j-1]_q[j-2]_q$, thus

$$\begin{aligned}\Sigma_6 &= \frac{-2\lambda}{[s]_q^2-1}\sum_{j=0}^{s}\frac{[j]_q^3}{[s]_q^2}b_{s+1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&= \frac{-2\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{[s]_q^2([s]_q^2-1)}\sum_{j=0}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda) - \frac{2q(2+q)\lambda[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2}{[s]_q^2([s]_q^2-1)}\sum_{j=0}^{s-2}b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&-\frac{2q^3\lambda[s+1]_q[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^3}{[s]_q^2([s]_q^2-1)}\sum_{j=0}^{s-3}b_{s-2,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&= \frac{-2\lambda[s+1]_q}{[s]_q^2([s]_q^2-1)}\Bigg[\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\right\}\\
&+q(2+q)[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\\
&+q^3[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^3\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-2}\right\}\Bigg].\end{aligned}$$

For finding Σ_7, we use $[j]_q^2 = \frac{[j+1]_q[j]_q}{q} - \frac{[j+1]_q+1}{q^2}$, thus

$$\begin{aligned}\Sigma_7 &= \frac{-\lambda}{[s]_q+1}\sum_{j=1}^{s-1}\frac{[j]_q^2}{[s]_q^2}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&= \frac{-\lambda[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2}{q[s]_q^2([s]_q+1)}\sum_{j=0}^{s-2}b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda) + \frac{\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{q^2[s]_q^2([s]_q+1)}\sum_{j=1}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&-\frac{\lambda}{q^2[s]_q^2([s]_q+1)}\sum_{j=1}^{s-1}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&= \frac{-\lambda}{q^2[s]_q^2([s]_q+1)}\Bigg[q[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\\
&-[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^s\right\}\\
&+\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\Bigg\}\Bigg].\end{aligned}$$

Next, we obtain Σ_8, thus by a simple computation we use $[j]_q^3 = [j+1]_q[j]_q[j-1]_q - \frac{1-q}{q^2}[j+1]_q[j]_q + \frac{[j+1]_q}{q^3} - \frac{1}{q^3}$, thus

$$\Sigma_8 = \frac{2q\lambda}{[s]_q^2-1}\sum_{j=1}^{s-1}\frac{[j]_q^3}{[s]_q^2}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda) + \frac{2q\lambda[s+1]_q[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^3}{[s]_q^2([s]_q^2-1)}\sum_{j=0}^{s-3}b_{s-2,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)$$

$$
\begin{aligned}
&-\frac{2(1-q)\lambda[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2}{[s]_q^2([s]_q^2-1)}\sum_{j=0}^{s-2}b_{s-1,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&+\frac{2\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{q^2[s]_q^2([s]_q^2-1)}\sum_{j=1}^{s-1}b_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)-\frac{2\lambda}{q^2[s]_q^2([s]_q^2-1)}\sum_{j=1}^{s-1}b_{s+1,j+1}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\\
&=\frac{2\lambda}{q^2[s]_q^2([s]_q^2-1)}\Bigg[q^3[s+1]_q[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^3\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-2}\right\}\\
&-q(1-q)[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s-1}\right\}\\
&+[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)^{s}\right\}\\
&-\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^wy\right)-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\Bigg\}\Bigg].
\end{aligned}
$$

□

Lemma 2.4. *Operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)$ have the central moments of order one given by:*

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q)$$

$$
\begin{aligned}
&=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg[\frac{\lambda\left(y-\frac{[s]_q}{[s]_q+\varkappa_2}\right)[s+1]_q}{[s]_q([s]_q-1)}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}\Bigg]\\
&-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{2\left(y-\frac{[s]_q}{[s]_q+\varkappa_2}\right)\lambda[s+1]_q}{([s]_q^2-1)[s]_q}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\\
&+q[s]_q\left(y-\frac{[s]_q}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\Bigg]\\
&+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda}{q([s]_q+1)[s]_q}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\\
&-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^wy\right)-[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}\Bigg]\\
&+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda}{q[s]_q([s]_q^2-1)}\Bigg[2q[s]_q[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\Bigg]\\
&-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg[2[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}\Bigg]
\end{aligned}
$$

$$+2\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\Bigg\}\Bigg]$$

$$:=\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)\ (suppose).$$

Lemma 2.5. *Operators* $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)$ *have the central moments of order two is given by:*

