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The Euler like operators on tuples of Lagrangians and functions on total spaces

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Abstract. We describe all Euler like operators C, i.e. natural operators transforming tuples (λ, g) of Lagrangians $\lambda : J^sY \to \bigwedge^m T^*M$ on a fibred manifold $Y \to M$ and functions $g : Y \to \mathbf{R}$ into Euler maps $C(\lambda, g) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$ on $Y \to M$, where m is the dimension of M. The most important example of such operators is the Euler operator E (from the variational calculus) being the one in question depending only on Lagrangians. We describe all formal Euler like operators, too.

1. Introduction

All manifolds and maps between manifolds considered in this paper are assumed to be smooth (i.e. C^∞).

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred diffeomorphisms onto open images.

Given a $\mathcal{FM}_{m,n}$-object $Y \to M$, we have the s-jet prolongation J^sY of $Y \to M$ and the jet projection $\pi^{2s}_s : J^{2s}Y \to J^sY$ for any positive integer s. If $f : Y \to Y^1$ is a $\mathcal{FM}_{m,n}$-map with the base map $\underline{f} : M \to M_1$, then we have the map $J^s f : J^sY \to J^sY_1$ given by $J^s f(j^s_{x_o}\sigma) = j^s_{\underline{f}(x_o)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^s_{x_o}\sigma \in J^s_{x_o}Y$, $x_o \in M$.

Given a fibred manifold $Y \to M$, we have the vertical bundle $VY \to Y$ and its dual bundle $V^*Y \to Y$ and the cotangent bundle T^*M of M and its m-th inner product $\bigwedge^m T^*M$.

Given fibred manifolds $Z_1 \to M$ and $Z_2 \to M$ with the same basis M, let $C^\infty_M(Z_1, Z_2)$ denotes the space of all base preserving fibred maps of Z_1 into Z_2. Given a $\mathcal{FM}_{m,n}$-object $Y \to M$, elements from the space $C^\infty_M(J^sY, \bigwedge^m T^*M)$ are called (s-th order) Lagrangians on $Y \to M$. Elements from the space $C^\infty_Y(J^qY, V^*Y \otimes \bigwedge^m T^*M)$ are called Euler maps on $Y \to M$.

We inform that the concept of natural operators can be found in [3].

By Proposition 49.3 of [3], any s-th order Lagrangian $\lambda : J^sY \to \bigwedge^m T^*M$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ induces the Euler map $E(\lambda) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$. So, we have the $\mathcal{FM}_{m,n}$-natural operator

$$E : C^\infty_M(J^sY, \bigwedge^m T^*M) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M) .$$

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It is called the Euler operator.

In [2] (see, [4]), I. Kolář proved that given integers $m \geq 2$ and $n \geq 1$ and $s \geq 1$, any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator $C^\infty_M(J^sY, \bigwedge^m T^*M) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$ is of the form cE, $c \in \mathbf{R}$, where E is the Euler operator.

In [5] we generalized the result of [2]. Namely, in [5], if $m \geq 2$, we described all regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operators

$$C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(M, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

transforming a tuple of a s-th order Lagrangian on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a real valued map on the basis M of $Y \to M$ into a Euler map on $Y \to M$. A reformulation of the mentioned description of [5] will be presented in Theorem 7.3.

In the present paper, if $m \geq 2$, we describe all Euler like operators, i.e. regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operators

$$C : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

transforming a tuple (λ, g) of a Lagrangian $\lambda \in C^\infty_M(J^sY, \bigwedge^m T^*M)$ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a real valued map $g \in C^\infty(Y, \mathbf{R})$ on the total space Y of $Y \to M$ into a Euler map $C(\lambda, g) \in C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$.

We describe all formal Euler like operators, too.

2. The Euler like operators

Example 2.1. *Let $l = 0, 1, ..., s$. We define $E^{(l)}(\lambda, g) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$ by*

$$E^{(l)}(\lambda, g)_{|j^{2s}_{x_o}\sigma} := \frac{1}{l!} E((-1)^l (g - g(\sigma(x_o)))^l \cdot \lambda)_{|j^{2s}_{x_o}\sigma}$$

for any $\lambda \in C^\infty_M(J^sY, \bigwedge^m T^*M)$ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and any $g \in C^\infty(Y, \mathbf{R})$ and any $j^{2s}_{x_o}\sigma \in J^{2s}_{x_o}Y$ and any $x_o \in M$, where E is the Euler operator. So, we have the $\mathcal{FM}_{m,n}$-natural (i.e. invariant with respect to $\mathcal{FM}_{m,n}$-morphisms) operator

