Filomat 39:3 (2025), 1051–1068
https://doi.org/10.2298/FIL2503051Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bounds and monotonicity results for means involving the q-polygamma functions

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Abstract. Let $\psi_{q,n} = (-1)^{n-1}\psi_q^{(n)}$ for $n \in \mathbb{N} \cup \{0\}$ and $q \in (0,1)$, where $\psi_q^{(n)}$ is the q-polygamma functions. In this paper, by means of the monotonicity of means and two classes of completely monotonicity functions, we establish lower and upper bounds for the means $I_{\psi_{q,n},w}(a,b)$ defined, for $b > a > 0$ and w being positive and integrable on $[a,b]$, by

$$I_{\psi_{q,n},w}(a,b) = \psi_{q,n}^{-1}\left(\frac{\int_a^b w(x)\,\psi_{q,n}(x)\,dx}{\int_a^b w(x)\,dx}\right);$$

and prove that the sequence $\left\{I_{\psi_{q,n},w}(a,b)\right\}_{n\geq 0}$ is decreasing with

$$\lim_{n\to\infty} I_{\psi_{q,n},w}(a,b) = a.$$

Moreover, we show that, for $a, b \in \mathbb{R}$ with $a \neq b$, the function

$$x \mapsto \psi_{q,n}^{-1}\left(\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)\,dt}{\int_a^b w(t)\,dt}\right) - x$$

is increasing from $(-\min\{a,b\},\infty)$ onto $(\min\{a,b\},\beta_2)$, where

$$\beta_2 = \log_q\left(\frac{\int_a^b w(t)\,q^t dt}{\int_a^b w(t)\,dt}\right).$$

These generalize some known results.

2020 *Mathematics Subject Classification*. Primary 33D05, 26A48; Secondary 26E60, 26D15.
Keywords. q-polygamma function, mean, complete monotonicity, monotonicity, inequality.
Received: 01 April 2024; Revised: 19 July 2024; Accepted: 06 October 2024
Communicated by Miodrag Spalević
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1. Introduction

The q-gamma function is defined **[11], [20]** for $x > 0$ and $q \neq 1$ by

$$\Gamma_q(x) = \begin{cases} (1-q)^{1-x} \prod_{n=0}^{\infty} \dfrac{1-q^{n+1}}{1-q^{n+x}} & \text{if } 0 < q < 1, \\ (q-1)^{1-x} q^{x(x-1)/2} \prod_{n=0}^{\infty} \dfrac{1-q^{-(n+1)}}{1-q^{-(n+x)}} & \text{if } q > 1. \end{cases} \tag{1}$$

It follows from (1) that, for all $q > 0$,

$$\Gamma_q(x) = q^{(x-1)(x-2)/2}\Gamma_{1/q}(x), \quad x > 0. \tag{2}$$

It is easy to see that

$$\lim_{x\to 0}\Gamma_q(x) = \infty \text{ and } \lim_{x\to\infty}\Gamma_q(x) = \infty.$$

The logarithmic derivative of the q-gamma function $\psi_q(x) = \Gamma_q'(x)/\Gamma_q(x)$ is known as q-psi or q-digamma function, which has a series representation:

$$\psi_q(x) = -\ln(1-q) + (\ln q)\sum_{k=0}^{\infty}\frac{q^{k+x}}{1-q^{k+x}} \tag{3}$$

$$= -\ln(1-q) + (\ln q)\sum_{k=1}^{\infty}\frac{q^{kx}}{1-q^{k}} \text{ for } 0 < q < 1. \tag{4}$$

The $\psi_q', \psi_q'', ..., \psi_q^{(n)}$ are called q-polygamma functions. For convenience, we denote by $\psi_{q,n} = (-1)^{n-1}\psi_q^{(n)}$ for $n \in \mathbb{N}$ and $\psi_{q,0} = -\psi_q$. From (4) $\psi_{q,n}$ has a series representation:

$$\psi_{q,n}(x) = (-1)^{n-1}\psi_q^{(n)}(x) = (-\ln q)^{n+1}\sum_{k=1}^{\infty}\frac{k^n q^{kx}}{1-q^k} \quad \text{if } 0 < q < 1 \tag{5}$$

for $x > 0$ and $n \in \mathbb{N}$, which shows that $\psi_{q,n}$ for $n \in \mathbb{N}$ is completely monotonic on $(0, \infty)$.

Remark 1.1. *A function f is called completely monotonic on an interval I, if f has the derivative of any order on I and satisfies*

$$(-1)^k (f(x))^{(k)} \geq 0$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ on I, see [7, 41] and recent papers [22, 39, 42, 43, 46, 47].

It is readily seen from (4) and (5) that, for $n \in \mathbb{N}$ and $q \in (0, 1)$,

$$\begin{aligned} &\lim_{x\to 0+}\psi_q(x) = -\infty, \quad \lim_{x\to\infty}\psi_q(x) = -\ln(1-q), \\ &\lim_{x\to 0+}\psi_{q,n}(x) = \infty, \quad \lim_{x\to\infty}\psi_{q,n}(x) = 0. \end{aligned} \tag{6}$$

Usually, Γ_q and ψ_q are respectively called a q-analogue of the ordinary gamma Γ and digamma ψ functions since

$$\lim_{q\to 1^-}\Gamma_q(x) = \lim_{q\to 1^+}\Gamma_q(x) = \Gamma(x) \text{ and } \lim_{q\to 1^-}\psi_q(x) = \lim_{q\to 1^+}\psi_q(x) = \psi(x) \tag{7}$$

for $x > 0$, where the first and second limit relations were proved in [5] (see also [11, p. 17], [23]) and in [21]. Similarly, as shown in Appendix, $\psi_q^{(n)}$ is a q-analogue of the ordinary polygamma functions $\psi^{(n)}$. We thus have

$$\lim_{q\to 1}\psi_{q,n}(x)=\lim_{q\to 1}\left[(-1)^{n-1}\psi_q^{(n)}(x)\right]=(-1)^{n-1}\psi^{(n)}(x)=\psi_n(x).$$

This allows us to deduce the properties of the polygamma functions $\psi^{(n)}$ via $\psi_q^{(n)}$, or, to generalize the properties of polygamma functions $\psi^{(n)}$ to $\psi_q^{(n)}$.