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\left((t-y)^2;y,q\right)$$

$$=\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2+\frac{1}{[s]_q}\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{[s]_q^2([s]_q-1)}\Bigg[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s$$

$$+q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\Bigg]$$

$$-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{2\lambda[s+1]_q}{[s]_q^2([s]_q^2-1)}\Bigg[\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}$$

$$+q(2+q)[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}$$

$$+q^3[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^3\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-2}\right\}\Bigg]$$

$$-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda}{q^2[s]_q^2(1+[s]_q)}\Bigg[q^2[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}$$

$$-[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\left\{\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\right\}\Bigg]$$

$$+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{2\lambda}{q^2[s]_q^2([s]_q^2-1)}\Bigg[q^3[s+1]_q[s]_q[s-1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^3\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-2}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-2}\right\}$$

$$+q(q-1)[s+1]_q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}$$

$$+[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$-\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right\}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\Bigg]$$

$$-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2}{([s]_q-1)[s]_q}\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$+\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{2\lambda[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2}{[s]_q([s]_q^2-1)}\left[\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right.$$

$$\left.+q[s]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\right]$$

$$-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{q(1+[s]_q)[s]_q}\left[\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\right.$$

$$\left.-[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}\right]$$

$$-\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\frac{\lambda\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)}{q[s]_q([s]_q^2-1)}\left[2q[s]_q[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s-1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s-1}\right\}\right.$$

$$+2[s+1]_q\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^s\right\}$$

$$\left.+2\left\{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^{s+1}-\left(y-\frac{\varkappa_1}{[s]_q+\varkappa_2}\right)_q^{s+1}\right\}-\prod_{w=0}^{s}\left(\frac{[s]_q+\varkappa_1}{[s]_q+\varkappa_2}-q^w y\right)\right\}\right]$$

$$:=\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)\ (suppose).$$

3. Convergence properties of operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)$

Here, for the operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)$ by (8), we derive various global and local approximation theorems. Using the Ditzian-Totik uniform modulus of smoothness, we first define the uniformly convergence property for our operators and then obtain the local and global approximations. Next, we derive some straightforward theorems based on the Lipschitz type maximal approximation property and Peetre's K-functional property. Given a continuous function g in $C[0,1]$ on $[0,1]$, we can replace g with a real-valued function equipped with the norm $\| g \|_{C[0,1]}=\sup_{y\in[0,1]}\left|g(y)\right|$.

Theorem 3.1. *([17, 24]) Any series of positive linear operators K_s that acts uniformly on $[u,v]$ for all $\rho=0,1,2$ will fulfill $\lim_{s\to\infty}K_s(t^\rho;y)=y^\rho$, for all $C[u,v]$. Then, for each compact subset of $[u,v]$, the operators $\lim_{s\to\infty}K_s(g)=g$ are uniformly convergent for each $g\in C[u,v]$.*

Theorem 3.2. *Let $q=q_s$ be any real numbers satisfying $0<q_s<1$, then for all $g\in C[0,1]$ we get*

$$\lim_{s\to\infty}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q)=g(y)$$

is converges uniformly on $[0,1]$, specifically, $C[0,1]$ represents the set of all continuous functions on $[0,1]$).

Proof. It is evident from Lemma 2.2 and Lemma 2.3 that in order to achieve

$$\lim_{s\to\infty}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^\rho;y,q)=y^\rho\quad(\rho=0,1,2).$$

It follows that the operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q)$ are uniformly convergent to set $g \in C[0,1]$ by considering the Bohman-Korovkin-Popoviciu theorem. We finish the needed Theorem proof. □

Theorem 3.3. *[19, 20]* $\{P_s\}_{s\geq 1}$ *for any operator taking action from* $C[0,1]$ *to* $C[0,1]$. *Let* $\lim_{s\to\infty} \|P_s(t^\rho) - y^\rho\|_{C[0,1]} = 0$, $\rho = 0,1,2$, *therefore for every* $g \in C[0,1]$, *to obtain*

$$\lim_{s\to\infty} \|P_s(g) - g\|_{C[0,1]} = 0.$$

Theorem 3.4. *Consider* $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}$ *as an actuator that moves from* $C[0,1]$ *to* $C[0,1]$ *and is associated with the property* $\lim_{n\to\infty} \|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^\rho;y,q) - y^\rho\|_{C[0,1]} = 0$. *Moreover, take the positive number sequences* $q = q_s$ *such that* $q_s \in (0,1)$. *Then for any* $g \in C[0,1]$, *we get*

$$\lim_{s\to\infty} \| \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) - g \|_{C[0,1]} = 0.$$