$$E^{(l)} : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M) .$$

One can easily see that $E^{(l)}$ is π^{2s}_s-local, i.e. $E^{(l)}(\lambda, g)_\rho$ depends on $\mathrm{germ}_{\pi^{2s}_s(\rho)}(\lambda, g)$ for any $\rho \in J^{2s}Y$ and $\lambda \in C^\infty_M(J^sY, \bigwedge^m T^*M)$ and any $g \in C^\infty(Y, \mathbf{R})$. One can also see that $E^{(l)}$ is regular, i.e. it sends smoothly parametrized families of tuples of Lagrangians and maps into smoothly parametrized families of Euler maps.

Example 2.2. *We define $D(\lambda, g) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$ by*

$$D(\lambda, g)_{|j^{2s}_{x_o}\sigma} := d_{\sigma(x_o)} g_{|V_{\sigma(x_o)}Y} \otimes \lambda(j^s_{x_o}\sigma)$$

for any $\lambda \in C^\infty_M(J^sY, \bigwedge^m T^*M)$ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and any $g \in C^\infty(Y, \mathbf{R})$ and any $j^{2s}_{x_o}\sigma \in J^{2s}_{x_o}Y$ and any $x_o \in M$. So, we have the $\mathcal{FM}_{m,n}$-natural operator

$$D : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M) .$$

It is regular and π^{2s}_s-local.

We have the following

Theorem 2.3. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is of the form $C = h \cdot D + \sum_{l=0}^s h_l \cdot E^{(l)}$ for some (uniquely determined by C) maps $h, h_l : \mathbf{R} \to \mathbf{R}$, $l = 0, ..., s$, where $h \cdot C$ is defined by $(h \cdot C)(\lambda, g)_{|j^{2s}_{x_o}\sigma} = h(g(\sigma(x_o))) \cdot C(\lambda, g)_{|j^{2s}_{x_o}\sigma}$ for any $h : \mathbf{R} \to \mathbf{R}$ and any C in question and any $\lambda, g, j^{2s}_{x_o}\sigma$ as above.

So, the space $\mathbf{C}_{m,n,s}$ of all C (as in Theorem 2.3) is the free $(s + 2)$-dimensional $C^\infty(\mathbf{R})$-module and the operators $D, E^{(l)}$ for $l = 0, 1, ..., s$ form the basis in this module.

The proof of Theorem 2.3 will be given in Section 4.

3. A preparatory lemma

Let $\mathbf{N}$ be the set of non-negative integers and let $\mathbf{R}^{m,n}$ be the trivial $\mathcal{FM}_{m,n}$-object $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and let $x^1, ..., x^m, y^1, ..., y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$. Given $i = 1, ..., m$ let $1_i := (0, ..., 0, 1, 0, ..., 0) \in \mathbf{N}^m$, 1 in i-th position.

We have the induced coordinates $((x^i), (y^j_\alpha))$ on $J^s(\mathbf{R}^{m,n})$, where $i = 1, ..., m$ and $j = 1, ..., n$ and $\alpha = (\alpha_1, ...\alpha_m) \in \mathbf{N}^m$ are such that $|\alpha| = \alpha_1 + ... + \alpha_m \leq s$. They are given by

$$x^i(j^s_{x_o}\sigma) = x^i_o \text{ and } y^j_\alpha(j^s_{x_o}\sigma) = (\partial_\alpha \sigma^j)(x_o)$$

for any $j^s_{x_o}\sigma = j^s_{x_o}(\sigma^1, ..., \sigma^n) \in J^s_{x_o}(\mathbf{R}^{m,n}) = J^s_{x_o}(\mathbf{R}^m, \mathbf{R}^n)$, $x_o \in \mathbf{R}^m$, where ∂_α is the iterated partial derivative as indicated multiplied by $\frac{1}{\alpha!}$.