Let $f: I\to\mathbb{R}$ be (strictly) monotonic, $a,b\in I$. Elezović and Pečarić [9] (see also [24]) introduced the so-called integral f-mean of a and b defined as

$$I_f(a,b)=f^{-1}\left(\frac{\int_a^b f(x)\,dx}{b-a}\right)\text{ if } a\neq b \text{ and } I_f(a,a)=a. \tag{8}$$

The authors proved that for $x,a,b>0$, $I_{\psi'}(a,b)\le I_\psi(a,b)$, and the function $x\mapsto I_\psi(x+a,x+b)-x$ is increasing concave with

$$\lim_{x\to\infty}\left(I_\psi(x+a,x+b)-x\right)=\frac{a+b}{2}.$$

Yang and Zheng in [48, Theorems 1.2, 1.3] showed that for $x,a,b>0$, the sequence $\{I_{\psi_n}(a,b)\}_{n\ge 0}$ is strictly decreasing with

$$\lim_{n\to\infty}I_{\psi_n}(a,b)=\min\{a,b\};$$

the function $x\mapsto I_{\psi_n}(x+a,x+b)-x$ is strictly increasing from $(-\min\{a,b\},\infty)$ onto $(\min\{a,b\},(a+b)/2)$. Moreover, Qi [31, Theorem 1] established lower and upper bounds for $I_{\psi_n}(a,b)$ in terms of generalized logarithmic mean, that is, the double inequality

$$L_{p_2}(a,b)<I_{\psi_n}(a,b)=\psi_n^{-1}\left(\frac{\int_a^b\psi_n(t)\,dt}{b-a}\right)\le L_{p_1}(a,b) \tag{9}$$

holds for $a,b>0$ with $a\neq b$ if $p_1\ge -n+1$ and $p_2\le -n$, where

$$L_p(a,b)=\left(\frac{1}{p}\frac{a^p-b^p}{a-b}\right)^{1/(p-1)}\quad\text{if } p(p-1)\neq 0 \tag{10}$$

is the generalized logarithmic mean of a and b (see [37]). This improved Batir's results in [6].

In what follows, we always suppose that $w: I\to\mathbb{R}_+=(0,\infty)$ is integrable. For $a,b\in I$, it is easy to check that

$$I_{f,w}(a,b):=f^{-1}\left(\frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}\right)\text{ if } a\neq b \text{ and } I_{f,w}(a,a)=a \tag{11}$$

is also a mean of a and b. We call $I_{f,w}(a,b)$ an integral f-mean with weight w of a and b. Similarly, if $f:\mathbb{R}\to\mathbb{R}$ is (strictly) monotonic, then we can verify that

$$A_{f,w}(x):=f^{-1}\left(\frac{\int_a^b w(t)f(x+t)\,dt}{\int_a^b w(t)\,dt}\right)-x\text{ if } a\neq b \text{ and } A_{f,w}(x)=a \text{ if } a=b \tag{12}$$

is a mean of a and b.

Motivated by the results mentioned above, the aims of this paper are to

(i) find the lower and upper bounds for the mean $I_{f,w}(a,b)$ for $f=\psi_{q,n}$;

(ii) prove that the sequence $\left\{I_{\psi_{q,n},w}(a,b)\right\}_{n\geq 0}$ is decreasing;

(iii) prove that the function $x\mapsto A_{f,w}(x)$ for $f=\psi_{q,n}$ is increasing on $(-\min\{a,b\},\infty)$.

The paper is organized as follows. In Section 2, we recall the monotonicity results for the one-parameter mean and power mean of a function on an interval with a weight, and the latter is crucial to prove Lemma 4 and Theorem 1. In Section 3, we present two classes of completely monotonic functions involving q-polygamma functions, which are of independent interest. The main results are stated and proved in Section 4, in which several consequences are also listed. In the last section, we summarize the conclusions of this paper and propose two problems.

2. Preliminaries

2.1. One-parameter mean and power mean of a function

Let $a,b>0$ with $a\neq b$ and $r\in\mathbb{R}$. The one parameter mean $J_r(a,b)$ is defined as

$$J_r(a,b)=\begin{cases}\dfrac{r}{r+1}\dfrac{a^{r+1}-b^{r+1}}{a^r-b^r} & \text{if } r\neq -1,0,\\ \dfrac{a-b}{\ln a-\ln b}=L(a,b) & \text{if } r=0,\\ ab\dfrac{\ln a-\ln b}{a-b}=\dfrac{G^2(a,b)}{L(a,b)} & \text{if } r=-1,\end{cases}$$

where $L(a,b)$ and $G(a,b)$ are the logarithmic and geometric means of a and b. It was shown in [30, Theorem 1] (see also [44]) that

Lemma 2.1. *Let $a,b>0$ and $r\in\mathbb{R}$. The function $r\mapsto J_r(a,b)$ is increasing on $\mathbb{R}$.*

This monotonicity result will be used in the proof of Lemma 3.1.

Recall that the r-th power mean in the discrete case. Let $\boldsymbol{a}=(a_1,a_2,...,a_n)$, $\boldsymbol{w}=(w_1,w_2,...,w_n)$ be two positive n-tuples, $r\in\mathbb{R}$., Then the r-th power mean of $\boldsymbol{a}$ with weight $\boldsymbol{w}$ is defined by

$$M_n^{[r]}(\boldsymbol{a};\boldsymbol{w})=\left(\frac{\sum_{k=0}^n w_k a_k^r}{\sum_{k=0}^n w_k}\right)^{1/r} \text{ if } r\neq 0 \text{ and } M_n^{[0]}(\boldsymbol{a};\boldsymbol{w})=\prod_{k=0}^n a_k^{w_k'}$$

where $w_k'=w_k/\sum_{k=0}^n w_k$. It is known that the function $r\mapsto M_n^{[r]}(\boldsymbol{a};\boldsymbol{w})$ is increasing on $\mathbb{R}$ with

$$\lim_{r\to-\infty}M_n^{[r]}(\boldsymbol{a};\boldsymbol{w})=\min\{\boldsymbol{a}\} \text{ and } \lim_{r\to\infty}M_n^{[r]}(\boldsymbol{a};\boldsymbol{w})=\max\{\boldsymbol{a}\}.$$

Let f,w be two positive and integrable functions on $[a,b]$ $(a<b)$. The r-th power mean of f on $[a,b]$ with the weight w is defined [25, Defintion 2.3] by

$$M_{f,w}^{[r]}(a,b)=\begin{cases}\left(\dfrac{\int_a^b w(x)f(x)^r dx}{\int_a^b w(x)dx}\right)^{1/r} & \text{if } r\neq 0,\\ \exp\left(\dfrac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}\right) & \text{if } r=0.\end{cases} \tag{13}$$

Evidently, $M_{f,w}^{[r]}(a,b)$ is completely analogous to the discrete case. The following monotonicity result will be used to prove Lemma 3.4.

Lemma 2.2 ([32, Theorem 1.1], [8, p. 375, Theorem 6]). *Let f, w be positive and integrable on $[a,b]$ $(a<b)$. Then the function $r \mapsto M_{f,w}^{[r]}(a,b)$ is increasing on $\mathbb{R}$. Moreover, if f is monotonic on $[a,b]$, then*

$$\lim_{r\to-\infty} M_{f,w}^{[r]}(a,b) = \min\{f(a), f(b)\} \quad \text{and} \quad \lim_{r\to\infty} M_{f,w}^{[r]}(a,b) = \max\{f(a), f(b)\}.$$

2.2. Simple properties of $\psi_{q,n}$

The following lemma gives five simple properties of $\psi_{q,n}$.

Lemma 2.3. *Let $\psi_{q,n} = (-1)^{n-1}\psi_q^{(n)}$ for $q \in (0,1)$ and $n \in \mathbb{N}$. Then the following statements are true and equivalent to each other:*

(i) The inequality

$$\psi_{q,n}(x)\,\psi_{q,n+2}(x) > \psi_{q,n+1}^2(x) \tag{14}$$

holds for $x>0$.

(ii) The sequence $\{\psi_{q,n+1}/\psi_{q,n}\}_{n\in\mathbb{N}}$ is strictly increasing for each fixed $x>0$.

(iii) The function $x \mapsto \psi_{q,n+1}(x)/\psi_{q,n}(x)$ is strictly decreasing on $(0,\infty)$.