Proof. We take in account Theorem 3.3 and Korovkin's theorem then easily we lead to show that

$$\lim_{n\to\infty} \|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^\rho;y,q) - y^\rho\|_{C[0,1]} = 0, \quad \rho = 0,1,2.$$

In the view of Lemma 2.2, easy to obtain $\|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y,q) - 1\|_{C[0,1]} = \sup_{y\in[0,1]} \left|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y,q) - 1\right| = 0$. For $\rho = 1$, easy to see

$$\begin{aligned} \|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) - y\|_{C[0,1]} &= \sup_{y\in[0,1]} \left|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) - y\right| \\ &= \sup_{y\in[0,1]} \Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y). \end{aligned}$$

Since $s \to \infty$ then $\frac{1}{[s]_q} \to 0$, $\frac{[s]_q+\varkappa_1]_q}{[s]_q+\varkappa_2} \to 1$ therefore we get $\|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) - y\|_{C[0,1]} \to 0$. Similarly for $\rho = 2$, we see

$$\begin{aligned} \|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y,q) - y^2\|_{C[0,1]} &= \sup_{y\in[0,1]} \left|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y,q) - y^2\right| \\ &= \sup_{y\in[0,1]} \Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y). \end{aligned}$$

which leads to get $\|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y,q) - y^2\|_{C[0,1]} \to 0$ as $s \to \infty$. These observations leads to get our proof. □

Using the Ditzian-Totik uniform modulus of smoothness, we present some results on global approximations. We go over the fundamental characteristic of the uniform modulus of smoothness for orders first and second, which is

$$\omega(g,\delta) := \sup_{0<|\rho|\leq\delta} \sup_{y,y+\rho\gamma(y)\in[0,1]} \{|g(y+\beta\gamma(y)) - g(y)|\};$$

$$\omega_2^\gamma(g,\delta) := \sup_{0<|\beta|\leq\delta} \sup_{y,y\pm\beta\gamma(y)\in[0,1]} \{|g(y+\beta\gamma(y)) - 2g(y) + g(y-\beta\gamma(y))|\},$$

and the step-weight function γ on $[u,v]$ and suppose $\gamma(y) = [(y-u)(v-y)]^{1/2}$ if $y \in [u,v]$ (see [16]). Let we denote the set of all absolutely continuous functions be C^*, then the Peetre's K-functional property is given by

$$K_2^\gamma(g,\delta) = \inf_{\zeta\in\Theta^2(\gamma)} \left\{\|g-\zeta\|_{C[0,1]} + \delta\|\gamma^2\zeta''\|_{C[0,1]} : \zeta \in C^2[0,1]\right\},$$

for any $\delta > 0$, $\Theta^2(\gamma) = \{\zeta \in C[0,1] : \zeta' \in C^*[0,1],\ \gamma^2\zeta'' \in C[0,1]\}$ and $C^2[0,1] = \{\zeta \in C[0,1] : \zeta',\zeta'' \in C[0,1]\}$.

Remark 3.5. *([15]) For any absolute positive constant M one has*

$$M\omega_2^{\gamma}(g,\sqrt{\delta}) \leq K_2^{\gamma}(g,\delta) \leq M^{-1}\omega_2^{\gamma}(g,\sqrt{\delta}). \tag{9}$$

Theorem 3.6. *Let $\gamma(y)$ ($\gamma \neq 0$) be any step-weight function such that γ^2 is concave, then, for all $g \in C[0,1]$ and $s \in [0,1]$ operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}$ satisfying*

$$|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) - g(y)| \leq M\,\omega_2^{\gamma}\Big(g, \frac{[\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) + \Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)]^{1/2}}{2\gamma(y)}\Big) + \omega\Big(g, \frac{\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{\gamma(y)}\Big),$$

where the number $0<q<1$ and $\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) = \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q)$, and $\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) = \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\left((t-y)^2;y,q\right)$.