Lemma 3.1. *([5]) Let $i = 1, ..., m$ and $j = 1, ..., n$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{N}^m$ be such that $|\alpha| \leq s$.*

(i) For any $\tau = (\tau^1, ..., \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we have

$$(J^s\psi_\tau)_* y^j_\alpha = \tau^j y^j_\alpha ,$$

where $\psi_\tau = (x^1, ..., x^m, \frac{1}{\tau^1}y^1, ..., \frac{1}{\tau^n}y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(ii) For any $t \in \mathbf{R} \setminus \{0\}$, we have

$$(J^s\varphi^i_t)_* y^j_\alpha = t^{-\alpha_i} y^j_\alpha ,$$

where $\varphi^i_t = (x^1, ..., \frac{1}{t}x^i, ..., x^m, y^1, ..., y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(iii) If $\alpha_i \neq 0$, we have

$$(J^s\psi^{(i)})_* y^1_\alpha = y^1_\alpha + x^i y^1_\alpha + y^1_{\alpha-1_i} ,$$

where $\psi^{(i)} = (x^1, ..., x^m, y^1 + x^i y^1, y^2, ..., y^n)^{-1}$ is the $\mathcal{FM}_{m,n}$-map (defined over $0 \in \mathbf{R}^m$).

(iv) If $\alpha_1 \neq 0$, we have

$$(J^s\chi_t)_* y^1_{\alpha-1_1+1_2} = y^1_{\alpha-1_1+1_2} + c_1 t y^1_\alpha + ... + c_{\alpha_2+1} t^{\alpha_2+1} y^1_{(\alpha_1+\alpha_2,0,\alpha_3,...,\alpha_m)}$$

for some $c_1, ... \in \mathbf{R}$ with $c_1 \neq 0$, where $\chi_t = (x^1 + tx^2, x^2, ..., x^m, y^1, ..., y^n)$ is the $\mathcal{FM}_{m,n}$-map (defined if $m \geq 2$).

4. Proof of Theorem

We will use the notations as in the previous sections. Additionally, let $dx^\mu := dx^1 \wedge ... \wedge dx^m$ and let $x^\alpha := (x^1)^{\alpha_1} \cdot ... \cdot (x^m)^{\alpha_m}$ for any $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{N}^m$. We are now in position to prove Theorem 2.3.

Proof. Clearly, C is determined by the collection of values

$$< C(\lambda, g)_\rho, v > \in \bigwedge^m T_0^*\mathbf{R}^m$$

for all $\lambda \in C^\infty_{R^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$ and $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$ and $\rho = j_0^{2s}(\sigma) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$ and $g : \mathbf{R}^{m,n} \to \mathbf{R}$. It means that if C' is an another such operator giving the same as C collection of values in question then $C = C'$. This fact follows from the invariance of C with respect to $\mathcal{FM}_{m,n}$-charts.

We can additionally assume $\rho = \theta := j_0^{2s}(0)$. Indeed, for any element $\rho = j_0^{2s}(\sigma) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$, there exists a $\mathcal{FM}_{m,n}$-map $\nu : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ transforming $j_0^{2s}(\sigma)$ into $\theta := j_0^{2s}(0) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$. (For example, we can use $\nu := (x, y - \sigma(x))$, where $x = (x^1, ..., x^m)$ and $y = (y^1, ..., y^n)$.)

Because of the regularity of C, we can additionally assume that $\frac{\partial}{\partial y^1} g(0,0) \neq 0$. Then by the invariance of C with respect to $\mathcal{FM}_{m,n}$-map

$$(x^1, ..., x^m, g(x^1, ..., x^m, y^1, ..., y^n) - g(x^1, ..., x^m, 0, ..., 0), y^2, ..., y^n)^{-1}$$

(it preserves θ) we may additionally assume $g = y^1 + h(x^1, ..., x^m)$, where $h : \mathbf{R}^m \to \mathbf{R}$ is a map.

Similarly, because of the regularity of C, we can additionally assume $d_{(0,0)}y^1(v) \neq 0$. Then using the invariance of C with respect to an $\mathcal{FM}_{m,n}$-map $\mathrm{id}_{\mathbf{R}^m} \times \phi$ for a respective linear isomorphism $\phi : \mathbf{R}^n \to \mathbf{R}^n$ (it preserves θ), we can additionally assume $v = \frac{\partial}{\partial y^1}_{|(0,0)}$.

Similarly, because of the regularity of C, we can additionally assume $\frac{\partial}{\partial x^m} h(0) \neq 0$. Then using the invariance of C with respect to $\mathcal{FM}_{m,n}$-map

$$(x^1, ..., x^{m-1}, h(x^1, ..., x^m) - h(0, ..., 0), y^1, ..., y^n)^{-1} ,$$

we may additionally assume $h = x^m + c$, where c is a real number.