(iv) The function $x \mapsto \psi_{q,n}(x)$ is log-convex on $(0,\infty)$.

(v) The function $x \mapsto \psi_{q,n+1}\circ\psi_{q,n}^{-1}(x)$ is convex on $(0,\infty)$.

Proof. (i) Using the series representation (5) we obtain

$$\begin{aligned} U_{q,n}(x) &:= \frac{\psi_{q,n}(x)\,\psi_{q,n+2}(x) - \psi_{q,n+1}^2(x)}{(-\ln q)^{2n+4}} \\ &= \left(\sum_{k=1}^{\infty}\frac{k^n q^{kx}}{1-q^k}\right)\left(\sum_{k=1}^{\infty}\frac{k^{n+2} q^{kx}}{1-q^k}\right) - \left(\sum_{k=1}^{\infty}\frac{k^{n+1} q^{kx}}{1-q^k}\right)^2 \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{j^n k^{n+2} - j^{n+1}k^{n+1}}{(1-q^k)(1-q^j)} q^{(k+j)x}. \end{aligned} \tag{15}$$

Since the indices k and j are symmetric, interchanging them yields

$$U_{q,n}(x) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{k^n j^{n+2} - k^{n+1}j^{n+1}}{(1-q^j)(1-q^k)} q^{(j+k)x}. \tag{16}$$

An addition of (15) and (16) gives

$$2U_{q,n}(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{k^n j^n (j-k)^2}{(1-q^k)(1-q^j)} q^{(k+j)x} > 0 \tag{17}$$

for $x>0$, which proves the inequality (14).

(ii) The inequality (14) implies that $\psi_{q,n+2}/\psi_{q,n+1} > \psi_{q,n+1}/\psi_{q,n}$ for $n \in \mathbb{N}$, which is the required increasing property.

(iii) Since $\psi_{q,n}' = -\psi_{q,n+1}$, we have

$$\left(\frac{\psi_{q,n+1}}{\psi_{q,n}}\right)' = \frac{\psi_{q,n+1}'\psi_{q,n} - \psi_{q,n+1}\psi_{q,n}'}{\psi_{q,n}^2} = \frac{-\psi_{q,n+2}\psi_{q,n} + \psi_{q,n+1}^2}{\psi_{q,n}^2} < 0,$$

that is, the third assertion is true.

(iv) Differentiation gives

$$\left(\ln \psi_{q,n}\right)' = \frac{\psi'_{q,n}}{\psi_{q,n}} = -\frac{\psi_{q,n+1}}{\psi_{q,n}}, \quad \left(\ln \psi_{q,n}\right)'' = -\left(\frac{\psi_{q,n+1}}{\psi_{q,n}}\right)' > 0.$$

(v) Differentiation yields

$$\begin{aligned}\left[\psi_{q,n+1} \circ \psi_{q,n}^{-1}(x)\right]' &= \frac{\psi'_{n+1}\left(\psi_{q,n}^{-1}(x)\right)}{\psi'_{q,n}\left(\psi_{q,n}^{-1}(x)\right)} = \frac{\psi_{q,n+2}(y)}{\psi_{q,n+1}(y)}, \text{ where } y = \psi_{q,n}^{-1}(x) \\ \left[\psi_{q,n+1} \circ \psi_{q,n}^{-1}(x)\right]'' &= \left[\frac{\psi_{q,n+2}(y)}{\psi_{q,n+1}(y)}\right]' \frac{1}{\psi'_{q,n}(y)} = -\left[\frac{\psi_{q,n+2}(y)}{\psi_{q,n+1}(y)}\right]' \frac{1}{\psi_{q,n+1}(y)} > 0,\end{aligned}$$

which completes the proof. □

Remark 2.4. *From the relation (17), we have*

$$2(-1)^m U_{q,n}^{(m)}(x) = (-\ln q)^m \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{k^n j^n (j-k)^2 (k+j)^m}{(1-q^k)(1-q^j)} q^{(k+j)x} > 0,$$

for $x > 0$ and $m \in \mathbb{N}_0$, which implies that the function $x \mapsto \psi_{q,n}(x)\psi_{q,n+2}(x) - \psi_{q,n+1}^2(x)$ is completely monotonic on $(0, \infty)$. Some completely monotonic functions involving the q-gamma and q-polygamma functions can be found in [12, 13, 16–18, 33–36] and recent papers [1, 38].

Remark 2.5. *The function $\psi_{q,n}$ has another interesting property proved in [26], which states that the functions*

$$\begin{aligned} x &\mapsto \left[\psi'_q(x) - \ln q\right]^2 + \psi''_q(x) \text{ for } q \in (0,1), \\ x &\mapsto \left[\psi'_q(x)\right]^2 + \psi''_q(x) \text{ for } q \in (1,\infty)\end{aligned}$$

are completely monotonic on $(0, \infty)$. Clearly, this is a generalization of the complete monotonicity of the function $x \mapsto [\psi'(x)]^2 + \psi''(x)$ on $(0, \infty)$(see [27].

3. Two classes of completely monotonic functions

To prove Theorem 4.1 below, we have to determine the values of the parameter α such that the function

$$g_{q,n}(x;\alpha) = \frac{q^x - 1}{\ln q}\psi_{q,n+1}(x) - (1 + \alpha q^x)\psi_{q,n}(x) \tag{18}$$

is positive or negative for $x > 0$. In fact, we can find the sufficient conditions under which the function $x \mapsto \pm g_{q,n}(x;\alpha)$ are completely monotonic on $(0, \infty)$ for all $q \in (0, 1)$ and $n \in \mathbb{N}$, which reads as follows.

Lemma 3.1. *Let $q \in (0, 1)$ and $n \in \mathbb{N}$. The following statements are valid:*
(i) If $\alpha \le \alpha_1(n, q) = n - 2q/(q + 1)$, then the function $x \mapsto g_{q,n}(x;\alpha)$ is completely monotonic on $(0, \infty)$.
(ii) If $\alpha \ge \alpha_2(n, q) = (2^n + q - 1)/(q + 1)$ then the function $x \mapsto -g_{q,n}(x;\alpha)$ is completely monotonic on $(0, \infty)$.

Remark 3.2. *Letting $q \to 1^-$ yields*

$$g_n(x;\alpha) := g_{1,n}(x;\alpha) = x\psi_{n+1}(x) - (1+\alpha)\psi_n(x).$$

Qi and his coauthors [28] proved that the function $\pm g_n(x;\alpha)$ is completely monotonic on $(0, \infty)$ if and only if $0 \le 1 + \alpha \le n$ $(1 + \alpha \ge n + 1)$. Obviously, Lemma 3.1 is a generalization of Qi et al.'s result, and the proof of Lemma 3.1 is more difficult than the case of $q \to 1^-$.

The following corollary is a direct consequence of Lemma 3.1.

Corollary 3.3. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. The double inequality*

$$1+\alpha_1 q^x < \frac{q^x-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)} < 1+\alpha_2 q^x \tag{19}$$

holds for $x>0$, where $\alpha_1=\alpha_1(n,q)$ and $\alpha_2=\alpha_2(n,q)$ are as in Lemma 3.1.

