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# Complete convergence and complete $q$-th moment convergence for weighted sums of random vectors in Hilbert spaces

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**Abstract.** We investigate the complete convergence and complete $q$-th moment convergence for weighted sums of coordinatewise negatively associated random vectors taking values in a real separable Hilbert space without assumption of identical distribution. The results obtained in the paper generalize and improve some known ones.

## 1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [10] as follows: a sequence $\{X_n, n \geqslant 1\}$ of random variables *converges completely* to a constant $C$ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - C| > \varepsilon) < \infty.$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to C$ almost surely. The converse is true if $\{X_n, n \geqslant 1\}$ is independent. Hsu and Robbins proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of the variables, provided their variance is finite. The necessity was proved by Erdös [6, 7]. The result of Hsu, Robbins and Erdös is a fundamental theorem in probability theory and was later generalized and extended during a process which led to the now classical paper by Baum and Katz [3].

The concept of complete moment convergence was introduced by Chow [5] as follows: let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n \, \mathbb{E}\Big((b_n^{-1}|X_n| - \varepsilon)^+\Big)^q < \infty \quad \text{for every } \ \varepsilon > 0,$$

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where $x^+ = \max\{0, x\}$, then $\{X_n, n \geqslant 1\}$ is said to be *complete q-th moment convergence*. As we know, the complete $q$-th moment convergence implies the complete convergence. Moreover, the complete $q$-th moment convergence can describe the convergence rate of a sequence of random variables more exactly than the complete convergence.

The concept of negative association for random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [14]. A finite family $\{Y_i, 1 \leqslant i \leqslant n\}$ of random variables is said to be *negatively associated* (NA) if for any disjoint subsets $A$, $B$ of $\{1, 2, ..., n\}$ and any real coordinatewise nondecreasing functions $f$ on $\mathbb{R}^{|A|}$, $g$ on $\mathbb{R}^{|B|}$,

$$\mathrm{Cov}\Big(f(Y_i, i \in A), g(Y_j, j \in B)\Big) \leqslant 0$$

whenever the covariance exists, where $|A|$ denotes the cardinality of $A$. An infinite family of random variables is NA if every finite subfamily is NA.

Afterwards, the concept of negative association was extended to finite dimensional random vectors and to Hilbert space valued random vectors (for details see Zhang [23], Ko et al. [17]). Let $H$ be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$, let $\{e_j, j \geqslant 1\}$ be an orthonormal basis in $H$, let $X$ be an $H$-valued random vector, and $\langle X, e_j\rangle$ will be denoted by $X^{(j)}$. In [13], the authors introduced the concept of coordinatewise negative association for $H$-valued random vectors which is more general than the concept of negative association of Ko et al. [17].

**Definition 1.1 ([13], Definition 1.3).** A sequence $\{X_n, n \geqslant 1\}$ of $H$-valued random vectors is said to be *coordinatewise negatively associated* (CNA) if for each $j \geqslant 1$, the sequence $\{X_n^{(j)}, n \geqslant 1\}$ of random variables is NA.

Our results are related to the following two theorems. The first one is a part of the main results of Baum and Katz (for details see [3, Theorem 1 and Theorem 2]).

**Theorem 1.2.** *Let $\alpha$ be a real number $(1/2 < \alpha \leqslant 1)$, let $\beta$ be a fixed number that can only take the value 0 or 1, and let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. mean zero random variables with the n-th partial sum $S_n$ $(n \geqslant 1)$. Then the following statements are equivalent:*

$$\mathbb{E}(|X_1|^{1/\alpha} (\log_+ |X_1|)^\beta) < \infty; \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}(|S_n| > \varepsilon n^\alpha) < \infty \text{ for every } \varepsilon > 0, \tag{2}$$

*here and afterwards the logarithms are to the base 2, $\log_+ x = \log(\max\{2, x\})$.*

Note that the equivalence ((1)⇔(2)) was proved by Spitzer [19] for the case of $\beta = 0$ and $\alpha = 1$. Obviously, the condition

$$\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}(\max_{1\leqslant k\leqslant n} |S_k| > \varepsilon n^\alpha) < \infty \text{ for every } \varepsilon > 0 \tag{3}$$

implies (2). The criterion (3) is more interesting than (2). As in Bai et al. [2], the criterion (3) implies $S_n/n^\alpha \to 0$ a.s. In [11], the author extended Theorem 1.2 to random vectors taking values in a real separable Hilbert space as follows.