Summing up, we see that C is determined by the collection of values

$$< C(\lambda, x^m + y^1 + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > \in \bigwedge^m T_0^*\mathbf{R}^m$$

for all $\lambda \in C^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$ and all $c \in \mathbf{R}$, where $\theta := j_0^{2s}(0)$.

Next, we can write $\lambda = L((x^i), (y^j_\alpha))dx^\mu + f(x^1, ..., x^m)dx^\mu$, where L and f are arbitrary real valued maps with $L((x^i), (0)) = 0$, and by the regularity of C we can assume $f(0) \neq 0$. Then, using the invariance of C with respect to $\mathcal{FM}_{m,n}$-map $b = (F(x^1, .., x^m), x^2, ..., x^m, y^1, ..., y^n)^{-1}$, where $\frac{\partial}{\partial x^1} F = f$ and $F(0, x^2, ..., x^m) = 0$, we may additionally assume $f = 1$ because b preserves θ and $x^m + y^1 + c$ (as $m \geq 2$) and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends dx^μ into $f dx^\mu$. Consequently, we can write $\lambda = L((x^i), (y^j_\alpha))dx^\mu + dx^\mu$, where L is a arbitrary real valued map with $L((x^i), (0)) = 0$.

Next, because of the π^{2s}_s-locality of C, using the main result of [7], we may additionally assume that L is a arbitrary polynomial in $((x^i), (y^j_\alpha))$ of degree $\leq q$, where q is an arbitrary positive integer.

Next, by the invariance of C with respect to $\psi_\tau = (x^1, ..., x^m, \frac{1}{\tau^1} y^1, ..., \frac{1}{\tau^n} y^n)$ being $\mathcal{FM}_{m,n}$-map for any $(\tau^1, ..., \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we get the homogeneity condition

$$< C(L((x^i), (\tau^j y^j_\alpha))dx^\mu + e dx^\mu, b x^m + \tau^1 a y^1 + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

$$= \tau^1 < C(L((x^i), (y^j_\alpha))dx^\mu + e dx^\mu, b x^m + a y^1 + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > ,$$

see Lemma 3.1 (i), where a and b and c and e are arbitrary real numbers.

Then using the homogeneous function theorem ([3]), we derive that the value

$$< C(Ldx^{\mu} + edx^{\mu}, bx^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >$$

is linear in L (for L satisfying the additional conditions) for any real numbers b and c and e, and that C is determined by the collection of values

$$< C(x^{\beta} y^{1}_{\alpha} dx^{\mu} + dx^{\mu}, x^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m} \text{ and } < C(dx^{\mu}, x^{m} + y^{1} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m}$$

for $\alpha, \beta \in \mathbf{N}^{m}$ with $|\beta| \leq q$ and $|\alpha| \leq s$ and $c \in \mathbf{R}$.

Now, using Lemma 4.1, we derive that C is determined by the collection of values

$$< C(dx^{\mu}, x^{m} + y^{1} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m} \text{ and } < C(y^{1}_{(0,\dots,0,k)} dx^{\mu} + dx^{\mu}, x^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m}$$

for all $c \in \mathbf{R}$ and $k = 0, 1, \dots, s$.

So, C is determined by the collection of (smooth as C is regular) maps $C^{(o)}, C^{<k>} : \mathbf{R} \to \mathbf{R}$ for $k = 0, \dots, s$ defined by

$$C^{<k>}(c) dx^{\mu}_{|0} :=< C(y^{1}_{(0,\dots,0,k)} dx^{\mu} + dx^{\mu}, x^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} > , \; c \in \mathbf{R} ,$$

$$C^{(o)}(c) dx^{\mu}_{|0} :=< C(dx^{\mu}, x^{m} + y^{1} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} > , \; c \in \mathbf{R} .$$

It means that if C' is an another such operator with $C^{<k>} = (C')^{<k>}$ for $k = 0, \dots, s$ and $C^{(o)} = (C')^{(o)}$, then $C = C'$.

Conversely, using the coordinate expression of the Euler map $E(\lambda)$ from [3] one can verify that given a collection of maps $h_l, h : \mathbf{R} \to \mathbf{R}$ we have

$$(h \cdot D + \sum_{l=0}^{s} h_l \cdot E^{(l)})^{<k>} = h_k \text{ and } (h \cdot D + \sum_{l=0}^{s} h_l \cdot E^{(l)})^{(o)} = h$$

for $k = 0, \dots, s$.

