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Novel Hermite-Hadamard type integral inequalities for twice differentiable functions by weighted integrals

Juan E. Nápoles Valdés[a,b], Çetin Yıldız[c,*]

[a]*National University of the Northeast, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina*
[b]*UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina*
[c]*Atatürk University, K.K. Education Faculty, Deparment of Mathematics, 25240, Campus, Erzurum, Turkey*

Abstract. For convex, $s-$convex and $h-$convex functions, the Hermite-Hadamard inequality is already well known in the theory of inequalities. In this regard, this work presents new inequalities associated with the weighted integrals for (h, m)-convex modified functions by utilizing a different technique. These novel inequalities are also obtained by means of Power-mean, Young, and Hölder inequalities.

1. Introduction

A fundamental principle in mathematics, convexity has a wide range of applications in both applied and pure mathematics. Convexity is used to evaluate functions and sets, to prove inequality, and to model and solve real-life problems. This concept is crucial for estimating integrals and defining limits in many branches of mathematics and beyond (see [1]-[5]).

In convex analysis, we thus remember the basic notation as follows:

Definition 1.1. *([6]) A set $\mathcal{I} \subset \mathbb{R}$ is said to be convex function if*

$$\tau\xi + (1-\tau)\varsigma \in \mathcal{I}$$

for each $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$.

Definition 1.2. *([6]) The mapping $\psi : \mathcal{I} \to \mathbb{R}$, is said to be convex function if the following inequality holds:*

$$\psi\left(\tau\xi + (1-\tau)\varsigma\right) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$$

for all $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$. If $(-\psi)$ is convex, then ψ is said to be concave. This suggests that $\mathcal{U}$ is geometrically on or below the chord $\mathcal{BZ}$ if $\mathcal{B}$, $\mathcal{U}$ and $\mathcal{Z}$ are three different places on the graph of ψ, with $\mathcal{U}$ between $\mathcal{B}$ and $\mathcal{Z}$.

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* Corresponding author: Çetin Yıldız
Email addresses: jnapoles@exa.unne.edu.ar (Juan E. Nápoles Valdés), cetin@atauni.edu.tr (Çetin Yıldız)
ORCID iDs: https://orcid.org/0000-0003-2470-1090 (Juan E. Nápoles Valdés), https://orcid.org/0000-0002-8302-343X (Çetin Yıldız)

Convex functions are essential in many different areas. For example, a convex function applied to the expected value of a random variable is never bounded below the expected value of the convex function, according to probability theory [[7],[8],[9],[10]]. This result, called Jensen's inequality, may be used to generate additional inequalities, such as Hölder's inequality and the geometric-arithmetic mean inequality. Because of its many applications, its robustness, and its widely held views, the concept of convexity has become a rich source of inspiration for researchers [[11],[12],[13]]. The concept of convexity has allowed mathematicians to create amazing tools and numerical techniques that they use to tackle and solve a huge variety of problems that arise in the pure and applied sciences. Because of its many views, adaptability, and wide range of applications, the concept of convexity has grown to be a rich source of inspiration and an intriguing subject for researchers [[14],[15]]. Mathematicians have focused on this theory for more than a century because of its long and significant history. Nevertheless, there are many novel problems in applied mathematics for which the idea of convexity is not sufficient to characterise them sufficiently to have useful consequences. For this reason, the idea of convexity has been extended and developed in various research projects [[16],[17]].

Convex mappings and sets have been extensively studied in mathematics due to their robustness (as mentioned above), particularly in convexity theory, which has been used to prove numerous inequalities found in the literature.To the best of our knowledge, the Hermite-Hadamard type integral inequality, often known as the Hadamard inequality, is a well-known, significant and immensely useful inequality in the practical literature on theory of inequalities [[18],[19],[20]]. The classical Hermite-Hadamard integral inequality is closely related to a number of classical inequalities, including the arithmetic-geometric, Ostrowski, Opial, Young, Hardy, Hölder, Simpson, Minkowski, and Grüss inequalities [[21],[22],[23],[24]]. These inequalities are quite important. Following is a statement of this double inequality: Assume that ψ is a convex mapping on $[\xi,\varsigma]\subset\mathbb{R}$, where $\xi\neq\varsigma$. Therefore

$$\psi\left(\frac{\xi+\varsigma}{2}\right)\leq\frac{1}{\varsigma-\xi}\int_{\xi}^{\varsigma}\psi(\varkappa)d\varkappa\leq\frac{\psi(\xi)+\psi(\varsigma)}{2}.$$