**Theorem 1.3 ([11], Theorems 3.1 and 3.3).** *Let $\alpha$ be a positive real number $(1/2 < \alpha < 1)$, let $\beta$ be a fixed number that can only take the value 0 or 1, and let $\{X_n, n \geqslant 1\}$ be a sequence of H-valued CNA mean zero random vectors with the n-th partial sum $S_n$ $(n \geqslant 1)$. Suppose that $\{X_n, n \geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector X. Then the condition*

$$\sum_{j=1}^{\infty} \mathbb{E}\Big(|X^{(j)}|^{1/\alpha} (\log_+ |X^{(j)}|)^\beta\Big) < \infty \tag{4}$$

*implies*

$$\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \|S_k\| > \varepsilon n^\alpha\Big) < \infty \text{ for every } \varepsilon > 0. \tag{5}$$

In the present paper, we study Theorem 1.3 for weighted sums of CNA random vectors in Hilbert spaces, where $1/2 < \alpha \leqslant 1$ and $\beta$ is any nonnegative real number. We also investigate the complete $q$-th moment convergence for weighted sums of CNA random vectors without assumption of identical distribution.

Let us note that our results still hold if the function $x \mapsto (\log_+(x))^\beta$ is replaced by a slowly varying function at infinity defined on $[0, \infty)$. For more details about some properties and applications of slowly varying functions, the reader may refer to [8, 18, 20, 22].

Throughout this paper, the symbol $C$ will denote a generic positive constant which is not necessarily the same one in each appearance. The logarithms are to the base 2, for $x \in \mathbb{R}$, $\log(\max\{2; x\})$ will be denoted by $\log_+ x$.

Inspired by the work of Gut [9], we introduce two concepts of stochastic domination for random vectors in Hilbert spaces. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of $H$-valued random vectors. We consider the following inequalities

$$\sup_{n\geqslant 1} \sum_{j=1}^{\infty} \mathbb{P}(|X_n^{(j)}| > t) \leqslant C \sum_{j=1}^{\infty} \mathbb{P}(|X^{(j)}| > t); \tag{6}$$

$$\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{\infty} \mathbb{P}(|X_k^{(j)}| > t) \leqslant C \sum_{j=1}^{\infty} \mathbb{P}(|X^{(j)}| > t). \tag{7}$$

If there exists a positive constant $C$ such that (6) (respectively, (7)) is satisfied for every $t \geqslant 0$, then the sequence $\{X_n, n \geqslant 1\}$ is said to be *stochastically dominated* (respectively, *weakly mean dominated*) by $X$. It is well known that (6) and (7) are, of course, automatic with $X = X_1$ and $C = 1$ if $\{X_n, n \geqslant 1\}$ is a sequence of identically distributed random vectors. If $X$ dominates the sequence $\{X_n, n \geqslant 1\}$ in the stochastically dominated sense, then it also dominates the sequence $\{X_n, n \geqslant 1\}$ in the weakly mean dominated sense. For more details about stochastically dominating random variables, one can refer to Gut [9] and Thanh [21].

## 2. Preliminary lemmas

In this section, we give the following lemmas which will be used to prove our main results.

**Lemma 2.1 ([13], Lemma 1.7).** *Let $\{X_n, n \geqslant 1\}$ be a sequence of $H$-valued CNA random vectors with $\mathbb{E}X_n = 0$ and $\mathbb{E}\|X_n\|^2 < \infty$ for all $n \geqslant 1$. Then, we have*

$$\mathbb{E}\Big(\max_{1\leqslant k\leqslant n} \Big\|\sum_{l=1}^{k} X_l\Big\|^2\Big) \leqslant C \sum_{k=1}^{n} \mathbb{E}\|X_k\|^2 \quad \text{for all} \quad n \geqslant 1.$$

The techniques used in the proofs of the following two lemmas can be founded in Gut [9, Lemma 2.1].