Proof of Lemma 3.1. Using the representation (4) yields

$$\begin{aligned}\frac{g_{q,n}(x;\alpha)}{(-\ln q)^{n+1}} &= (1-q^x)\sum_{k=1}^{\infty}\frac{k^{n+1}q^{kx}}{1-q^k}-(1+\alpha q^x)\sum_{k=1}^{\infty}\frac{k^n q^{kx}}{1-q^k}\\ &= \sum_{k=1}^{\infty}\left(\frac{k^{n+1}}{1-q^k}-\frac{(k-1)^{n+1}}{1-q^{k-1}}-\alpha\frac{(k-1)^n}{1-q^{k-1}}-\frac{k^n}{1-q^k}\right)q^{kx}\\ &= \sum_{k=2}^{\infty}(u_k(n)-\alpha)\frac{(k-1)^n}{1-q^{k-1}}q^{kx},\end{aligned}$$

where

$$u_k(n)=\frac{k^n}{(k-1)^{n-1}}\frac{1-q^{k-1}}{1-q^k}-(k-1).$$

Then, for $m\in\mathbb{N}_0$,

$$(-1)^m\left(g_{q,n}(x;\alpha)\right)^{(m)}=(-\ln q)^{m+n+2}\sum_{k=2}^{\infty}(u_k(n)-\alpha)\frac{(k-1)^n k^m}{1-q^{k-1}}q^{kx}.$$

Note that

$$u_k(n+1)-u_k(n)=\frac{k^n}{(k-1)^n}\frac{1-q^{k-1}}{1-q^k}=\frac{1}{(1-1/k)^{n-1}}\frac{1}{J_{k-1}(1,q)}:=\Delta_k(n),$$

where $J_{k-1}(1,q)$ is the one-parameter mean of 1 and q.

By Lemma 2.1 we see that the sequence $\{J_{k-1}(1,q)\}_{k\geq 2}$ is positive and increasing; the sequence $\left\{(1-1/k)^{n-1}\right\}_{k\geq 2}$ is clearly positive and increasing; these yield that the sequence $\{\Delta_k(n)\}_{k\geq 2}$ is decreasing, and therefore,

$$1=\lim_{k\to\infty}\Delta_k(n)\leq u_k(n+1)-u_k(n)\leq\Delta_2(n)=\frac{2^n}{q+1}. \tag{20}$$

The first inequality of (20) implies that

$$u_k(n+1)-\left(n+1-\frac{2q}{q+1}\right)\geq u_k(n)-\left(n-\frac{2q}{q+1}\right)$$

for all $n\in\mathbb{N}$, that is, the sequence $\{u_k(n)-(n-2q/(q+1))\}_{n\geq 1}$ is increasing, and hence,

$$\begin{aligned}u_k(n)-\left(n-\frac{2q}{q+1}\right) &\geq u_k(1)-\left(1-\frac{2q}{q+1}\right)\\ &= k\frac{1-q^{k-1}}{1-q^k}-k+\frac{2q}{q+1}\\ &= -(1-q)\frac{kq^{k-1}}{1-q^k}+\frac{2q}{q+1}\geq 0,\end{aligned}$$

where the second inequality holds due to the fact that sequence $\left\{kq^{k-1}/\left(1-q^k\right)\right\}_{k\geq 2}$ is decreasing. Then $u_k(n)-\alpha \geq u_k(n)-(n-2q/(q+1)) \geq 0$ if $\alpha \leq n-2q/(q+1)=\alpha_1(n,q)$, and we thus conclude that $g_{q,n}(x;\alpha)$ is completely monotonic on $(0,\infty)$ for $\alpha \leq \alpha_1(n,q)$.

The second inequality of (20) implies that

$$u_k(n+1)-\frac{2^{n+1}+q-1}{q+1} \leq u_k(n)-\frac{2^n+q-1}{q+1}$$

for all $n \in \mathbb{N}$, that is, the sequence $\{u_k(n)-(2^n+q-1)/(q+1)\}_{n\geq 1}$ is decreasing, and therefore,

$$\begin{aligned} u_k(n)-\frac{2^n+q-1}{q+1} &\leq u_k(1)-\frac{2+q-1}{q+1} \\ &= k\frac{1-q^{k-1}}{1-q^k}-(k-1)-1=-kq^{k-1}\frac{1-q}{1-q^k}<0. \end{aligned}$$

This yields that

$$u_k(n)-\alpha \leq u_k(n)-(2^n+q-1)/(q+1) \leq 0$$

if $\alpha \geq (2^n+q-1)/(q+1)=\alpha_2(n,q)$, and we conclude that $-g_{q,n}(x;\alpha)$ is completely monotonic on $(0,\infty)$ for $\alpha \geq \alpha_2(n,q)$. This completes the proof. □

To prove Theorem 4.6 below, we need to determine the positivity or negativity of the function

$$h_{q,n}(x;\beta)=\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)\,dt}{\int_a^b w(t)\,dt}-\psi_{q,n}(x+\beta) \tag{21}$$

on $(-\min\{a,b,\beta\},\infty)$. The following lemma offers the necessary and sufficient conditions for the function $x\mapsto \pm h_{q,n}(x;\beta)$ to be completely monotonic on $(0,\infty)$ for $q\in(0,1)$ and $n\in\mathbb{N}_0$.

Lemma 3.4. *Let $q\in(0,1)$, $n\in\mathbb{N}_0$, $a,b\in\mathbb{R}$ with $a\neq b$ and $\beta\geq\min\{a,b\}$. The function $x\mapsto h_{q,n}(x;\beta)$ is completely monotonic on $(-\min\{a,b\},\infty)$ if and only if*

$$\beta \geq \beta_2=\log_q M_{q^t,w}^{[1]}(a,b)=\log_q\left(\frac{\int_a^b w(t)\,q^t dt}{\int_a^b w(t)\,dt}\right); \tag{22}$$

while the function $x\mapsto -h_{q,n}(x;\beta)$ is completely monotonic on $(-\min\{a,b\},\infty)$ if and only if $\beta=\beta_1=\min\{a,b\}$.

Remark 3.5. *The above lemma for $w(t)=1$ was proved by Tian and Yang in [38, Corollary 3].*

Taking $\beta=\beta_1,\beta_2$ in Lemma 3.4 we have the following corollary.

Corollary 3.6. *Let $q\in(0,1)$, $n\in\mathbb{N}_0$ and $a,b\in\mathbb{R}$ with $a\neq b$. The double inequality*

$$\psi_{q,n}(x+\beta_2)<\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)dt}{\int_a^b w(t)\,dt}<\psi_{q,n}(x+\beta_1) \tag{23}$$

holds for $x>-\min\{a,b\}$ with the best constants $\beta_1=\min\{a,b\}$ and β_2 given in (22).