The proof of the theorem is complete. □

Lemma 4.1. *The collection of values*

$$< C(x^{\beta} y^{1}_{\alpha} dx^{\mu} + dx^{\mu}, x^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m}$$

for $\alpha, \beta \in \mathbf{N}^{m}$ with $|\beta| \leq q$ and $|\alpha| \leq s$ and $c \in \mathbf{R}$ is determined by the one of values

$$< C(y^{1}_{(0,\dots,0,k)} dx^{\mu} + dx^{\mu}, x^{m} + c)_{\theta}, \frac{\partial}{\partial y^{1}}_{|(0,0)} >\in \bigwedge^{m} T^{*}_{0}\mathbf{R}^{m}$$

for all $c \in \mathbf{R}$ and $k = 0, 1, \dots, s$.

Proof. We will proceed quite similarly as in the respective part of the proof of the main result of [5].

By the invariance of C with respect to $\varphi^i_t = (x^1, ..., \frac{1}{t}x^i, ..., x^m, y^1, ..., y^m)$ being $\mathcal{FM}_{m,n}$-map for any $t \in \mathbf{R}\setminus\{0\}$ and any $i = 1, ..., m$ and using the fact that $< C(Ldx^\mu + edx^\mu, bx^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$ depends linearly in L (as above) for any real numbers b and c and e, we get the condition

$$t^{\beta_i-\alpha_i} < C(x^\beta y^1_\alpha dx^\mu + tdx^\mu, t^{\delta_{im}}x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >=< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

because φ^i_t preserves C and θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends x^β into $t^{\beta_i}x^\beta$ and it sends x^m into $t^{\delta_{im}}x^m$ (the Kronecker delta) and it sends y^1_α into $t^{-\alpha_i}y^1_\alpha$ and it sends dx^μ into tdx^μ, see Lemma 3.1 (ii). Then putting $t \to 0$, we get

$$< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >= 0$$

for any $\beta, \alpha \in \mathbf{N}^m$ with both $|\alpha| \le s$ and $\beta_i > \alpha_i$ for some $i = 1, ..., m$.

So, our collection is determined by the one of values

$$< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >\in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \le s$ and $\beta_1 \le \alpha_1$ and...and $\beta_m \le \alpha_m$.

Now, consider $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \le s$ and $\beta_1 \le \alpha_1$ and ...and $\beta_m \le \alpha_m$. Assume $\beta \neq (0)$. For example, let $\beta_i \neq 0$ for some $i = 1, ..., m$. Using the invariance of C with respect to

$$\psi^{(i)} = (x^1, ..., x^m, y^1 + x^i y^1, y^2, ..., y^n)^{-1}$$

(being $\mathcal{FM}_{m,n}$-map defined over some neighborhood of $0 \in \mathbf{R}^m$), we get

$$< C(x^{\beta-1_i}(y^1_\alpha + x^i y^1_\alpha + y^1_{\alpha-1_i})dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >=< C(x^{\beta-1_i}y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

because $\psi^{(i)}$ preserves C and $x^{\beta-1_i}$ and θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and dx^μ and $x^m + c$ and it sends y^1_α into $y^1_\alpha + x^i y^1_\alpha + y^1_{\alpha-1_i}$, see Lemma 3.1 (iii). Then

$$< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >= - < C(x^{\beta-1_i}y^1_{\alpha-1_i}dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

because $< C(Ldx^\mu + dx^\mu)_\theta, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$ depends linearly in L. Repeating,

$$< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >= (-1)^{|\beta|} < C(y^1_{(\alpha-\beta)}dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > .$$

Consequently, our collection is determined by the one of values

$$< C(y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >\in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha \in \mathbf{N}^m$ with $|\alpha| \le s$.