**Lemma 2.2.** *Let $p$ be a positive real number, and let $\{X_n, n \geqslant 1\}$ be a sequence of random vectors which are stochastically dominated by a random vector $X$. Then for all $t \in \mathbb{R}$,*

(i) $\sup_{n\geqslant 1} \sum_{j=1}^{\infty} \mathbb{E}\Big(|X_n^{(j)}|^p \,\mathbb{I}(|X_n^{(j)}| > t)\Big) \leqslant C \sum_{j=1}^{\infty} \mathbb{E}\Big(|X^{(j)}|^p \,\mathbb{I}(|X^{(j)}| > t)\Big)$;

(ii) $\sup_{n\geqslant 1} \sum_{j=1}^{\infty} \mathbb{E}\Big(|X_n^{(j)}|^p \,\mathbb{I}(|X_n^{(j)}| \leqslant t)\Big) \leqslant C \sum_{j=1}^{\infty} \Big(\mathbb{E}(|X^{(j)}|^p \,\mathbb{I}(|X^{(j)}| \leqslant t)) + t^p \,\mathbb{P}(|X^{(j)}| > t)\Big)$.

**Lemma 2.3.** *Let $p$ be a positive real number, and let $\{X_n, n \geqslant 1\}$ be a sequence of random vectors which are weakly mean dominated by a random vector $X$. Then for all $t \in \mathbb{R}$,*

(i) $\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{\infty} \mathbb{E}\Big(|X_k^{(j)}|^p \, \mathbb{I}(|X_k^{(j)}| > t)\Big) \leqslant C \sum_{j=1}^{\infty} \mathbb{E}\Big(|X^{(j)}|^p \, \mathbb{I}(|X^{(j)}| > t)\Big);$

(ii) $\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{\infty} \mathbb{E}\Big(|X_k^{(j)}|^p \, \mathbb{I}(|X_k^{(j)}| \leqslant t)\Big) \leqslant C \sum_{j=1}^{\infty} \Big(\mathbb{E}(|X^{(j)}|^p \, \mathbb{I}(|X^{(j)}| \leqslant t)) + t^p \, \mathbb{P}(|X^{(j)}| > t)\Big).$

The proof of the next lemma is, except for details, the same as the proof of Lemma 2.3 in [12] and will be omitted.

**Lemma 2.4.** *Let $\alpha, \beta, \gamma$ be real numbers ($\alpha > 0, \beta \geqslant 0, \gamma \geqslant 0$), and let $X$ be a random vector satisfying (4). Then*

(i) $\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha\gamma}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| > n^\alpha)\Big) < \infty \quad \text{if } \alpha\gamma < 1;$

(ii) $\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha\gamma}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant n^\alpha)\Big) < \infty \quad \text{if } \alpha\gamma > 1.$

**Lemma 2.5.** *Let $\alpha, \beta, \gamma, q$ be real numbers ($q > 0, 0 < \alpha < 1/q, \beta \geqslant 0, \gamma \geqslant 0$), and let $X$ be a random vector satisfying (4). Then*

(i) $\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q}} \int_{n^{\alpha q}}^{\infty} \frac{1}{t^{\gamma/q}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| > t^{1/q})\Big) dt < \infty \quad \text{if } \alpha\gamma < 1;$

(ii) $\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q}} \int_{n^{\alpha q}}^{\infty} \frac{1}{t^{\gamma/q}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant t^{1/q})\Big) dt < \infty \quad \text{if } \alpha\gamma > 1.$

*Proof.* The proofs of the above statements are quite similar, so we will only give that of statement (ii). It is well known that $g(x) = x^{-\alpha q} (\log_+ x)^\beta$ is a regularly varying function of index $\rho = -\alpha q > -1$. Then by Remark A.5 in [15] and Lemma 2.4, we obtain the following estimations

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q}} \int_{n^{\alpha q}}^{\infty} \frac{1}{t^{\gamma/q}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant t^{1/q})\Big) dt \\
&\leqslant \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q}} \sum_{k=n}^{\infty} \int_{k^{\alpha q}}^{(k+1)^{\alpha q}} \frac{1}{k^{\alpha\gamma}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant (k+1)^\alpha)\Big) dt \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha\gamma - \alpha q + 1}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant (k+1)^\alpha)\Big) \sum_{n=1}^{k} \frac{(\log_+ n)^\beta}{n^{\alpha q}} \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\log_+ k)^\beta}{k^{\alpha\gamma}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant (k+1)^\alpha)\Big) \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{k=2}^{\infty} \frac{\big(\log_+ k)\big)^\beta}{k^{\alpha\gamma}} \mathbb{E}\Big(|X^{(j)}|^\gamma \, \mathbb{I}(|X^{(j)}| \leqslant k^\alpha)\Big) < \infty.
\end{aligned}
$$