Proof of Lemma 3.4. First, we prove sufficiency. Using the series representation (5) we obtain that, for $n \in \mathbb{N}$,

$$h_{q,n}(x;\beta) = \frac{\int_a^b w(t)\left(\psi_{q,n}(x+t) - \psi_{q,n}(x+\beta)\right)dt}{\int_a^b w(t)\,dt} = (-\ln q)^{n+1}\sum_{k=1}^{\infty}\frac{d_k(\beta)}{1-q^k}k^n q^{kx}, \tag{24}$$

where

$$d_k(\beta) = \frac{\int_a^b w(t)\,q^{kt}dt}{\int_a^b w(t)\,dt} - q^{k\theta} = M_{q^t,w}^{[k]}(a,b)^k - q^{k\theta}, \tag{25}$$

and $M_{f,w}^{[r]}(a,b)$ is defined by (13). Clearly, the relation(24) is also true for $n = 0$. Then, for $m \in \mathbb{N}_0$,

$$h_{q,n}^{(m)}(x;\beta) = (-\ln q)^{m+n+1}\sum_{k=1}^{\infty}\frac{d_k(\beta)}{1-q^k}k^{m+n}q^{kx}.$$

Since

$$\begin{aligned} d_k(\beta) &= \frac{M_{q^t,w}^{[k]}(a,b)^k - q^{k\beta}}{\ln M_{q^t,w}^{[k]}(a,b)^k - \ln q^{k\beta}}\left(\ln M_{q^t,w}^{[k]}(a,b)^k - \ln q^{k\beta}\right) \\ &= L\left(M_{q^t,w}^{[k]}(a,b)^k, q^{k\beta}\right)\times(-k\ln q)\left(\beta - \log_q M_{q^t,w}^{[k]}(a,b)\right), \end{aligned}$$

where $L(x,y) = (x-y)/(\ln x - \ln y)$ is the logarithmic mean of $x, y > 0$ with $x \neq y$, we see that

$$\operatorname{sgn} d_k = \operatorname{sgn}\left(\beta - \log_q M_{q^t,w}^{[k]}(a,b)\right).$$

By Lemma 2.2 we see that the sequence $\left\{M_{q^t,w}^{[k]}(a,b)\right\}_{k\geq 1}$ is increasing with

$$\begin{aligned} \inf_{k\in\mathbb{N}}\left\{M_{q^t,w}^{[k]}(a,b)\right\} &= M_{q^t,w}^{[1]}(a,b), \\ \sup_{k\in\mathbb{N}}\left\{M_{q^t,w}^{[k]}(a,b)\right\} &= \lim_{k\to\infty}M_{q^t,w}^{[k]}(a,b) = \max\left\{q^a, a^b\right\} \end{aligned}$$

due to that $t \mapsto q^t$ is decreasing on $\mathbb{R}$. Consequently, $d_k \geq 0$ for all $k \in \mathbb{N}$ if and only if $\beta \geq \log_q M_{q^t,w}^{[1]}(a,b) = \beta_2$; $d_k < 0$ for all $k \in \mathbb{N}$ if and only if $\beta \leq \log_q\left(\max\left\{q^a, a^b\right\}\right) = \min\{a,b\}$. This proves sufficiency.

Second, we prove the necessity. Suppose that $x \mapsto -h_{q,n}(x;\beta)$ is completely monotonic on $(-\min\{a,b\},\infty)$. We prove that $\beta = \min\{a,b\}$. In view of the symmetry of a and b, we let $b > a$. Then $\beta_1 = \min\{a,b\} = a$. If $\beta \neq a$, that is, $\beta > a$, then $\lim_{x\to -a^+}\psi_{q,n}(x+\beta) = \psi_{q,n}(\beta - a)$ is a constant. If we prove that

$$\lim_{x\to -a^+}\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)\,dt}{\int_a^b w(t)\,dt} = \infty, \tag{26}$$

then $\lim_{x\to -a^+}h_{q,n}(x;\beta) = \infty$, which yields a contradiction, and the necessity follows. In fact, since $w(t)$ is positive and integrable on $[a,b]$, we have

$$\begin{aligned} \int_a^b w(t)\,\psi_{q,n}(x+t)\,dt &> \min_{t\in[a,b]}\{w(t)\}\int_a^b \psi_{q,n}(x+t)\,dt \\ &= \min_{t\in[a,b]}\{w(t)\}\left(\psi_{q,n-1}(x+a) - \psi_{q,n-1}(x+b)\right), \end{aligned}$$

which, by (6), clearly tends to infinity as $x \to -a^{+}$. This proves the necessary condition for $x \mapsto -h_{q,n}(x;\beta)$ to be completely monotonic on $(-\min\{a,b\},\infty)$ is that: $\beta = \min\{a,b\}$.

The necessary condition for $x \mapsto h_{q,n}(x;\beta)$ to be completely monotonic on $(-\min\{a,b\},\infty)$ follows from the limit relation

$$\lim_{x\to\infty} \frac{h_{q,n}(x;\beta)}{q^x} = (-\ln q)^{n+1} d_1 \geq 0,$$

which implies that $\beta \geq \log_q M^{[1]}_{q^t,w}(a,b) = \beta_2$, thereby completing the proof. □

4. Main results

4.1. Bounds for $I_{\psi_{q,n},w}(a,b)$

In this subsection, we establish the lower and upper bounds for the mean

$$I_{\psi_{q,n},w}(a,b) = \psi^{-1}_{q,n}\left(\frac{\int_a^b w(x)\,\psi_{q,n}(x)\,dx}{\int_a^b w(x)\,dx}\right). \tag{27}$$

Theorem 4.1. *For $q \in (0,1)$, $r \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $a,b > 0$ with $a \neq b$, let $M^{[r]}_{f,w}(a,b)$ be defined by (13), where $f(x) = 1 - q^x$ and $w(x)$ is positive and integrable on $[\min\{a,b\},\max\{a,b\}]$. The double inequality*

$$\log_q\left(1 - M^{[r_1]}_{f,w}(a,b)\right) < I_{\psi_{q,n},w}(a,b) < \log_q\left(1 - M^{[r_2]}_{f,w}(a,b)\right) \tag{28}$$

holds if

$$r_1 \leq -\alpha_2(n+1,q) = -\frac{2^{n+1}+q-1}{q+1}$$

and

$$r_2 \geq -\alpha_1(n+1,q) = -n - \frac{1-q}{q+1}.$$

In particular, we have

$$\lim_{n\to\infty} I_{\psi_{q,n},w}(a,b) = \min\{a,b\}. \tag{29}$$

Proof. Let $f_r(x) = f(x)^r = (1-q^x)^r$ $(r<0)$ and $g(x) = \psi_{q,n}(x)$. Then $f_r^{-1}(x) = \log_q\left(1 - x^{1/r}\right)$. Since

$$f_r'(x) = -(r\ln q)\,q^x(1-q^x)^{r-1} < 0,$$

the desired inequalities are equivalent to

$$f_{r_1}\circ g^{-1}\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right) < \frac{\int_a^b w(x)\,f_{r_1}(x)\,dx}{\int_a^b w(x)\,dx},$$

$$f_{r_2}\circ g^{-1}\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right) > \frac{\int_a^b w(x)\,f_{r_2}(x)\,dx}{\int_a^b w(x)\,dx},$$

which suffice to check that the function $f_r \circ g^{-1}$ is convex if $r = r_1$ and is concave if $r = r_2$. Differentiation yields

$$\begin{aligned}\left(f_r\left(g^{-1}(x)\right)\right)' &= \frac{f_r'(y)}{g'(y)}, \text{ where } y = g^{-1}(x),\\ \left(f_r\left(g^{-1}(x)\right)\right)'' &= \frac{f_r''(y)\,g'(y) - f_r'(y)\,g''(y)}{g'(y)^3} := \frac{D(y)}{g'(y)^3}.\end{aligned}$$