Proof. If we consider an auxiliary operators

$$\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) = \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) + g(y) - g\Big(\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) + y\Big), \tag{10}$$

given $g \in C[0,1]$ and $s \in [0,1]$, it is simple to obtain the following relations using Lemma 2.2.

$$\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y,q) = 1 \text{ and } \tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) = y,$$
$$\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q) = 0.$$

Let $y = \beta y + (1-\beta)t$ for $\beta \in [0,1]$. Following the property $\gamma^2(y) \geq \rho\gamma^2(y) + (1-\beta)\gamma^2(t)$ as γ^2 is concave on $[0,1]$ and

$$\frac{|t-y|}{\gamma^2(y)} \leq \frac{\beta|y-t|}{\rho\gamma^2(y) + (1-\beta)\gamma^2(t)} \leq \frac{|t-y|}{\gamma^2(y)}. \tag{11}$$

We get the following identities:

$$\begin{aligned}|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) - g(y)| &\leq |\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g-\zeta;y,q)| + |\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y,q) - \zeta(y)| + |g(y) - \zeta(y)| \\ &\leq 4\|g-\zeta\|_{C[0,1+\varkappa_1]} + |\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y,q) - \zeta(y)|. \end{aligned} \tag{12}$$

By use of Taylor's series, we can conclude that

$$\begin{aligned}|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y,q) - \zeta(y)| &\leq \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\Big(\Big|\int_y^t |t-y|\,|\zeta''(y)|\mathrm{d}_q y\Big|;y,q\Big) + \Big|\int_y^{\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)} \Big|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) - y\Big|\,|\zeta''(y)|\,\mathrm{d}_q y\Big| \\ &\leq \|\gamma^2\zeta''\|_{C[0,1]}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\Big(\Big|\int_y^t \frac{|t-y|}{\gamma^2(y)}\mathrm{d}_q y\Big|;y,q\Big) + \|\gamma^2\zeta''\|_{C[0,1]} \\ &\quad \times \Big|\int_y^{\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q)} \Big|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y,q) - y\Big|\,\frac{\mathrm{d}_q y}{\gamma^2(y)}\Big| \\ &\leq \gamma^{-2}(y)\|\gamma^2\zeta''\|_{C[0,1]}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q) + \gamma^{-2}(y)\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)\|\gamma^2\zeta''\|_{C[0,1]}. \end{aligned} \tag{13}$$

We use the Peetre's K-functional properties and the relations (9), (12), and (13), then easy to get

$$\begin{aligned}\Big|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q) - g(y)\Big| &\leq 4\|g-\zeta\|_{C[0,1]} + \gamma^{-2}(x)\|\gamma^2\zeta''\|_{C[0,1]}\Big(\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) + \Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)\Big) \\ &\leq M\,\omega_2^{\gamma}\Big(g, \frac{1}{2}\sqrt{\frac{\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y) + \Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{\gamma(y)}}\Big).\end{aligned}$$

Hence, from the order one uniform smoothness property, it is obvious that

$$\left|g\left(\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)+y\right)-g(y)\right|=\left|g\Big(\gamma(y)\frac{\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{\gamma(y)}+y\Big)-g(y)\right|\le\omega\Big(g,\frac{\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{\gamma(y)}\Big).$$

Thus, finally, we get the inequality

$$\begin{aligned}|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q)-g(y)|&\le|\tilde{\Omega}_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y,q)-g(y)|+\left|g\left(\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)+y\right)-g(y)\right|\\&\le M\,\omega_2^{\gamma}\Big(g,\frac{1}{2}\sqrt{\frac{\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)+\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{(y-u)(v-y)}}\Big)+\omega\Big(g,\frac{\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}{\gamma(y)}\Big),\end{aligned}$$

which completes the desired proof of Theorem 3.6. □

Theorem 3.7. *Suppose $f'\in C[0,1]$ and $y\in[0,1]$, then for any real $0<q<1$ we get the inequality:*

$$|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f;y,q)-f(y)|\le\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)\,|f'(y)|+2\sqrt{\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}\,\omega\Big(f',\sqrt{\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)}\Big).$$

Proof. We know the relation

$$f(t)=f(y)+f'(y)(t-y)+\int_y^t(f'(z)-f'(y))\mathrm{d}_qz, \tag{14}$$

for all $t,y\in[0,1]$. On apply the operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}$ to equality (14), we obtain