Thus, the proof is completed. □

**Lemma 2.6 ([11], Lemma 2.4).** *Let $\alpha$ be a positive real number, and let $X$ be an $H$-valued random vector such that $\sum_{j=1}^{\infty}\mathbb{E}|X^{(j)}|^{1/\alpha}<\infty$. Then*

$$\sum_{j=1}^{\infty}\mathbb{E}\Big(|X^{(j)}|^{1/\alpha}I(|X^{(j)}|^{1/\alpha}>n)\Big)\to 0 \quad as \ \ n\to\infty.$$

## 3. Main Results

With the preliminaries accounted for, the main results may now be established. In the following theorem, we obtain the complete convergence and complete $q$-th moment convergence for weighted sums of CNA random vectors taking values in a real separable Hilbert space without assumption of identical distribution.

**Theorem 3.1.** *Let $\alpha,\beta,q$ be real numbers $(1/2<\alpha<1, 0<q<1/\alpha, \beta\geqslant 0)$, let $\{a_{ni}, n\geqslant 1, 1\leqslant i\leqslant n\}$ be an array of constants such that*

$$\sum_{i=1}^{n}a_{ni}^2\leqslant C n^{\delta} \quad \text{for all } n\geqslant 1, \ \text{for some } \delta\geqslant 1,$$

*and let $\{X_n, n\geqslant 1\}$ be a sequence of $H$-valued CNA mean zero random vectors. Suppose that $\{X_n, n\geqslant 1\}$ is stochastically dominated by a random vector $X$. Then the condition* (4) *implies*

$$\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{\delta}}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>\varepsilon n^{\alpha}\Big)<\infty \ \text{for every } \ \varepsilon>0; \tag{8}$$

$$\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{\alpha q+\delta}}\mathbb{E}\Big(\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|-\varepsilon n^{\alpha}\Big)^{+}\Big)^{q}<\infty \ \text{for every } \ \varepsilon>0. \tag{9}$$

*Proof.* Since $a_{ni}=a_{ni}^{+}-a_{ni}^{-}$, without loss of generality, we can assume that $a_{ni}\geqslant 0$ for all $n\geqslant 1, 1\leqslant i\leqslant n$. For each $n\geqslant 1, t\in\mathbb{R}$, set

$$Y_n^{(j)}(t)=X_n^{(j)}\,\mathbb{I}(|X_n^{(j)}|\leqslant t)+t\,\mathbb{I}(X_n^{(j)}>t)-t\,\mathbb{I}(X_n^{(j)}<-t);$$

$$Z_n^{(j)}(t)=X_n^{(j)}-Y_n^{(j)}(t), \ \ j\geqslant 1; \quad Z_n(t)=\sum_{j=1}^{\infty}Z_n^{(j)}(t)\,e_j; \ \ Y_n(t)=\sum_{j=1}^{\infty}Y_n^{(j)}(t)\,e_j.$$

Then for every $\varepsilon>0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{\delta}}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>\varepsilon n^{\alpha}\Big)\\
&\leqslant\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{\delta}}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Y_i(n^{\alpha})-\mathbb{E}(a_{ni}Y_i(n^{\alpha}))\Big)\Big\|>\varepsilon n^{\alpha}/2\Big)\\
&+\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{\delta}}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Z_i(n^{\alpha})-\mathbb{E}(a_{ni}Z_i(n^{\alpha}))\Big)\Big\|>\varepsilon n^{\alpha}/2\Big)\\
&=I_1+I_2.
\end{aligned}$$