A particular area of study in the calculus of non-integer order, or fractional calculus, is inequality using fractional integrals. Generalizing integrals and derivative operators is the focus of this topic. In the literature, there are several definitions for fractional integral operators, such as Riemann-Liouville fractional integral, Hadamard integral, conformable fractional integral, k-Riemann-Liouville fractional integral, and Caputo-Fabrizio fractional integral [[25],[26],[27]]. The fractional operators can be made more generic by giving new parameters to such fractional integral operators, leading to the following inequalities: Ostrowski, Hermite-Hadamard, Jensen-Mercer, Grüss, Minkowski, and others. [[28],[29],[30]]. Future work should aim to present original concepts using unified fractional operators and derive inequalities employing such operators. These generalizations have motivated further research. Inequalities related to fractional integral operators have practical applications across various fields of study. An understanding of fractional calculus theory is necessary for solving many additional special function problems, such as those that require the solution of differential equations, integral equations, and integral-differentiable equations. In order to obtain remarks and corollaries, it is important to recall the preliminary formulae and notations of well-known Riemann-Liouville fractional integral operators.

Various varieties of fractional integrals have been described in literature, with the most traditional being Riemann-Liouville fractional integrals. These are defined as follows:

Definition 1.3. *([31])Let $\psi\in L[\xi,\varsigma]$. The Riemann-Liouville integrals $J_{\xi^+}^{\alpha}\psi$ and $J_{\varsigma^-}^{\alpha}\psi$ of order $\alpha>0$ with $\varkappa_1\geq 0$ are defined by*

$$J_{\xi^+}^{\alpha}\psi(\phi)=\frac{1}{\Gamma(\alpha)}\int_{\xi}^{\phi}\left(\phi-\sigma\right)^{\alpha-1}\psi(\sigma)d\sigma,\ \ \phi>\xi$$

and

$$J_{\varsigma^-}^{\alpha}\psi(\phi)=\frac{1}{\Gamma(\alpha)}\int_{\phi}^{\varsigma}\left(\sigma-\phi\right)^{\alpha-1}\psi(\sigma)d\sigma,\ \ \phi<\varsigma$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$. *Here is* $J^\alpha_{\xi^+}\psi(\phi) = J^\alpha_{\varsigma^-}\psi(\phi) = \psi(\phi)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [32], Sarıkaya *et al*. proved the following Hadamard type inequalities for fractional integrals as follows:

Theorem 1.4. *Let* $\psi : [\xi, \varsigma] \to \mathbb{R}$ *be positive function with* $0 \le \xi < \varsigma$ *and* $\psi \in L[\xi, \varsigma]$. *If* ψ *is convex function on* $[\xi, \varsigma]$, *then the following inequalities for fractional integrals hold:*

$$\psi\left(\frac{\xi+\varsigma}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\varsigma-\xi)^\alpha}\left[J^\alpha_{\xi^+}\psi(\varsigma) + J^\alpha_{\varsigma^-}\psi(\xi)\right] \le \frac{\psi(\xi)+\psi(\varsigma)}{2}$$

with $\alpha > 0$.

The famous Young inequality is defined as follows:

Theorem 1.5. *([33]) Let* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\xi\varsigma \le \frac{1}{p}\xi^p + \frac{1}{q}\varsigma^q \tag{1}$$

where ϖ *and* ϕ *are nonnegative numbers. The reversed version of inequality (1) reads*

$$\xi\varsigma \ge \frac{1}{p}\xi^p + \frac{1}{q}\varsigma^q, \quad \xi, \varsigma > 0, \quad 0 < p < 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The following is the most commonly used version of Young's inequality, which is often used to demonstrate the well-known inequality for L_p functions:

$$\xi^\tau\varsigma^{1-\tau} \le \tau\xi + (1-\tau)\varsigma,$$

where $\xi, \varsigma > 0$ and $0 \le \tau \le 1$.

Important inequalities have been identified using different types of convexity. (h, m)-convex modified is one of several varieties of convexity. The definition of this class in [34] is as follows:

The mapping $\psi : \mathcal{I} \to \mathbb{R}$, is said to be convex function if the following inequality holds:

$$\psi(\tau\xi + (1-\tau)\varsigma) \le \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$$

for all $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0, 1]$.