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f(t)-f(y);y,q)=f'(y)\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q)+\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\Big(\int_y^t(f'(z)-f'(y))\mathrm{d}_qz;y,q\Big).$$

For all $f\in C[0,1]$ and $y\in[0,1]$, one has

$$|f(t)-f(y)|\le\Big(1+\frac{|t-y|}{\delta}\Big)\omega(f,\delta),\quad\delta>0.$$

From the above inequality, we have

$$\left|\int_t^y(f'(y)-f'(z))\mathrm{d}_qz\right|\le\Big(|t-y|+\frac{(t-y)^2}{\delta}\Big)\omega(f',\delta).$$

Therefore, easy to obtain

$$\begin{aligned}|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f;y,q)-f(y)|&\le|f'(y)|\,|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q)|\\&\quad+\omega(f',\delta)\Big\{\frac{1}{\delta}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)+\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|;y,q)\Big\}.\end{aligned}$$

The Cauchy-Schwarz inequality yields,

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|;y,q)\le\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)^{\frac{1}{2}}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)^{\frac{1}{2}}=\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)^{\frac{1}{2}}.$$

Thus, we have

$$|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f;y,q)-f(y)|\le f'(y)\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)+\Big\{\frac{1}{\delta}\sqrt{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)}+1\Big\}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)^{\frac{1}{2}}\omega(f',\delta),$$

by use of $\delta=\sqrt{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y,q)}$, then we get the desired results. □

Next, we estimate the local direct approximation by use of Lipschitz-type maximal function thus we recall from [37]

$$Lip_M(\kappa) := \left\{g \in C[0,1] : |g(t) - g(y)| \leq M \frac{|t-y|^{\kappa}}{(\beta_1 y^2 + \beta_2 y + t)^{\frac{\kappa}{2}}};\ s,t \in [0,1]\right\}$$

where $\beta_1 \geq 0,\ \beta_2 > 0,\ \kappa \in (0,1]$ and $M > 0$ be any constant (see [37]).

Theorem 3.8. *Let $g \in Lip_M(\kappa)$, then for any $\kappa \in (0,1]$ and $0 < q < 1$, we obtain*

$$|\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(g;y,q) - g(y)| \leq M \sqrt{\frac{[\Phi^{\varkappa_1,\varkappa_2}_s(\lambda;q,y)]^{\kappa}}{(\beta_1 y^2 + \beta_2 y)^{\kappa}}}.$$

Proof. Take $g \in Lip_M(\kappa)$ for any $\kappa \in (0,1]$. First, we want to show that our result is valid for $\kappa = 1$. Thus for any $g \in Lip_M(1)$ we have

$$\begin{aligned}|\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(g;y,q) - g(y)| &\leq |\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(|g(t) - g(y)|;y,q)| + g(y)\,|\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(1;y,q) - 1| \\ &\leq \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)\Big|g(t) - g(y)\Big| \\ &\leq M \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \frac{\tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)|t-y|}{(\beta_1 y^2 + \beta_2 y + t)^{\frac{1}{2}}}.\end{aligned}$$

By using

$$(\beta_1 y^2 + \beta_2 y + t)^{-1/2} \leq (\beta_1 y^2 + \beta_2 y)^{-1/2} \quad (\beta_1 \geq 0, \beta_2 > 0)$$

and by Cauchy-Schwarz inequality, we see

$$\begin{aligned}|\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(g;y,q) - g(x)| &\leq M(\beta_1 y^2 + \beta_2 y)^{-1/2} \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)\Big|t-y\Big| \\ &= M(\beta_1 y^2 + \beta_2 y)^{-1/2} |\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(t-y;y,q)| \\ &\leq M(\beta_1 y^2 + \beta_2 y)^{-1/2} \tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(|t-y|;y,q) \\ &\leq M\left(\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}((t-y)^2;y,q)\right)^{\frac{1}{2}} (\beta_1 y^2 + \beta_2 y)^{-1/2}.\end{aligned}$$