It is well known that $\{Y_k^{(j)}(n^{\alpha}), k\geqslant 1\}$ is NA for all $j\geqslant 1$ and $n\geqslant 1$, and so $\{a_{nk}Y_k(n^{\alpha}), k\geqslant 1\}$ is CNA for all $n\geqslant 1$. By the Markov inequality, Lemmas 2.1, 2.2 and 2.4, we have

$$I_1\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^{\beta}}{n^{2\alpha+\delta}}\mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Y_i(n^{\alpha})-\mathbb{E}(a_{ni}Y_i(n^{\alpha}))\Big)\Big\|\Big)^2$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{2\alpha+\delta}}\ \sum_{k=1}^{n} a_{nk}^2\,\mathbb{E}\|Y_k(n^\alpha)-\mathbb{E}Y_k(n^\alpha)\|^2\\
&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{2\alpha+\delta}}\ \sum_{k=1}^{n} a_{nk}^2\,\mathbb{E}\|Y_k(n^\alpha)\|^2\\
&= C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\delta}}\sum_{k=1}^{n} a_{nk}^2\,\mathbb{P}(|X_k^{(j)}|>n^\alpha)\\
&+ C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{2\alpha+\delta}}\sum_{k=1}^{n} a_{nk}^2\,\mathbb{E}\Big((X_k^{(j)})^2 I(|X_k^{(j)}|\leqslant n^\alpha)\Big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(\log_+ n)^\beta\,\mathbb{P}(|X^{(j)}|>n^\alpha)\\
&+ C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{2\alpha}}\,\mathbb{E}\Big((X^{(j)})^2 I(|X^{(j)}|\leqslant n^\alpha)\Big)<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha+\delta}}\ \mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Z_i(n^\alpha)-\mathbb{E}(a_{ni}Z_i(n^\alpha))\Big)\Big\|\Big)\\
&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha+\delta}}\sum_{k=1}^{n}\mathbb{E}\|a_{nk}Z_k(n^\alpha)\|\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha+\delta}}\sum_{k=1}^{n}|a_{nk}|\,\mathbb{E}|X_k^{(j)}\,\mathbb{I}(|X_k^{(j)}|>n^\alpha)|\\
&+ C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\delta}}\sum_{k=1}^{n}|a_{nk}|\,\mathbb{P}(|X_k^{(j)}|>n^\alpha)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha+(\delta-1)/2}}\,\mathbb{E}|X^{(j)}\,\mathbb{I}(|X^{(j)}|>n^\alpha)|\\
&+ C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{(\delta-1)/2}}\,\mathbb{P}(|X^{(j)}|>n^\alpha)<\infty.
\end{aligned}
$$

Therefore (8) holds.

On the other hand, it follows from (8) that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\,\mathbb{E}\Big(\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|-\varepsilon n^\alpha\Big)^+\Big)^q\\
&\leqslant \sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_0^{n^{\alpha q}}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>\varepsilon n^\alpha\Big)dt\\
&+ \sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>t^{1/q}\Big)dt\\
&\leqslant \sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\delta}}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>\varepsilon n^\alpha\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Y_i(t^{1/q})-\mathbb{E}(a_{ni}Y_i(t^{1/q}))\Big)\Big\|>t^{1/q}/2\Big)\,dt\\
&+\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}\Big((X_i-Y_i(t^{1/q}))-\mathbb{E}(X_i-Y_i(t^{1/q}))\Big)\Big\|>t^{1/q}/2\Big)\,dt\\
&=C+I_3+I_4.
\end{aligned}
$$

By the Markov inequality, Lemmas 2.1, 2.2 and 2.5, we have

$$
\begin{aligned}
I_3&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{2/q}}\,\mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}\Big(a_{ni}Y_i(t^{1/q})-\mathbb{E}(a_{ni}Y_i(t^{1/q}))\Big)\Big\|\Big)^2\,dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{2/q}}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{E}\|Y_k(t^{1/q})-\mathbb{E}Y_k(t^{1/q})\|^2\,dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{2/q}}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{E}\|Y_k(t^{1/q})\|^2\,dt\\
&=C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{2/q}}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{E}\Big(X_k^{(j)}\,\mathbb{I}(|X_k^{(j)}|\leqslant t^{1/q})\Big)^2\,dt\\
&+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{P}(|X_k^{(j)}|>t^{1/q})\,dt\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{2/q}}\,\mathbb{E}\Big(X^{(j)}\,\mathbb{I}(|X^{(j)}|\leqslant t^{1/q})\Big)^2\,dt\\
&+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q}}\int_{n^{\alpha q}}^{\infty}\mathbb{P}(|X^{(j)}|>t^{1/q})\,dt<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4&\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{1/q}}\sum_{k=1}^{n}|a_{nk}|\,\mathbb{E}\|X_k-Y_k(t^{1/q})\|\,dt\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{1/q}}\sum_{k=1}^{n}|a_{nk}|\,\mathbb{E}|X_k^{(j)}\,\mathbb{I}(|X_k^{(j)}|>t^{1/q})|\,dt\\
&+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}}\int_{n^{\alpha q}}^{\infty}\sum_{k=1}^{n}|a_{nk}|\,\mathbb{P}(|X_n^{(j)}|>t^{1/q})\,dt\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+(\delta-1)/2}}\int_{n^{\alpha q}}^{\infty}\frac{1}{t^{1/q}}\,\mathbb{E}|X^{(j)}\,\mathbb{I}(|X^{(j)}|>t^{1/q})|\,dt\\
&+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+(\delta-1)/2}}\int_{n^{\alpha q}}^{\infty}\mathbb{P}(|X^{(j)}|>t^{1/q})\,dt<\infty.
\end{aligned}
$$