Definition 1.6. *Let* $h : [0, 1] \to \mathbb{R}$ *be a nonnegative function,* $h \ne 0$. *The mapping* $\psi : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be* (h, m)*-convex modified of the first type on I if the following inequality holds:*

$$\psi(\tau\xi + m(1-\tau)\varsigma) \le h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi(\varsigma) \tag{2}$$

for all $\xi, \varsigma \in I$ *and* $\tau \in [0, 1]$, *where* $m \in [0, 1]$, $s \in [-1, 1]$.

Definition 1.7. *Let* $h : [0, 1] \to \mathbb{R}$ *nonnegative functions,* $h \ne 0$. *The mapping* $\psi : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be* (h, m)*-convex modified of the second type on I if the following inequality holds:*

$$\psi(\tau\xi + m(1-\tau)\varsigma) \le h^s(\tau)\psi(\xi) + m(1-h(\tau))^s\psi(\varsigma) \tag{3}$$

for all $\xi, \varsigma \in I$ *and* $\tau \in [0, 1]$, *where* $m \in [0, 1]$, $s \in [-1, 1]$.

Next, the weighted integral operators are presented ([34–36]), which will serve as a starting point for our investigation.

Definition 1.8. *Let $\phi \in L[\xi, \varsigma]$ and let w be a continuous and positive function, $w : [0, 1] \to \mathbb{R}^+$, with second order derivatives integrable on I. Then (left and right, respectively) define the weighted fractional integrals as follows:*

$$J^{w}_{\xi+}\psi(r) = \int_{\xi}^{r} w''\left(\frac{r-\sigma}{\varsigma-\xi}\right)\psi(\sigma)d\sigma, \quad r > \xi$$

and

$$J^{w}_{\varsigma-}\psi(r) = \int_{r}^{\varsigma} w''\left(\frac{\sigma-r}{\varsigma-\xi}\right)\psi(\sigma)d\sigma, \quad r < \varsigma.$$

Remark 1.9. *In order to have a better understanding of the amplitude of Definition 1.8, let us examine a few specific instances of the kernel w'':*

a) Taking $w''(\tau) \equiv 1$, we obtain the classical Riemann integral.

b) If we have $w''(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}$, then we have the right fractional integral of Riemann-Liouville, and the left integral can be obtained in the same manner.

c) With convenient kernel choices w'' we can get the k-Riemann-Liouville fractional integral right and left of ([37]), the right-sided fractional integrals of a function ψ with respect to another function h on $[\xi, \varsigma]$ (see [38]), the right and left integral operator of [39], the right and left sided generalized fractional integral operators of [40] and the integral operators of [41] and [42], can also be obtained from above Definition by imposing similar conditions to w''.

Of course, readers interested in such matters may find other known integral operators, fractional or not, which can be produced as special instances of the previous one. (see [43]-[46])

In this paper we derive several versions of the Hermite-Hadamard inequality using generalised operators of Definition 1.8 within the framework of (h, m)-convex modified functions.

2. Main Results

The following equality, which will be basic in obtaining the other results.

Lemma 2.1. *Let ψ be a real function defined on some interval $[\xi, \varsigma] \subset \mathbb{R}$, twice differentiable on (ξ, ς). If $\psi'' \in L[\xi, \varsigma]$, then we obtain the following equality:*

$$\begin{aligned}
& -\frac{\varkappa-\xi}{\varpi+1}\left[w(1)\psi'(\xi) - w(0)\psi'\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] - \left[w'(1)\psi(\xi) - w'(0)\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
+ & \ \frac{\varsigma-\varkappa}{\varpi+1}\left[w(1)\psi'(\varsigma) - w(0)\psi'\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right] - \left[w'(1)\psi(\varsigma) - w'(0)\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right] \\
+ & \ \frac{\varpi+1}{\varkappa-\xi}J^{w}_{\xi+}\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) \\
= & \ \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)d\tau + \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)d\tau
\end{aligned}$$

with $\varkappa \in [\xi, \varsigma]$ and $\varpi \in \mathbb{N}$.