As a result, we get our result is true for $\kappa = 1$. Furthermore, using the monotonicity property to operators $\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(g;y,q)$ and introducing the Hölder's property easy to get the required statement is also satisfy for $\kappa \in (0,1]$ as follows:

$$\begin{aligned}\Big|\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(g;y,q) - g(y)\Big| &\leq \frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)\Big|g(t) - g(y)\Big| \\ &\leq \left(\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)\Big|g(t) - g(y)\Big|\right)^{\frac{\kappa}{2}} \left(\tilde{Z}^{\varkappa_1,\varkappa_2}_{s,\lambda}(1;y,q)\right)^{\frac{2-\kappa}{2}} \\ &\leq M\Bigg(\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)^s_q} \frac{\sum_{j=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,j}(y;q,\lambda)\Big(t-y\Big)^2}{\beta_1 y^2 + \beta_2 y + t}\Bigg)^{\frac{\kappa}{2}}\end{aligned}$$

$$\leq M\left\{\frac{1}{\left(\frac{[s]_q}{[s]_q+\varkappa_2}\right)_q^s}\sum_{j=0}^{s}\tilde{b}_{s,j}^{\varkappa_1,\varkappa_2}(y;q,\lambda)\Big(t-y\Big)^2\right\}^{\frac{\kappa}{2}}(\beta_1 y^2+\beta_2 y+t)^{-\frac{\kappa}{2}}$$
$$\leq M(\beta_1 y^2+\beta_2 y)^{-\kappa/2}\left[\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\left((t-y)^2;y,q\right)\right]^{\frac{\kappa}{2}}$$
$$= M\sqrt{\frac{[\Phi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)]^{\kappa}}{(\beta_1 y^2+\beta_2 y)^{\kappa}}}.$$

□

4. Voronovskaja type asymptotic formula

Based on the article [8, 35], we examine the quantitative Voronovskaja-type approximation theorem and derive the Voronovskaja-type approximation properties for our novel operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}$. Use of the definition of the modulus of smoothness discussed in the section before allows for this. An explanation of this smoothness modulus is

$$\omega_{\Theta}(\zeta,\delta) := \sup_{0<|\rho|\leq\delta}\left\{\left|f\Big(y+\frac{\rho\Theta(y)}{2}\Big)-\zeta\Big(y-\frac{\rho\Theta(y)}{2}\Big)\right|, y\pm\frac{\rho\Theta(y)}{2}\in[0,1]\right\}.$$

Here $\zeta\in C[0,]$ and $\Theta(y)=(y-y^2)^{1/2}$, and related Peetre's K-functional given by:

$$K_{\Theta}(\zeta,\delta) = \inf_{f\in\omega_{\Theta}[0,1]}\left\{\|f-\zeta\|+\delta\|\Theta f'\| : f'\in C[0,1], \delta>0\right\},$$

where $\omega_{\Theta}[0,1]=\{f : f'\in C^*[0,1], \|\Theta f'\|<\infty\}$ and $C^*[0,1]$, which represents completely absolutely continuous functions on the intervals $[a,b]\subset[0,1]$. A positive constant M exists such that

$$K_{\Theta}(f,\delta)\leq M\,\omega_{\Theta}(f,\delta).$$

Theorem 4.1. *For all $\zeta,\zeta',\zeta''\in C[0,1]$, it verify that*

$$\left|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y)-\zeta(y)-\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)\zeta'(y)-\frac{\delta_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+1}{2}\zeta''(y)\right|\leq M\frac{\Theta^2(y)}{s}\omega_{\chi}\Big(\zeta'',\frac{1}{\sqrt{s}}\Big),$$

where $y\in[0,1]$, $C>0$ a constant, $\Psi_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)=\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y,q)$ and $\delta_s^{\varkappa_1,\varkappa_2}(\lambda;q,y)=\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\left((t-y)^2;y,q\right)$.

Proof. Let $\zeta\in C[0,1]$, then from the Taylor series expansion we examine the results as follows:

$$\zeta(t)-\zeta(y)-\zeta'(y)(t-y)=\int_y^t\zeta''(\theta)(t-\theta)\mathrm{d}_q\theta,$$

then easy to get

$$\zeta(t)-\zeta(y)-(t-y)\zeta'(y)-\frac{\zeta''(y)}{2}\Big((t-y)^2+1\Big)\leq\int_y^t(t-\theta)[\zeta''(\theta)-\zeta''(y)]\mathrm{d}_q\theta. \tag{15}$$

Therefore, (15) give us,

$$\left|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y)-\zeta(y)-\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)\zeta'(y)-\frac{\zeta''(y)}{2}\Big(\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)+\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)\Big)\right|$$
$$\leq\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}\Big(\Big|\int_y^t|\zeta''(\theta)|t-\theta|-\zeta''(y)|\,\mathrm{d}_q\theta\Big|;y\Big). \tag{16}$$