From the above arguments, we obtain (9). □

**Remark 3.2.** Under the assumptions of Theorem 3.1, (4) implies (8). However, the reverse is not true in general. In the special case when $\beta=0$, $\delta=1$ and $a_{ni}=1$ ($n\geqslant 1, 1\leqslant i\leqslant n$), it was shown in [11, Example 3.2] that (8) is strictly weaker than (4).

**Remark 3.3.** Suppose that $\alpha, \beta, \delta, q$ be nonnegative real numbers ($q > 0$), $\{a_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ be an array of constants, $\{X_n, n \geqslant 1\}$ be a sequence of $H$-valued random vectors. Then for every $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > \varepsilon n^\alpha\Big) \\
&= \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \mathbb{P}\Big(\Big(\frac{1}{n^\alpha}\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| - \varepsilon/2\Big)^+ > \varepsilon/2\Big) \\
&\leqslant C\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}} \mathbb{E}\Big(\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| - \varepsilon n^\alpha/2\Big)^+\Big)^q.
\end{aligned}$$

It is sufficient to show that (9) implies (8). Furthermore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^{\alpha q+\delta}} \mathbb{E}\Big(\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| - \varepsilon n^\alpha\Big)^+\Big)^q \\
&= \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \int_0^\infty \mathbb{P}\Big(\Big(\frac{1}{n^\alpha}\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| - \varepsilon\Big)^+ > x^{1/q}\Big)dx \\
&\geqslant \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \int_{\varepsilon^q}^\infty \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > (\varepsilon + x^{1/q})n^\alpha\Big)dx \\
&= \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \int_{(2\varepsilon)^q}^\infty \Big(\frac{t^{1/q}-\varepsilon}{t^{1/q}}\Big)^{q-1} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > t^{1/q}n^\alpha\Big)dt \\
&\geqslant C\int_{(2\varepsilon)^q}^\infty \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > t^{1/q}n^\alpha\Big)dt.
\end{aligned}$$

The above arguments ensure that (9) implies the following form of complete integral convergence (see also Proposition 1.1 of Chen and Wang [4])

$$\int_\varepsilon^\infty \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^\delta} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > t^{1/q}n^\alpha\Big)dt < \infty \text{ for every } \varepsilon > 0.$$

In the following theorem, the complete convergence in Theorem 3.1 will be established for the case $\alpha = 1$. It is interesting to observe that we cannot prove Theorem 3.4 by using the same method as in the proof of Theorem 3.1, and vice versa.

**Theorem 3.4.** *Let $\beta$ be a nonnegative real number, let $\{a_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ be an array of constants such that*

$$\sum_{i=1}^{n} a_{ni}^2 \leqslant Cn \quad \textit{for all } n \geqslant 1, \tag{10}$$

*and let $\{X_n, n \geqslant 1\}$ be a sequence of $H$-valued CNA mean zero random vectors. Suppose that $\{X_n, n \geqslant 1\}$ is stochastically dominated by a random vector $X$. Then the condition*

$$\sum_{j=1}^{\infty} \mathbb{E}\Big(|X^{(j)}|(\log_+|X^{(j)}|)^\beta\Big) < \infty \tag{11}$$

*implies*

$$\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} a_{ni}X_i\Big\| > \varepsilon n\Big) < \infty \quad \textit{for every } \varepsilon > 0. \tag{12}$$