Now, let $\alpha \in \mathbf{N}^m$, where $|\alpha| \le s$, and assume that $\alpha_i \neq 0$ for some $i = 1, ..., m-1$. For example, let $\alpha_1 \neq 0$. For any $t \in \mathbf{R}$, the $\mathcal{FM}_{m,n}$-map

$$\chi_t = (x^1 + tx^2, x^2, ..., x^m, y^1, ..., y^n)$$

(defined if $m \geq 2$) preserves dx^μ and θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and $x^m + c$ and it sends $y^1_{\alpha-1_1+1_2}$ into

$$y^1_{\alpha-1_1+1_2} + c_1 t y^1_\alpha + ... + c_{\alpha_2+1} t^{\alpha_2+1} y^1_{(\alpha_1+\alpha_2,0,\alpha_3,...,\alpha_m)}$$

for some $c_1, ... \in \mathbf{R}$ with $c_1 \neq 0$, see Lemma 3.1 (iv). Then using the invariance of C with respect to χ_t we get

$$< C((y^1_{\alpha-1_1+1_2} + c_1 t y^1_\alpha + ...)dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >=< C(y^1_{\alpha-1_1+1_2} dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

for any $t \in \mathbf{R}$. Then since $< C(L dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$ depends linearly in L, we get

$$< C(y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >= 0 \, .$$

That is why, our collection is determined by the collection of values

$$< C(y^1_{(0,...,0,k)} dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >\in \bigwedge^m T^*_0 \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and $k = 0, 1, ..., s$.

The proof of the lemma is complete. □

Remark 4.2. *In the proof of the main result of [5], we used true but non-correctly justified formula*

$$t^{\beta_i-\alpha_i} < C(x^\beta y^1_\alpha dx^\mu + t dx^\mu, t^{\delta_{im}} x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >=< C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > \, .$$

In [5], we suggested that this formula follows from the fact that $< C(L dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$ depends linearly in L for any real number c. However, it is not true. Now, in the proof of Lemma 4.1, we observed that this formula follows from the fact that $< C(L dx^\mu + e dx^\mu, b x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$ depends linearly in L for any real numbers e, b, c. So, we have eliminated this gap from the proof of the main result of [5].

5. The formal Euler like operators

In [1] (see, [4]), I. Kolář introduced the so called formal Euler operator

$$\mathbf{E} : C^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

for all $\mathcal{FM}_{m,n}$-objects $Y \to M$.

In this section, we describe all regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operators

$$C : C^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M) \, .$$

Example 5.1. *Let $l = 0, 1, ..., s$. We define $\mathbf{E}^{(l)}(\lambda, g) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$ by*

$$\mathbf{E}^{(l)}(\lambda, g)_{|j^{2s}_{x_o}\sigma} := \frac{1}{l!}\mathbf{E}((-1)^l (g - g(\sigma(x_o)))^l \cdot \lambda)_{|j^{2s}_{x_o}\sigma}$$

for any $\lambda \in C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$ on a $\mathcal{FM}_{m,n}$-object $Y \to M$ and any $g \in C^\infty(Y, \mathbf{R})$ and any $j^{2s}_{x_o}\sigma \in J^{2s}_{x_o}Y$ and any $x_o \in M$, where $\mathbf{E}$ is the formal Euler operator. So, we have the $\mathcal{FM}_{m,n}$-natural operator

$$\mathbf{E}^{(l)} : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M) \, .$$

It is regular and π^{2s}_s-local.

Theorem 5.2. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is of the form $C = \sum_{l=0}^s h_l \cdot \mathbf{E}^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \to \mathbf{R}$, $l = 0, ..., s$, where $h \cdot C$ is defined by $(h \cdot C)(\lambda, g)_{|j^{2s}_{x_o}\sigma} = h(g(\sigma(x_o))) \cdot C(\lambda, g)_{|j^{2s}_{x_o}\sigma}$ for any $h : \mathbf{R} \to \mathbf{R}$ and any C in question and any $\lambda, g, j^{2s}_{x_o}\sigma$ as above.

So, the space $\mathbf{F}_{m,n,s}$ of all C (as in Theorem 5.2) is the free $(s+1)$-dimensional $C^\infty(\mathbf{R})$-module and the $\mathbf{E}^{(l)}$ for $l = 0, 1, ..., s$ form the basis in this module.

6. Schema of the proof of Theorem 5.2

The proof of Theorem 5.2 is the following modification of the one of Theorem 2.3.