Proof. First denote

$$I_1 = \int_0^1 w(\tau)\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)d\tau$$

$$I_2 = \int_0^1 w(\tau)\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)d\tau.$$

Integrating by parts twice, we obtain

$$\begin{aligned}
& I_1 \\
= & -\frac{\varpi+1}{\varkappa-\xi}\left[w(1)\psi'(\xi)-w(0)\psi'\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
- & \frac{(\varpi+1)^2}{(\varkappa-\xi)^2}\left[w'(1)\psi(\xi)-w'(0)\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
+ & \frac{(\varpi+1)^2}{(\varkappa-\xi)^2}\int_0^1 w''(\tau)\psi\left(\frac{\varpi+\tau}{\varpi+1}\xi+\frac{1-\tau}{\varpi+1}\varkappa\right)d\tau \\
= & -\frac{\varpi+1}{\varkappa-\xi}\left[w(1)\psi'(\xi)-w(0)\psi'\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
- & \frac{(\varpi+1)^2}{(\varkappa-\xi)^2}\left[w'(1)\psi(\xi)-w'(0)\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
+ & \frac{(\varpi+1)^3}{(\varkappa-\xi)^3}J_{\xi+}^{w}\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right),
\end{aligned}$$

since

$$\begin{aligned}
& \int_0^1 w''(\tau)\psi\left(\frac{\varpi+\tau}{\varpi+1}\xi+\frac{1-\tau}{\varpi+1}\varkappa\right)d\tau \\
= & \frac{\varpi+1}{\varkappa-\xi}\int_{\xi}^{\frac{\varpi\xi+\varkappa}{\varpi+1}} w''\left[\frac{\frac{\varpi\xi+\varkappa}{\varpi+1}-u}{\frac{\varpi\xi+\varkappa}{\varpi+1}-\xi}\right]\psi(u)du,
\end{aligned}$$

and $\frac{\varpi\xi+\varkappa}{\varpi+1}-\xi=\frac{\varkappa-\xi}{\varpi+1}$.
Analogously,

$$\begin{aligned}
& I_2 \\
= & \frac{\varpi+1}{\varsigma-\varkappa}\left[w(1)\psi'(\varsigma)-w(0)\psi'\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right] \\
- & \frac{(\varpi+1)^2}{(\varsigma-\varkappa)^2}\left[w'(1)\psi(\varsigma)-w'(0)\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right] \\
+ & \frac{(\varpi+1)^3}{(\varsigma-\varkappa)^3}J_{\varsigma-}^{w}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right).
\end{aligned}$$

Adding I_1 and I_2 and reordering, we obtain the desired result. □

The breadth and generality of the previous Lemma can be verified with the following points. So we have

Remark 2.2. *With $w(\tau)=\tau(1-\tau)^{\alpha}$, $\varpi=0$, put $\varkappa=\varsigma$ in I_1 and $\varkappa=\xi$ en I_2 we obtain the Lemma 1.5 of [34].*

Remark 2.3. *If $w(\tau)=(1-\tau^2)$, $\varpi=0$, and working only with I_1 ($\varkappa=\varsigma$) and using the argument $\tau\xi+m(1-\tau)\varsigma$, the Lemma 2 of [47] is obtained.*

Remark 2.4. *Considering $w(\tau)=\tau(1-\tau)$, $\varpi=0$, putting $\varkappa=\frac{\xi+\varsigma}{2}$ in I_1 and working in I_2 with argument $\tau\frac{\xi+\varsigma}{2}+(1-\tau)\varsigma$ we get Lemma 2.1 of [48].*

Remark 2.5. *Making $w(\tau)=\tau(1-\tau)$, $\varpi=1$, put $\varkappa=\varsigma$ in I_1 and $\varkappa=\xi$ en I_2 we obtain the Lemma 2.1 of [49].*

Remark 2.6. *The Lemma 2.1 of [50] is derived from preceding result making $w(\tau) = (1-\tau)^{\alpha+1}$, $\varpi = 1$, and put $\varkappa = \varsigma$ en I_1 and $\varkappa = \xi$ in I_2.*

Remark 2.7. *If we take $w(\tau) = (1-\tau)^{\alpha+1}$, and considering $\varkappa = \varsigma$ en I_1 and $\varkappa = \xi$ in I_2, we obtain the Lemma 1.3 of [51].*