A direct computation gives

$$\frac{D(x)}{-r(\ln q)^2 q^x (1-q^x)^{-n-2}} = \frac{1-q^x}{-\ln q}\psi_{q,n+2}(x) - (1-rq^x)\,\psi_{q,n+1}(x) = g_{q,n+1}(x;-r),$$

where $g_{q,n}(x;\alpha)$ is as in (18). Using Lemma 3.1 we see that $g_{q,n+1}(x;-r) < (>)\,0$ if $-r \geq \alpha_2(n+1,q)$ $(0 < -r \leq \alpha_1(n+1,q))$, where $\alpha_1(n,q)$ and $\alpha_2(n,q)$ are as in Lemma 3.1. This together with $g'(y) = -\psi_{q,n+1}(x) < 0$ yields that $\left(f_r\left(g^{-1}(x)\right)\right)'' > (<)\,0$ if $r \leq -\alpha_2(n+1,q)$ $(-\alpha_1(n+1,q) \leq r < 0)$, which proves the double inequality (28) if $r_1 \leq -\alpha_2(n+1,q)$ and $-\alpha_1(n+1,q) \leq r_2 < 0$. The right hand side inequality in (28) if $r_2 \geq 0$ follows from the increasing property of the function $r \mapsto M_{f,w}^{[r]}(a,b)$ given by Lemma 2.2.

Taking $r_1 = -\alpha_2(n+1,q)$ and $r_2 = -\alpha_1(n+1,q)$ in (28) and letting $n \to \infty$, then we see that $r_1, r_2 \to -\infty$. Since $f(x) = 1 - q^x$ is increasing on $(0,\infty)$, an application of Lemma 2.2 yields

$$\lim_{n\to\infty} M_{f,w}^{[r_i]}(a,b) = \lim_{r_i\to\infty} M_{f,w}^{[r_i]}(a,b) = \min\left\{1-q^a, 1-q^b\right\},\ i = 1,2,$$

then

$$\lim_{n\to\infty} \log_q\left(1 - M_{f,w}^{[r_1]}(a,b)\right) = \lim_{n\to\infty}\left(1 - M_{f,w}^{[r_2]}(a,b)\right) = \min\{a,b\},$$

which implies the limit relation (29), and the proof is done. □

The double inequality (28) is equivalent to

$$\psi_{q,n}\left(\log_q\left(1 - M_{f,w}^{[r_2]}(a,b)\right)\right) < \frac{\int_a^b w(x)\,\psi_{q,n}(x)\,dx}{\int_a^b w(x)\,dx} < \psi_{q,n}\left(\log_q\left(1 - M_{f,w}^{[r_1]}(a,b)\right)\right).$$

Taking $n = 0$ and $w(x) = 1$ in Theorem 4.1, and noting that

$$\int_a^b \frac{dx}{1-q^x} = b - a - \frac{1}{\ln q}\ln\frac{1-q^b}{1-q^a},$$

we obtain the following corollary.

Corollary 4.2. *Let $q \in (0,1)$ and $a, b > 0$ with $a \neq b$. The double inequality*

$$\psi_q\left(\log_q\left(1 - M_{f,1}^{[r_1]}(a,b)\right)\right) < \ln\left[\frac{\Gamma_q(b)}{\Gamma_q(a)}\right]^{1/(b-a)} < \psi_q\left(\log_q\left(1 - M_{f,1}^{[r_2]}(a,b)\right)\right) \tag{30}$$

holds if $r_1 \leq -1$ and $r_2 \geq (q-1)/(q+1)$, where

$$M_{f,1}^{[r]}(a,b) = \left(\frac{\int_a^b (1-q^x)^r\,dx}{b-a}\right)^{1/r}.$$

In particular, when $(r_1, r_2) = (-1, 0)$, *we have*

$$\psi_q\left(\log_q \frac{\ln\left(1-q^b\right)-\ln\left(1-q^a\right)}{\ln\left(q^{-b}-1\right)-\ln\left(q^{-a}-1\right)}\right) < \frac{\ln\Gamma_q(b)-\ln\Gamma_q(a)}{b-a} < \psi_q\left(\log_q\left(1-\exp\frac{\int_a^b \ln\left(1-q^x\right)dx}{b-a}\right)\right). \tag{31}$$

Remark 4.3. *Since* $1-q^x = 1-e^{x\ln q} \sim -x\ln q$ *as* $q \to 1^-$, *we have*

$$M_{f,1}^{[r]}(a,b) = \left(\frac{\int_a^b \left(1-q^x\right)^r dx}{b-a}\right)^{1/r} \sim (-\ln q)\left(\frac{\int_a^b x^r dx}{b-a}\right)^{1/r} = (-\ln q)\, L_{r+1}(a,b),$$

$$\log_q\left(1-M_{f,1}^{[r]}(a,b)\right) \sim \frac{\ln\left(1-(-\ln q)\, L_{r+1}(a,b)\right)}{\ln q} \to L_{r+1}(a,b),$$

as $q \to 1^-$, *where* $L_p(a,b)$ *is the generalized logarithmic mean of* a *and* b *defined by (10). Then the double inequality (30) is reduced to*

$$L_{r_1+1}(a,b) < I_{\psi_n}(a,b) = \psi_n^{-1}\left(\frac{\int_a^b \psi_n(x)\,dx}{b-a}\right) < L_{r_2+1}(a,b)$$

if $r_1 \le -2^n$ *and* $r_2 \ge -n$. *This lower bound for* $I_{\psi_n}(a,b)$ *is weaker than Qi's in (9).*

Remark 4.4. *Letting* $q \to 1^-$ *in the double inequality (31) gives*

$$\psi(L(a,b)) < \frac{\ln\Gamma(b)-\ln\Gamma(a)}{b-a} < \psi(I(a,b)), \tag{32}$$

where $L(a,b) = (b-a)/(\ln b - \ln a)$ *and*

$$I(a,b) = \exp\left(\frac{b\ln b - a\ln a}{b-a} - 1\right)$$

are the logarithmic and exponential means of a *and* b. *Inequalities (32) can also be deduced by taking* $(p_1, p_2) = (-n+1, -n)$ *and* $n = 0$ *in (9).*

4.2. Monotonicity of the sequence $\{I_{\psi_{q,n},w}(a,b)\}_{n\ge0}$

Using Lemma 2.3 (v) and Theorem 4.1 we easily prove the monotonicity of the sequence $\{I_{\psi_{q,n},w}(a,b)\}_{n\ge0}$, where $I_{\psi_{q,n},w}(a,b)$ is explicitly given by (27).

Theorem 4.5. *For* $a, b > 0$ *with* $a \ne b$, *the sequence* $\{I_{\psi_{q,n},w}(a,b)\}_{n\ge0}$ *is strictly decreasing with*

$$\lim_{n\to\infty} I_{\psi_{q,n},w}(a,b) = \min\{a,b\}.$$

Proof. The inequality $I_{\psi_{q,n},w}(a,b) > I_{\psi_{q,n+1},w}(a,b)$ is equivalent to

$$\psi_{q,n+1}\circ\psi_{q,n}^{-1}\left(\frac{\int_a^b w(x)\,\psi_{q,n}(x)\,dx}{\int_a^b w(x)\,dx}\right) < \frac{\int_a^b w(x)\,\psi_{q,n+1}(x)\,dx}{\int_a^b w(x)\,dx},$$

which follows from the convexity of the function $\psi_{q,n+1}\circ\psi_{q,n}^{-1}$ on $(0,\infty)$ (given by Lemma 2.3 (v)) and Jensen inequality.