Proof. (*Schema of the proof*) Similarly as in the proof of Theorem 2.3, C is determined by the collection of values

$$< C(\lambda, g)_\rho, v > \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $\lambda \in C^\infty_{J^s\mathbf{R}^{m,n}}(J^s\mathbf{R}^{m,n}, V^*J^s\mathbf{R}^{m,n} \otimes \bigwedge^m T^*\mathbf{R}^m)$ and $g \in C^\infty(\mathbf{R}^{m,n}, \mathbf{R})$ and $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$ and $\rho \in J^{2s}_0(\mathbf{R}^m, \mathbf{R}^n) = J^{2s}_0(\mathbf{R}^{m,n})$. Moreover, we can assume that $\rho = \theta = j^{2s}_0(0)$ and $g = x^m + y^n + c$ and $v = \frac{\partial}{\partial y^1}_{|(0,0)}$.

Further we can write $\lambda = \sum L^\beta_k((x^i), (y^j_\alpha))\tilde{d}y^k_\beta \otimes dx^\mu$, where L^β_k are real valued maps for $k = 1, ..., n$ and all $\beta \in \mathbf{N}^m$ with $|\beta| \leq s$ and where $\tilde{d}h$ denotes the restriction to VJ^sY of the differential dh of $h : J^sY \to \mathbf{R}$. Moreover, quite similarly as in the proof of Theorem 2.3, we can assume that L^β_k are polynomials in $((x^i), (y^j_\alpha))$ of degree $\leq q$, where q is an arbitrary positive integer.

Further, quite similarly as in the proof of Theorem 2.3, using the invariance of C with respect to $\psi_\tau = (x^1, ..., x^m, \frac{1}{\tau^1}y^1, ..., \frac{1}{\tau^n}y^n)$ and the homogeneous function theorem, we derive that $< C(\lambda, bx^m + c)_\theta, \frac{\partial}{\partial y^1}_{(0,0)} >$ is linear in λ for any real numbers b and c, and that C is determined by the collection of values

$$< C(x^\beta \tilde{d}y^1_\alpha \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > \text{ and } < C(0, x^m + y^n + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

for all $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \leq s$ and all real numbers c.

Further, using the invariance of C with respect to $(tx^1, x^2, ..., x^m, y^1, ..., y^n)$ and the fact that $m \geq 2$, we get $< C(0, x^m + y^n + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >= 0$.

Then by Lemma 4.1 with $x^\beta \tilde{d}y^1_\alpha \otimes dx^\mu$ instead of $x^\beta y^1_\alpha dx^\mu + dx^\mu$ (the proof of such lemma is quite similar to the one of Lemma 4.1) and using $\tilde{d}(x^i y^1_\alpha) = x^i \tilde{d}y^1\alpha$ (being the consequence of $dh = 0$ on VJ^sY for any $h : M \to \mathbf{R}$), we derive that C is determined by the values

$$< C(\tilde{d}y^1_{(0,...,0,k)} \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} >$$

for all $c \in \mathbf{R}$ and $k = 0, 1, ..., s$.

Consequently, C is determined by the collection of maps $C^{<k>} : \mathbf{R} \to \mathbf{R}$ for $k = 0, ..., s$ defined by

$$C^{<k>}(c)dx^\mu_{|0} :=< C(\tilde{d}y^1_{(0,...,0,k)} \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} > , \quad c \in \mathbf{R} .$$

Conversely, given maps $h_l : \mathbf{R} \to \mathbf{R}$, where $l = 0, ..., s$, we have $(\sum_{l=0}^s h_l \cdot \mathbf{E}^{(l)})^{<k>} = h_k$ for $k = 0, 1, ..., s$.

The proof of Theorem 5.2 is complete. □

7. A reformulation of the results of [5]

Example 7.1. *Let $\pi : Y \to M$ be a $\mathcal{FM}_{m,n}$-object and let*

$$C : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

be a regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator. We have the inclusion $C^\infty(M, \mathbf{R}) \subset C^\infty(Y, \mathbf{R})$ given by $h \to h \circ \pi$. Thus we have the π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator

$$\tilde{C} : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(M, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

given by the restriction of C.

Example 7.2. *Let $\pi : Y \to M$ be a $\mathcal{FM}_{m,n}$-object and let*

$$C : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(Y, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

be a regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator. Thus similarly as in Example 7.1 we have the π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator

$$\hat{C} : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(M, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

given by the restriction of C.

Thus the results of [5] can be rewritten as follows.