Corollary 2.8. *If we take $w(\tau) = \frac{\tau^2}{2}$, $\varpi = 0$ we obtain the following equality, new for Riemann Integral:*

$$\begin{aligned}
&(\xi-\varkappa)\frac{\psi'(\xi)}{2} - \psi(\xi) + (\varsigma-\varkappa)\frac{\psi'(\varsigma)}{2} - \psi(\varsigma) + \frac{1}{\varkappa-\xi}\int_\xi^\varkappa \psi(u)du + \frac{1}{\varsigma-\varkappa}\int_\varkappa^\varsigma \psi(u)du \\
= \quad & \frac{(\varkappa-\xi)^2}{2}\int_0^1 \tau^2\psi''(\tau\xi+(1-\tau)\varkappa)d\tau + \frac{(\varsigma-\varkappa)^2}{2}\int_0^1 \tau^2\psi''(\tau\varsigma+(1-\tau)\varkappa)d\tau.
\end{aligned}$$

Corollary 2.9. *If we take $w(\tau) = \frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}$, $\varpi = 0$ we can derive a new result for Riemann-Liouville Integral:*

$$\begin{aligned}
&\frac{(\xi-\varkappa)}{\Gamma(\alpha+2)}\psi'(\xi) + \frac{(\varsigma-\varkappa)}{\Gamma(\alpha+2)}\psi'(\varsigma) - \frac{1}{\Gamma(\alpha+1)}\{\psi(\xi)+\psi(\varsigma)\} \\
&+\frac{1}{(\varsigma-\xi)^{\alpha-1}}\left[\frac{1}{(\varkappa-\xi)}J^\alpha_{\xi^+}\psi(\sigma) + \frac{1}{(\varsigma-\varkappa)}J^\alpha_{\varsigma^-}\psi(\sigma)\right] \\
= \quad & \frac{(\varkappa-\xi)^2}{\Gamma(\alpha+2)}\int_0^1 \tau^{\alpha+1}\psi''(\tau\xi+(1-\tau)\varkappa)d\tau + \frac{(\varsigma-\varkappa)^2}{\Gamma(\alpha+2)}\int_0^1 \tau^{\alpha+1}\psi''(\tau\varsigma+(1-\tau)\varkappa)d\tau.
\end{aligned}$$

Remark 2.10. *Throughout this paper, we use the following notation:*

$$\begin{aligned}
&\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi) = \\
&-\frac{\varkappa-\xi}{\varpi+1}\left[w(1)\psi'(\xi) - w(0)\psi'\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] - \left[w'(1)\psi(\xi) - w'(0)\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)\right] \\
&+\frac{\varsigma-\varkappa}{\varpi+1}\left[w(1)\psi'(\varsigma) - w(0)\psi'\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right] - \left[w'(1)\psi(\varsigma) - w'(0)\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right].
\end{aligned}$$

Theorem 2.11. *Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) such that $\psi'' \in L[\xi,\varsigma]$. If $|\psi''|$ is (h,m)-convex modified of the second type on I for some fixed $m \in [0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi) + \frac{\varpi+1}{\varkappa-\varsigma}J^w_{\varsigma^+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa}J^w_{\varsigma^-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right| \\
\leq \quad & \mathcal{H}_1\left\{\frac{(\varkappa-\varsigma)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right| + \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|\right\} \\
&+m\mathcal{H}_2\left\{\frac{(\varkappa-\varsigma)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''\left(\frac{\varkappa}{m}\right)\right|\right\}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_1 &= \int_0^1 w(\tau)h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)d\tau, \\
\mathcal{H}_2 &= \int_0^1 w(\tau)\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s d\tau.
\end{aligned}$$

Proof. Taking modulus in Lemma 2.1 and using the (h,m)-convex modified of the second type of $\left|\psi''\right|$, we have

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\xi+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau\\
&+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau\\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left\{h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\xi)\right|+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|\right\}d\tau\\
&+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left\{h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\varsigma)\right|+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|\right\}d\tau\\
=\;& \left[\int_0^1 w(\tau)h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)d\tau\right]\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|\right\}\\
&+m\left[\int_0^1 w(\tau)\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s d\tau\right]\left\{\frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''\left(\frac{\varkappa}{m}\right)\right|\right\}.
\end{aligned}$$

Thus, the proof is completed. □

Corollary 2.12. *Under the assumptions of Theorem 2.11,*

1. *If we choose $m=1$, then we obtain the following inequality;*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \mathcal{H}_1\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|\right\}\\
&+\mathcal{H}_2\left\{\frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varkappa)\right|\right\}.
\end{aligned}$$

   $\mathcal{H}_1$ and $\mathcal{H}_2$ are as Theorem 2.11.