The limit relation $\lim_{n\to\infty} I_{\psi_{q,n},w}(a,b) = \min\{a,b\}$ follows from (29), which completes the proof. □

4.3. Monotonicity of the function $A_{\psi_{q,n},w}(x)$

Theorem 4.6. *Let $q \in (0,1)$, $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ with $a \neq b$. Then the function*

$$x \mapsto A_{\psi_{q,n},w}(x) = \psi_{q,n}^{-1}\left(\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)dt}{\int_a^b w(t)\,dt}\right) - x \tag{33}$$

is strictly increasing from $(-\min\{a,b\},\infty)$ onto (β_1,β_2), where $\beta_1 = \min\{a,b\}$ and β_2 is given in (22).

Proof. Let

$$y_n = y_n(x) = \psi_{q,n}^{-1}\left(\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)dt}{\int_a^b w(t)\,dt}\right).$$

Then

$$\psi_{q,n}(y_n) = \frac{\int_a^b w(t)\,\psi_{q,n}(x+t)dt}{\int_a^b w(t)\,dt}.$$

Differentiation with respect to x yields

$$-\psi_{q,n+1}(y_n)\,y_n' = -\frac{\int_a^b w(t)\,\psi_{q,n+1}(x+t)dt}{\int_a^b w(t)\,dt} = -\psi_{q,n+1}(y_{n+1}),$$

which implies that

$$y_n' = \frac{\psi_{q,n+1}(y_{n+1})}{\psi_{q,n+1}(y_n)}.$$

Since $\psi_{q,n+1} \circ \psi_{q,n}^{-1}$ is convex on $(0,\infty)$ (given by Lemma 2.3 (v)), by Jensen inequality we have

$$\begin{aligned}\psi_{q,n+1}(y_n) &= \psi_{q,n+1}\left(\psi_{q,n}^{-1}\left(\frac{\int_a^b w(t)\,\psi_{q,n}(x+t)dt}{\int_a^b w(t)\,dt}\right)\right) \\ &< \frac{\int_a^b w(t)\,\psi_{q,n+1}(x+t)dt}{\int_a^b w(t)\,dt} = \psi_{q,n+1}(y_{n+1}),\end{aligned}$$

which leads to

$$y_n' = \frac{\psi_{q,n+1}(y_{n+1})}{\psi_{q,n+1}(y_n)} > \frac{\psi_{q,n+1}(y_{n+1})}{\psi_{q,n+1}(y_{n+1})} = 1.$$

It follows that

$$A'_{\psi_{q,n},w}(x) = y_n'(x) - 1 > 0,$$

which proves the required monotonicity.

It remains to compute the limit values of $A_{\psi_{q,n},w}(x)$ when $x \to -\min\{a,b\},\infty$. By the symmetry of a and b, we assume that $b > a$. Then $\beta_1 = \min\{a,b\} = a$. By (6) we see that $\psi_{q,n}(0^+) = \infty$ for $n \geq 0$, and so $\psi_{q,n}^{-1}(\infty) = 0$. Now, by (26) we see that

$$\lim_{x\to -a^+} \psi_{q,n}(y_n) = \lim_{x\to -a^+} \frac{\int_a^b w(t)\,\psi_{q,n}(x+t)\,dt}{\int_a^b w(t)\,dt} = \infty,$$

and then

$$\lim_{x\to -a^+} A_{\psi_{q,n},w}(x) = \lim_{x\to -a^+} y_n(x) + a = \psi_{q,n}^{-1}(\infty) + a = a.$$

By the double inequality (23) we see that

$$\psi_{q,n}(x+\beta_2) < \psi_{q,n}(y_n),$$

which, due to $\psi'_{q,n} = -\psi_{q,n+1} < 0$, implies that

$$y_n < x + \beta_2$$

for $x > -a$. On the other hand, by mean value theorem we have

$$\frac{\psi_{q,n}(y_n) - \psi_{q,n}(x+\beta_2)}{y_n - x - \beta_2} = \psi'_{q,n}(\xi) = -\psi_{q,n+1}(\xi),$$

where $y_n < \xi < x + \beta_2$. Noting that $\psi_{q,n}(y_n) - \psi_{q,n}(x+\beta_2) = h_{q,n}(x;\beta_2)$ and using the series representations (24) together with $d_1(\beta_2) = 0$ and (5), we derive that

$$\begin{aligned} 0 &< x + \beta_2 - y_n = \frac{\psi_{q,n}(y_n) - \psi_{q,n}(x+\beta_2)}{\psi_{q,n+1}(\xi)} < \frac{\psi_{q,n}(y_n) - \psi_{q,n}(x+\beta_2)}{\psi_{q,n+1}(x+\beta_2)} \\ &= \left((-\ln q)^{n+1} \sum_{k=1}^{\infty} \frac{d_k(\beta_2)}{1-q^k} k^n q^{kx}\right) \Bigg/ \left((-\ln q)^{n+2} \sum_{k=1}^{\infty} \frac{k^{n+1} q^{k\beta_2}}{1-q^k} q^{kx}\right) \to 0 \end{aligned}$$

as $x \to \infty$, which proves that $\lim_{x\to\infty}(y_n - x) = \beta_2$. This completes the proof. □

Taking $w(t) = 1$ in Theorem 4.6 and noting that

$$\beta_2 = \log_q\left(\frac{\int_a^b q^t dt}{b-a}\right) = \log_q L\left(q^a, q^b\right) = \beta_2^*,$$

where $L\left(q^a, q^b\right)$ is the logarithmic mean of q^a and q^b, the following corollary is immediate.

Corollary 4.7. *Let $q \in (0,1)$, $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}$ with $a \neq b$. Then the function*

$$x \mapsto A_{\psi_{q,n}}(x) = \psi_{q,n}^{-1}\left(\frac{\int_a^b \psi_{q,n}(x+t)dt}{b-a}\right) - x$$

is strictly increasing from $(-\min\{a,b\}, \infty)$ onto $\left(\min\{a,b\}, \beta_2^*\right)$, where $\beta_2^* = \log_q L\left(q^a, q^b\right)$. Consequently, the double inequality

$$\psi_{q,n}\left(x+\beta_2^*\right) < \frac{\int_a^b \psi_{q,n}(x+t)dt}{b-a} < \psi_{q,n}(x + \min\{a,b\}) \tag{34}$$

holds for $x > -\min\{a,b\}$ with the best constants $\min\{a,b\}$ and β_2^*. In particular, when $n = 0$, we have

$$\exp\left(\psi_q(x+\min\{a,b\})\right) < \left(\frac{\Gamma_q(x+b)}{\Gamma_q(x+a)}\right)^{1/(b-a)} < \exp\left(\psi_q\left(x+\beta_2^*\right)\right).$$

Remark 4.8. *Ismail and Muldoon [19] proved that, for* $0 < s < 1$, *the inequality*