We can estimate the following from the right part of equality (16):

$$\left| \int_y^t |t-\theta|\, |\zeta''(\theta) - \zeta''(y)|\, d_q\theta \right| \leq 2\|\zeta'' - f\|(t-y)^2 + 2\|\chi f'\|\Theta^{-1}(y)|t-y|^3, \tag{17}$$

where $\zeta \in \omega_\Theta[0,1]$. A positive constant suppose M exists such that

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y, q) \leq \frac{M}{2[s]_q}\Theta^2(y) \quad \text{and} \quad \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4; y) \leq \frac{M}{2[s]_q^2}\Theta^4(y) \tag{18}$$

We may determine by applying the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\left| \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta; y, q) - \zeta(y) - \zeta'(y)\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y; y) - \frac{\zeta''(y)}{2}\Big(\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y, q) + \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(1; y, q)\Big) \right| \\
&\leq 2\|\zeta'' - f\|\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y) + 2\|\Theta(y)f'\|\Theta^{-1}(y)\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|^3; y, q) \\
&\leq \frac{M}{[s]_q}\Theta^2(y)\|\zeta'' - f\| + 2\|\Theta(y)f'\|\chi^{-1}(y)\{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y, q)\}^{1/2}\{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4; y, q)\}^{1/2} \\
&\leq M\frac{\Theta^2(y)}{[s]_q}\Big\{\|\zeta'' - f\| + [s]_q^{-1/2}\|\Theta(y)f'\|\Big\}.
\end{aligned}$$

We determine that by taking the infimum over all $f \in \omega_\Theta[0,1]$.

$$\left| \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta; y, q) - \zeta(y) - \Psi_s^{\varkappa_1,\varkappa_2}(\lambda; q, y)\zeta'(y) - \frac{\delta_s^{\varkappa_1,\varkappa_2}(\lambda; q, y) + 1}{2}\zeta''(y) \right| \leq \frac{M}{[s]_q}\Theta^2(y)\omega_\Theta\Big(\zeta'', \frac{1}{\sqrt{[s]_q}}\Big),$$

which brings the proof. □

Theorem 4.2. *For any $\zeta \in C_B[0,1]$, we examine*

$$\lim_{s\to\infty}[s]_q\left[\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta; y, q) - \zeta(y) - \Psi_s^{\varkappa_1,\varkappa_2}(\lambda; q, y)\zeta'(y) - \frac{\delta_s^{\varkappa_1,\varkappa_2}(\lambda; q, y)}{2}\zeta''(y)\right] = 0.$$

Proof. If $\zeta \in C_B[0,1]$, we can write using Taylor's series expansion as follows:

$$\zeta(t) = \zeta(y) + (t-y)\zeta'(y) + \frac{1}{2}(t-y)^2\zeta''(y) + (t-y)^2\, Q_y(t). \tag{19}$$

Moreover, $Q_y(t) \to 0$ as $t \to y$, where $Q_y(t) \in C[0,1]$ and specified for Peano form of remainder. Using the operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\cdot; y, q)$ to the equality (19), then easy to see

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta; y, q) - \zeta(y) = \zeta'(y)\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y; y, q) + \frac{\zeta''(y)}{2}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y, q) + \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2 Q_y(t); y, q).$$

Cauchy-Schwarz inequality gives us

$$\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2 Q_y(t); y, q) \leq \sqrt{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(Q_y^2(t); y, q)}\sqrt{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4; y, q)}. \tag{20}$$

We clearly observe here $q \to 1$ and $\lim_{s\to\infty} \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(Q_y^2(t); y, q) = 0$ and therefore

$$\lim_{s\to\infty}[s]_q\{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2 Q_y(t); y, q)\} = 0.$$

Thus, we have

$$\begin{aligned}
\lim_{s\to\infty}[s]_q\{\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta; y, q) - \zeta(y)\} = \lim_{s\to\infty}[s]_q\Big\{&\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y; y, q)\zeta'(y) + \frac{\zeta''(y)}{2}\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2; y, q) \\
&+ \tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2 Q_y(t); y, q)\Big\}.
\end{aligned}$$

□

5. Graphical analysis

In this section, we will give some numerical examples with illustrative graphics with the help of MATLAB.