2. *If $s=m=1$, then*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \mathcal{H}_3\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|\right\}\\
&+\mathcal{H}_4\left\{\frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varkappa)\right|\right\}
\end{aligned}$$

   where

$$\begin{aligned}
\mathcal{H}_3 &= \int_0^1 w(\tau)h\left(\frac{\varpi+\tau}{\varpi+1}\right)d\tau,\\
\mathcal{H}_4 &= \int_0^1 w(\tau)\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]d\tau.
\end{aligned}$$

3. *If we take $w(\tau)=\frac{\tau^2}{2}$, $\varpi=0$ we obtain the following inequality, new for Riemann Integral.*
4. *With $w(\tau)=\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}$, $\varpi=0$ readers will have no difficulty in obtaining a new inequality for for Riemann-Liouville Integral.*

To understand the generality of Theorem 2.11, we have the following comments.

Remark 2.13. *If we work with s-convex functions, i. e., $m = 1$ and $h(z) = z$ Theorem 9 of [52] is a particular case of our result, the same for Theorem 2 of [50], Theorem 2.1 of [51]. If we consider h-convex functions ($s = m = 1$), Theorem 6 of [53] follows from our result. If the functions considered are (α, s, m)-convex, $h(z) = z^\alpha$, then we can derive Theorem 3.2 of [54]. Analogously, for Theorems 3.2 and 3.5 of [55] with $\varkappa = \frac{\xi+\varsigma}{2}$.*

Theorem 2.14. *Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) such that $\psi'' \in L[\xi, \varsigma]$. If $|\psi''|^q$ is (h, m)-convex modified of the second type on I for some fixed $m \in [0, 1]$, then we obtain the following inequality:*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi) + \frac{\varpi+1}{\varkappa-\varsigma} J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa} J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right| \\
\leq\;& \mathcal{H}_5 \left\{ \frac{(\varkappa-\xi)^2}{(\varpi+1)^2} \left[\mathcal{H}_6 \left|\psi''(\xi)\right|^q + m\mathcal{H}_7 \left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}} \right. \\
&\left. + \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2} \left[\mathcal{H}_6 \left|\psi''(\varsigma)\right|^q + m\mathcal{H}_7 \left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}} \right\}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_5 &= \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}} \\
\mathcal{H}_6 &= \int_0^1 h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)d\tau, \\
\mathcal{H}_7 &= \int_0^1 \left[1 - h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s d\tau
\end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and using Hölder Inequality, we obtain

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi) + \frac{\varpi+1}{\varkappa-\varsigma} J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa} J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right| \\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2} \int_0^1 w(\tau) \left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)\right| d\tau \\
&+ \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2} \int_0^1 w(\tau) \left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)\right| d\tau \\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2} \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+ \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2} \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore, $\left|\psi''\right|^q$ is the (h,m)-convex modified of the second type, we get

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\ & \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}}\\
&\times\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left(\int_0^1\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\xi)\right|^q+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left(\int_0^1\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\varsigma)\right|^q+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right\}\\
=\ & \mathcal{H}_5\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_6\left|\psi''(\xi)\right|^q+m\mathcal{H}_7\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_6\left|\psi''(\varsigma)\right|^q+m\mathcal{H}_7\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Thus, the desired result is obtained. □

Corollary 2.15. *Under the assumptions of Theorem 2.14,*

1. *If we choose $m = 1$, then we obtain the following inequality;*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\ & \mathcal{H}_5\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_6\left|\psi''(\xi)\right|^q+\mathcal{H}_7\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_6\left|\psi''(\varsigma)\right|^q+\mathcal{H}_7\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

   $\mathcal{H}_5, \mathcal{H}_6$ and $\mathcal{H}_7$ are as Theorem 2.14.