*Proof.* Without loss of generality, assume that $a_{ni} \geqslant 0$ for all $n \geqslant 1, 1 \leqslant i \leqslant n$. For each $n \geqslant 1, j \geqslant 1, t \in \mathbb{R}$, we define $Y_n^{(j)}(t)$ and $Y_n(t)$ as in the proof of Theorem 3.1. Then for every $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} a_{ni} X_i \Big\| > \varepsilon n\Big) \\
&\leqslant \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \sum_{j=1}^{\infty} \sum_{k=1}^{n} \mathbb{P}(|X_k^{(j)}| > n) \\
&+ \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} a_{ni} Y_i(n) \Big\| > \varepsilon n\Big) \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (\log_+ n)^\beta \, \mathbb{P}(|X^{(j)}| > n) \\
&+ \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\frac{1}{n}\max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} \mathbb{E}(a_{ni} Y_i(n)) \Big\| > \varepsilon/2\Big) \\
&+ \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} \Big(a_{ni} Y_i(n) - \mathbb{E}(a_{ni} Y_i(n))\Big) \Big\| > \varepsilon n/2\Big) \\
&= C + J_1 + J_2.
\end{aligned}$$

We now prove that $J_{1n} = o(1)$, where

$$J_{1n} = \frac{1}{n} \max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} \mathbb{E}(a_{ni} Y_i(n)) \Big\|.$$

Indeed, by the mean zero assumption, Lemma 2.2 and Lemma 2.6,

$$\begin{aligned}
J_{1n} &\leqslant \frac{1}{n} \max_{1\leqslant k\leqslant n} \sum_{j=1}^{\infty} \Big| \sum_{i=1}^{k} a_{ni} \, \mathbb{E}\Big(X_i^{(j)} I(|X_i^{(j)}| \leqslant n)\Big) \Big| + \sum_{j=1}^{\infty} \sum_{k=1}^{n} |a_{nk}| \, \mathbb{P}\Big(|X_k^{(j)}| > n\Big) \\
&\leqslant \frac{C}{n} \sum_{j=1}^{\infty} \sum_{k=1}^{n} |a_{nk}| \, \mathbb{E}\Big(|X_k^{(j)}| I(|X_k^{(j)}| > n)\Big) \\
&\leqslant C \sum_{j=1}^{\infty} \mathbb{E}\Big(|X^{(j)}| \, I(|X^{(j)}| > n)\Big) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Then there exists a positive integer $n_0$ such that $J_{1n} \leqslant \varepsilon/2$ for all $n > n_0$, and so

$$\begin{aligned}
J_1 &= \sum_{n=1}^{n_0} \frac{(\log_+ n)^\beta}{n} \mathbb{P}(J_{1n} > \varepsilon/2) + \sum_{n=n_0+1}^{\infty} \frac{(\log_+ n)^\beta}{n} \mathbb{P}(J_{1n} > \varepsilon/2) \\
&= \sum_{n=1}^{n_0} \frac{(\log_+ n)^\beta}{n} \mathbb{P}(J_{1n} > \varepsilon/2) < \infty.
\end{aligned}$$

We next show that $J_2 < \infty$. By the Markov inequality, Lemmas 2.1, 2.2 and 2.4, we have

$$J_2 \leqslant C \sum_{n=1}^{\infty} \frac{(\log_+ n)^\beta}{n^3} \mathbb{E}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{i=1}^{k} \Big(a_{ni} Y_i(n) - \mathbb{E}(a_{ni} Y_i(n))\Big) \Big\|\Big)^2$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^3}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{E}\|Y_k(n)-\mathbb{E}Y_k(n)\|^2$$
$$\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^3}\sum_{k=1}^{n}a_{nk}^2\,\mathbb{E}\Big(Y_k^{(j)}(n)\Big)^2$$
$$\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(\log_+ n)^\beta\,\mathbb{P}(|X^{(j)}|>n)$$
$$+\,C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^2}\,\mathbb{E}\Big((X^{(j)})^2 I(|X^{(j)}|\leqslant n)\Big)<\infty.$$

Therefore (12) holds. □

The following corollary follows immediately from Theorem 3.1 and Theorem 3.4.