Example 5.1. *Let* $g(y) = y^2 + \frac{22}{7}$, $\varkappa_1 = 4, \varkappa_2 = 5.5, \lambda = 0.7$ *and* $s \in \{18, 34, 80\}$. *The convergence of the operator towards the function* $g(y)$ *is shown in Figure 1.*

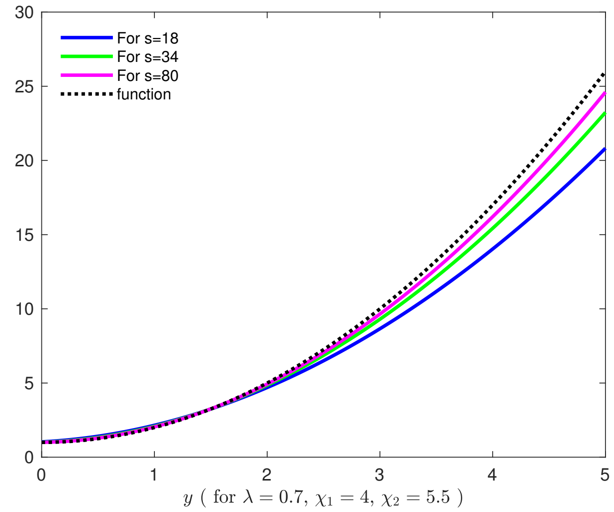

Figure 1: Convergence of the operator towards the function $g(y) = y^2 + \frac{22}{7}$

Example 5.2. *Let* $g(y) = (y - \frac{2}{9})(y - \frac{3}{5})$, $\varkappa_1 = 3, \varkappa_2 = 4, \lambda = 2.5$ *and* $s \in \{11, 25, 91\}$. *The convergence of the operator towards* $g(y)$ *is shown in Figure 2.*

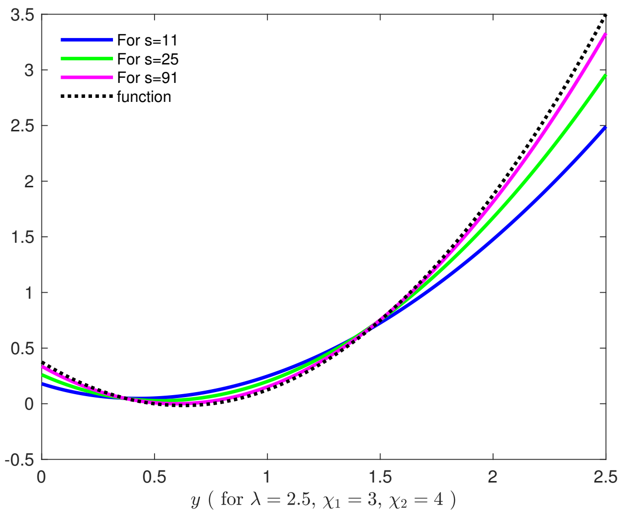

Figure 2: Convergence of the operator towards $g(y) = (y - \frac{2}{9})(y - \frac{3}{5})$

From these examples, we observe that approximation of function by the operators becomes better when we take larger values of s.

Notice that for $\varkappa_1 = \varkappa_2 = 0$, the operators (8) reduce to operators (7).

6. Conclusion & Observation

In the present article, we clearly conclude that our new operators (8) are the shifted knots variant of Bézier basis of (λ, q)-Bernstein operators. For $q = 1$ in the equality (8), then our operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f; y, q)$ have the approximation properties to the operators(3) by [6]. For $q = 1$ and $\varkappa_1 = \varkappa_2 = 0$ in the equality (8), then our operators $\tilde{Z}_{s,\lambda}^{\varkappa_1,\varkappa_2}(f; y, q)$ reduced to the approximation results of operators(7) defined by Cai et al. [11]. As a result, we can define our operators (8) as special examples of classical Bernstein-operators [9], Lupaş q-Bernstein-operators [25], λ-Bernstein operators [11], (λ, q)-Bernstein operators [10] and λ-Bernstein operators with Bézier basis [6]. Based on these data, we conclude that the new operators we have are more capable than the previous types of operators.

Conflicts of Interest
The authors declare that they have no conflicts of interest.
Authors' contributions
All authors contributed equally to writing this paper. All authors read and approved the manuscript.
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