2. *If $s = m = 1$, then*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\ & \mathcal{H}_5\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_8\left|\psi''(\xi)\right|^q+\mathcal{H}_9\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_8\left|\psi''(\varsigma)\right|^q+\mathcal{H}_9\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

   where

$$\begin{aligned}
\mathcal{H}_3 &= \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}}\\
\mathcal{H}_8 &= \int_0^1 h\left(\frac{\varpi+\tau}{\varpi+1}\right)d\tau,\\
\mathcal{H}_9 &= \int_0^1\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]d\tau.
\end{aligned}$$

3. *Taking into account the Corollary 2.12, items 3 and 4, we can derive new inequalities for Riemann and Riemann-Liouville Integrals, respectively.*

The breadth and generality of the previous Theorem can be verified with the following points:

Remark 2.16. *Following what was pointed out in Remark 2.13, it is easy to verify that the following results are particular cases of the previous one: Theorem 10 of [52], Theorem 7 of [53], Theorem 3 of [50], Theorem 2.2 of [51], Theorem 4 of [47], Theorem 2 of [56], Theorem 3.2 of [54] and Theorem 3.1 of [48].*

Theorem 2.17. *Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) such that $\psi'' \in L[\xi, \varsigma]$. If $\left|\psi''\right|^q$, $q \geq 1$, is (h, m)-convex modified of the second type on I for some fixed $m \in [0, 1]$, then we have*

$$\left|\mathcal{I}(w, \psi, \xi, \varsigma, \varkappa, \varpi) + \frac{\varpi+1}{\varkappa-\varsigma} J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa} J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|$$
$$\leq \mathcal{H}_{10}\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_1\left|\psi''(\xi)\right|^q + m\mathcal{H}_2\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}} + \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_1\left|\psi''(\varsigma)\right|^q + m\mathcal{H}_2\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}$$

where

$$\mathcal{H}_{10} = \left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}.$$

Proof. From Lemma 2.1 and using well-known power mean inequality, we get

$$\left|\mathcal{I}(w, \psi, \xi, \varsigma, \varkappa, \varpi) + \frac{\varpi+1}{\varkappa-\varsigma} J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa} J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|$$
$$\leq \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau + \frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau$$
$$\leq \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma + \frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right)^{\frac{1}{q}}.$$

By the (h, m)-convex modified of the second type of $\left|\psi''\right|^q$, we obtain

$$\left|\mathcal{I}(w, \psi, \xi, \varsigma, \varkappa, \varpi) + \frac{\varpi+1}{\varkappa-\varsigma} J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right) + \frac{\varpi+1}{\varsigma-\varkappa} J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|$$
$$\leq \left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left(\int_0^1 w(\tau)\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\xi)\right|^q + m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left(\int_0^1 w(\tau)\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\varsigma)\right|^q + m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right\}.$$

After the necessary modifications, the desired result is obtained, and the proof is completed. □

Corollary 2.18. *Under the assumptions of Theorem 2.17,*

1. *If we choose $m = 1$, then we obtain the following inequality;*

$$\begin{aligned}&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\ \leq\;&\mathcal{H}_{10}\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_1\left|\psi''(\xi)\right|^q+\mathcal{H}_2\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_1\left|\psi''(\varsigma)\right|^q+\mathcal{H}_2\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

   $\mathcal{H}_{10}$ is as Theorem 2.17.

2. *If $s = m = 1$, then*

$$\begin{aligned}&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\ \leq\;&\mathcal{H}_{10}\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left[\mathcal{H}_3\left|\psi''(\xi)\right|^q+\mathcal{H}_4\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left[\mathcal{H}_3\left|\psi''(\varsigma)\right|^q+\mathcal{H}_4\left|\psi''(\varkappa)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

   $\mathcal{H}_{10}$ is as Theorem 2.17 and $\mathcal{H}_3$, $\mathcal{H}_4$ are as Corollary 2.12.

3. *Corollary item 3 above is still valid.*

Remark 2.19. *Following the idea of Remark 2.16 we can derive the following as particular cases of this last result: Theorem 11 of [52], Theorem 8 of [53], Theorem 3.8 of [55], Theorem 4 of [50], Theorem 2.4 of [51], Theorem 6 of [47] and Theorem 5 of [56].*

Theorem 2.20. *Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) such that $\psi'' \in L[\xi,\varsigma]$. If $\left|\psi''\right|^q$ is (h,m)-convex modified of the second type on I for some fixed $m \in [0,1]$ with $\frac{1}{p}+\frac{1}{q}=1$, then we obtain the following inequality:*

$$\begin{aligned}&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\ \leq\;&\frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\mathcal{H}_{11}+\frac{m}{q}\mathcal{H}_7\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right\}\\ &+\frac{1}{q}\mathcal{H}_6\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|^q+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|^q\right\}\end{aligned}$$

where

$$\mathcal{H}_{11}=\int_0^1 w^p(\tau)d\tau$$

and $\mathcal{H}_6$, $\mathcal{H}_7$ are as Theoremy 2.14.