**Corollary 3.5.** *Let $\alpha,\beta$ be real numbers $(1/2<\alpha\leqslant 1,\beta\geqslant 0)$, let $\{a_{ni},n\geqslant 1,1\leqslant i\leqslant n\}$ be an array of constants satisfying* (10)*, and let $\{X_n,n\geqslant 1\}$ be a sequence of H-valued CNA mean zero random vectors. Suppose that $\{X_n,n\geqslant 1\}$ is stochastically dominated by a random vector X. Then the condition* (4) *implies*

$$\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k}a_{ni}X_i\Big\|>\varepsilon n^\alpha\Big)<\infty \text{ for every } \varepsilon>0.$$

In the following theorem, we obtain the complete convergence and complete $q$-th moment convergence for CNA random vectors which are weakly mean dominated by a random vector $X$. The first assertion of Theorem 3.6 is a generalization and improvement of Theorem 1.3. In the special case when $q=1$, $\beta$ is a fixed number that can only take the value 0 or 1, and $\{X_n,n\geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector $X$, the second assertion was proved by in [16, Theorem 3.1 and Theorem 3.3]. Ko [16, Theorem 3.3] also proved the second assertion of Theorem 3.6 for the case $q=1,\alpha=1,\beta=1$, and $\{X_n,n\geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector $X$. The key tool for proving this result is Lemma 2.5 in [16]. Let us note that there is a misprint in the proof of [16, Lemma 2.5]. For example, the estimation $\sum_{n=1}^{m}n^{-\alpha}\log n\leqslant C m^{1-\alpha}\log m$ $(m\geqslant 1)$ in [16, equation (2.11)] fails for $\alpha=1$.

**Theorem 3.6.** *Let $\alpha,\beta,q$ be real numbers $(\alpha>1/2, 0<q<1/\alpha,\beta\geqslant 0)$, and let $\{X_n,n\geqslant 1\}$ be a sequence of H-valued CNA mean zero random vectors with the n-th partial sum $S_n$ $(n\geqslant 1)$. Suppose that $\{X_n,n\geqslant 1\}$ is weakly mean dominated by a random vector X.*

(i) *If $\alpha\leqslant 1$, then the condition* (4) *implies* (5);

(ii) *If $\alpha<1$, then the condition* (4) *implies*

$$\sum_{n=1}^{\infty}\frac{(\log_+ n)^\beta}{n^{\alpha q+1}}\,\mathbb{E}\Big(\Big(\max_{1\leqslant k\leqslant n}\|S_k\|-\varepsilon n^\alpha\Big)^+\Big)^q<\infty \text{ for every } \varepsilon>0.$$

*Proof.* We can prove the above theorem by using Lemma 2.3 and the same techniques as in the proof of Theorem 3.4 and the second assertion of Theorem 3.1. □

To end the paper, we present an example to show that the conditions of the theorems above are satisfied.

**Example 3.7.** Let $\alpha,\beta,\gamma$ be real numbers $(1/2<\alpha\leqslant 1,\beta\geqslant 0,\gamma>\alpha)$, and let $a_{ni}=1$ for all $n\geqslant 1, 1\leqslant i\leqslant n$. We consider the space $\ell_2$ consisting of square summable real sequences $x=\{x_k,k\geqslant 1\}$ with norm $\|x\|=\Big(\sum_{k=1}^{\infty}x_k^2\Big)^{1/2}$. Let $\{X_n,n\geqslant 1\}$ be a sequence of $\ell_2$-valued i.i.d. random vectors with $\mathbb{P}\Big(X_1^{(j)}=\pm j^{-\gamma}\Big)=1/2$ for all $j \geqslant 1$. Then we have

$$\sum_{j=1}^{\infty} \mathbb{E}\Big(|X_1^{(j)}|^{1/\alpha} (\log_+ |X_1^{(j)}|)^{\beta}\Big) = \sum_{j=1}^{\infty} \mathbb{E}\Big((j^{-\gamma})^{1/\alpha} (\log_+ (j^{-\gamma}))^{\beta}\Big)$$
$$= \sum_{j=1}^{\infty} \mathbb{E}(j^{-\gamma})^{1/\alpha} = \sum_{j=1}^{\infty} \frac{1}{j^{\gamma/\alpha}} < \infty,$$

and therefore the conditions (4) and (11) are satisfied.

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