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < e^{(1-s)\psi_q(x+(s+1)/2)}$$

holds for $x > 0$. *Alzer [2] proposed an open problem: let* $0 < q \neq 1$ *and* $v \in (0,1)$ *be real numbers. Determine the best possible values* $r_1(q,v)$ *and* $r_2(q,v)$ *such that the inequalities*

$$e^{\psi_q(x+r_1(q,v))} < \left(\frac{\Gamma_q(x+1)}{\Gamma_q(x+v)}\right)^{1/(1-v)} < e^{\psi_q(x+r_2(q,v))} \tag{35}$$

hold for all $x > 0$. *This was solved by Gao in [14], and was resolved by Alzahrani, Salem and El-Shahed in [1, Corollary 3.1]. Clearly, Alzer's problem can be easily attacked by Corollary 4.7.*

5. Conclusions

In this paper, we established the lower and upper bounds for the mean $I_{\psi_{q,n},w}(a,b)$ (Theorem 1), which is a generalization of Qi's inequalities (9); proved that the sequence $\left\{I_{\psi_{q,n},w}(a,b)\right\}_{n\geq 0}$ is decreasing with $\lim_{n\to\infty} I_{\psi_{q,n},w}(a,b) = \min\{a,b\}$ (Theorem 2); and presented that, for $a,b \in \mathbb{R}$ with $a \neq b$, the function $A_{\psi_{q,n},w}(x)$ defined by (33) is strictly increasing from $(-\min\{a,b\},\infty)$ onto $(\min\{a,b\},\beta_2)$, where β_2 is given by (22) (Theoem 3). It is emphasized that some known results are direct consequences of Theorem 3.

Moreover, we presented several interesting properties of the q-polygamma functions, and found the conditions for which the functions $\pm g_{q,n}(x;\alpha)$ and $\pm h_{q,n}(x;\beta)$ are completely monotonic on $(0,\infty)$ and $(-\min\{a,b\},\infty)$, respectively. As consequences, we established two double inequalities (19) and (23), which are new.

Finally, we propose two problems. The first is inspired by the inequality (14) and the monotonicity of $x\psi_{n+1}(x)/\psi_n(x)$ proved by Alzer [3, Lemma 2] (see also [4, Lemma 2.1], [45, Corollary 2], [40]), which is stated as follows.

Problem 5.1. *Let* $q \in (0,1)$ *and* $n \in \mathbb{N}$. *Prove that the function*

$$x \mapsto \frac{q^x - 1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}$$

is decreasing on $(0,\infty)$.

The second is motivated by Lemma 3.4.

Problem 5.2. *Let* $q \in (0,1)$, $n \in \mathbb{N}_0$, $a,b,c,d \in \mathbb{R}$ *with* $(a-b)(c-d) \neq 0$ *and* $\rho = \min\{a,b,c,d\}$. *What are the conditions such that the function*

$$x \mapsto \frac{\int_a^b w(t)\psi_{q,n}(x+t)\,dt}{\int_a^b w(t)\,dt} - \frac{\int_c^d w(t)\psi_{q,n}(x+t)\,dt}{\int_c^d w(t)\,dt}$$

is completely monotonic on $(-\rho,\infty)$?

6. Appendix: $\lim_{q\to 1}\psi_q^{(n)}(x) = \psi^{(n)}(x)$

Proposition 6.1. *Let* $q \in \mathbb{R}$ *with* $q \neq 1$ *and* $n \in \mathbb{N}$. *For* $x > 0$, *we have*

$$\lim_{q\to 1}\psi_q^{(n)}(x) = \psi^{(n)}(x).$$

Proof. First assume that $q \in (0,1)$. By (3) we have, for $n \in \mathbb{N}$,

$$\psi_q^{(n)}(x) = \sum_{k=0}^{\infty} \frac{d^n}{dx^n}\left(\frac{\ln q}{1-q^{k+x}} - \ln q\right).$$

It thus suffices to prove that

$$\lim_{q\to 1^-} \frac{d^n}{dx^n}\left(\frac{\ln q}{1-q^{k+x}} - \ln q\right) = (-1)^{n+1}\frac{n!}{(k+x)^{n+1}}.$$

To this end, we use the formula for the n-th derivative of a composite function:

$$\frac{d^n f(g(x))}{dx^n} = \sum_{j=1}^{n} \frac{1}{j!} f^{(j)}(g(x))\, h_j(x),$$

where

$$h_j(x) = \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} g^i(x) \frac{d^n}{dx^n} g^{j-i}(x),$$

see [15, No. 0.430.1]. Let $f(y) = 1/(1-y)$ and $g(x) = q^{k+x}$. Then

$$f^{(n)}(y) = \frac{n!}{(1-y)^{n+1}}, \quad \frac{d^n}{dx^n} g^{j-i}(x) = \frac{d^n}{dx^n} q^{(j-i)(k+x)} = q^{(j-i)(k+x)}(j-i)^n \ln^n q,$$

$$h_j(x) = \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} q^{i(k+x)} q^{(j-i)(k+x)} (j-i)^n \ln^n q = (\ln^n q)\, q^{j(k+x)} j! S(n,j),$$

where

$$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j} (-1)^{j-i}\binom{j}{i} i^n$$

is the Stirling number of the second kind (see [10, p. 624]), and then,

$$\begin{aligned} \frac{d^n}{dx^n}\frac{\ln q}{1-q^{k+x}} &= \sum_{j=1}^{n}\left(\frac{1}{j!}\frac{j!\,(\ln q)^{n+1}}{(1-q^{k+x})^{j+1}} q^{j(k+x)} j! S(n,j)\right) \\ &= \left(\frac{\ln q}{1-q^{k+x}}\right)^{n+1} \sum_{j=1}^{n}\left[\left(1-q^{k+x}\right)^{n-j} q^{j(k+x)} j! S(n,j)\right] \\ :&= \left(\frac{\ln q}{1-q^{k+x}}\right)^{n+1} \sum_{j=1}^{n} F_{j,n}(k+x;q). \end{aligned}$$

Clearly, for $1 \le j \le n-1$, $F_{j,n}(k+x;q)$ tends to zero when $q \to 1^-$, and therefore,

$$\begin{aligned} \lim_{q\to 1^-} \frac{d^n}{dx^n}\left(\frac{\ln q}{1-q^{k+x}} - \ln q\right) &= \lim_{q\to 1^-}\left(\frac{\ln q}{1-q^{k+x}}\right)^{n+1} F_{n,n}(k+x;q) \\ &= \frac{(-1)^{n+1}}{(k+x)^{n+1}} n! S(n,n) = \frac{(-1)^{n+1} n!}{(k+x)^{n+1}}, \end{aligned}$$

where we have used the identity $S(n,n) = 1$ (see [10, p. 624]).

For $q > 1$, by (2) we have

$$\psi_q^{(n)}(x) = \left(\ln \Gamma_q(x)\right)^{(n+1)} = \frac{1}{2}\left((x-1)(x-2)\right)^{(n+1)} \ln q + \psi_{1/q}^{(n)}(x) \to \psi^{(n)}(x),$$

as $q \to 1^+$, thereby completing the proof. □

7. Ethics declarations

Conflict of interest The authors declare that they have no conflict of interest.
Ethical Approval The authors have approved the manuscript and the submission to this journal.
Availability of data and material Not applicable.

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