Proof. From Lemma 2.1 and using Young Inequality, we have

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau\\
&+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\int_0^1 w(\tau)\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|d\tau\\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\int_0^1 w^p(\tau)d\tau+\frac{1}{q}\int_0^1\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\xi+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right\}\\
&+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\int_0^1 w^p(\tau)d\tau+\frac{1}{q}\int_0^1\left|\psi''\left(\frac{\varpi+\tau}{\varpi+1}\varsigma+\frac{1-\tau}{\varpi+1}\varkappa\right)\right|^q d\tau\right\}.
\end{aligned}$$

Thus, $\left|\psi''\right|^q$ is the (h,m)-convex modified of the second type, we obtain

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\int_0^1 w^p(\tau)d\tau+\frac{1}{q}\int_0^1\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\xi)\right|^q+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right\}\\
&+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\int_0^1 w^p(\tau)d\tau+\frac{1}{q}\int_0^1\left[h^s\left(\frac{\varpi+\tau}{\varpi+1}\right)\left|\psi''(\varsigma)\right|^q+m\left[1-h\left(\frac{\varpi+\tau}{\varpi+1}\right)\right]^s\left|\psi''\left(\frac{\varkappa}{m}\right)\right|^q\right]d\tau\right\}.
\end{aligned}$$

After the necessary modifications, the desired result is obtained, and the proof is completed. □

Corollary 2.21. *Under the assumptions of Theorem 2.20,*

1. *If we choose $m=1$, then we obtain the following inequality;*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\mathcal{H}_{11}+\frac{1}{q}\mathcal{H}_7\left|\psi''(\varkappa)\right|^q\right\}\\
&+\frac{1}{q}\mathcal{H}_6\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|^q+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|^q\right\}
\end{aligned}$$

   where

$$\mathcal{H}_{11}=\int_0^1 w^p(\tau)d\tau$$

   and $\mathcal{H}_6$, $\mathcal{H}_7$ are as Theoremy 2.14.

2. *If $s=m=1$, then*

$$\begin{aligned}
&\left|\mathcal{I}(w,\psi,\xi,\varsigma,\varkappa,\varpi)+\frac{\varpi+1}{\varkappa-\varsigma}J^{w}_{\varsigma+}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)+\frac{\varpi+1}{\varsigma-\varkappa}J^{w}_{\varsigma-}\psi\left(\frac{\varpi\varsigma+\varkappa}{\varpi+1}\right)\right|\\
\leq\;& \frac{(\varkappa-\xi)^2+(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left\{\frac{1}{p}\mathcal{H}_{11}+\frac{1}{q}\mathcal{H}_9\left|\psi''(\varkappa)\right|^q\right\}+\frac{1}{q}\mathcal{H}_8\left\{\frac{(\varkappa-\xi)^2}{(\varpi+1)^2}\left|\psi''(\xi)\right|^q+\frac{(\varsigma-\varkappa)^2}{(\varpi+1)^2}\left|\psi''(\varsigma)\right|^q\right\}
\end{aligned}$$

   and $\mathcal{H}_8$, $\mathcal{H}_9$ are as Corollary 2.15.

3. *Taking into account the Corollary 2.12, items 3 and 4, we can derive new inequalities for Riemann and Riemann-Liouville Integrals, respectively.*

3. Conclusions

In this work we have obtained new inequalities of the Hermite-Hadamard type for weighted operators, via modified (h, m)-convex functions of the second type. Throughout the work we have shown how several results known from the literature are particular cases of ours. Aside from the Corollaries and Remarks presented, we can expand by saying that under certain values for $\varkappa$ in I_1 and I_2, new results can be derived. Situation that remains valid considering other weights or other notions of convexity. All of the above shows the breadth and generality of the results obtained and how new work directions are opened with the ideas presented here.

Data Availability

No data were used to support the findings of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors have read and approved the final manuscript.

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