Theorem 7.3. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$B : C^\infty_M(J^sY, \bigwedge^m T^*M) \times C^\infty(M, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is of the form $B = \sum_{l=0}^s h_l \cdot \widetilde{E^{(l)}}$ for some (uniquely determined by B) maps $h_l : \mathbf{R} \to \mathbf{R}$, $l = 0, ..., s$, where $h \cdot B$ is defined by $(h \cdot B)(\lambda, g)_{|j^{2s}_{x_o}\sigma} = h(g(x_o)) \cdot B(\lambda, g)_{|j^{2s}_{x_o}\sigma}$ for any $h : \mathbf{R} \to \mathbf{R}$ and any B in question and any $\lambda, g, j^{2s}_{x_o}\sigma$ as required.

So, the space $\mathbf{B}_{m,n,s}$ of all such B as above (in Theorem 7.3) is the free $(s+1)$-dimensional $C^\infty(\mathbf{R})$-module and the operators $\widetilde{E^{(l)}}$ for $l = 0, ..., s$ form the basis in this module.

Theorem 7.4. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$B : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \times C^\infty(M, \mathbf{R}) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is of the form $B = \sum_{l=0}^s h_l \cdot \widehat{\mathbf{E}^{(l)}}$ for some (uniquely determined by B) maps $h_l : \mathbf{R} \to \mathbf{R}$, $l = 0, ..., s$, where $h \cdot B$ is defined by $(h \cdot B)(\lambda, g)_{|j^{2s}_{x_o}\sigma} = h(g(x_o)) \cdot B(\lambda, g)_{|j^{2s}_{x_o}\sigma}$ for any $h : \mathbf{R} \to \mathbf{R}$ and any B in question and any $\lambda, g, j^{2s}_{x_o}\sigma$ as required.

So, the space $\mathbf{G}_{m,n,s}$ of all B as above (in Theorem 7.4) is the free $(s+1)$-dimensional $C^\infty(\mathbf{R})$-module and the $\widehat{\mathbf{E}^{(l)}}$ for $l = 0, 1, ..., s$ form the basis in this module.

8. Corollaries

From Theorem 2.3 it follows the following result by I. Kolář [2] (see also [4]).

Corollary 8.1. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : C^\infty_M(J^sY, \bigwedge^m T^*M) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is aE for some (uniquely determined by C) real number a, where E is the Euler operator.

Proof. Let C be a operator in question. By Theorem 2.3, we can write $C = h \cdot D + \sum_{l=0}^{s} h_l \cdot E^{(l)}$ for some (uniquely determined by C) maps $h, h_l : \mathbf{R} \to \mathbf{R}$, $l = 0, ..., s$. Then $C(\lambda) = C(\lambda, 1) = h_0(1)E(\lambda)$, i.e. $C = h_0(1)E$.

From Theorem 5.2 it follows the following result of [6].

Corollary 8.2. *Let m, n, s be positive integers. If $m \geq 2$, then any regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : C^\infty_M(J^sY, V^*J^sY \otimes \bigwedge^m T^*M) \to C^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$$

is $a\mathbf{E}$ for some (uniquely determined by C) real number a, where $\mathbf{E}$ is the formal Euler operator.

Proof. The proof is (almost) the same as the one of Corollary 8.1. We use Theorem 5.2 instead of Theorem 2.3. □

Let $\mathbf{B}_{m,n,s}$ and $\mathbf{C}_{m,n,s}$ be the $C^\infty(\mathbf{R})$-modules described after Theorems 7.3 and 2.3, respectively. From Theorems 2.3 and 7.3 it follows immediately

Corollary 8.3. *Let m, n, s be positive integers. If $m \geq 2$, then the correspondence $\tilde{(-)} : \mathbf{C}_{m,n,s} \to \mathbf{B}_{m,n,s}$ given by $C \to \tilde{C}$ (described in Example 7.1) is a epimorphism of $C^\infty(\mathbf{R})$-modules. The kernel of this epimorphism is the 1-dimensional $C^\infty(\mathbf{R})$-module spanned by the operator D from Example 2.2.*

Let $\mathbf{F}_{m,n,s}$ and $\mathbf{G}_{m,n,s}$ be the $C^\infty(\mathbf{R})$-modules described after Theorems 5.2 and 7.4, respectively. From Theorems 5.2 and 7.4 it follows immediately

Corollary 8.4. *Let m, n, s be positive integers. If $m \geq 2$, then the correspondence $\widehat{(-)} : \mathbf{F}_{m,n,s} \to \mathbf{G}_{m,n,s}$ given by $C \to \hat{C}$ (described in Example 7.2) is a isomorphism of $C^\infty(\mathbf{R})$-modules.*

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