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A note on EP element and Hermitian element in rings

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Abstract. Let $a \in R^{\dagger} \cap R^{\#}$ in this paper. We give several characterizations for a to be an EP element or an Hermitian element in terms of reverse order laws of two elements from the given set.

1. Introduction

Let R be a ring with an involution. If $a \in R$, then the Moore-Penrose inverse $a^{\dagger}$ of a is the unique solution of the system of equations

$$(1)\ axa = a, (2)\ xax = x, (3)\ (ax)^* = ax, (4)\ (xa)^* = xa.$$

Also, let $a\{i, j, \cdots, l\}$ denote the set of elements x which satisfy equations $(i), (j), \cdots, (l)$ from among equations (1)-(4), and in this case, x is called the $\{i, j, \cdots, l\}$-inverse of a. An element $a \in R$ is group invertible if there is an $x \in a\{1, 2\}$ that commutes with a, the group inverse of a is unique if it exists and is denoted by $a^{\#}$. We use $R^{\dagger}$ and $R^{\#}$ to denote the set of all Moore-Penrose invertible and group invertible elements in R, respectively.

Recall that in [1, 5] an element $a \in R$ is called an EP element if $a \in R^{\dagger} \cap R^{\#}$ with $a^{\dagger} = a^{\#}$. An element $a \in R$ is called Hermitian (or symmetric) if $a = a^*$. We use R^{EP} and R^{Her} to denote the set of all EP and Hermitian elements in R, respectively.

As is known to all, if $a, b \in R^{-1}$, then $ab \in R^{-1}$ and

$$(ab)^{-1} = b^{-1}a^{-1}.$$

This equation is called the reverse order law. In 1966, Greville [6] studied the problem when $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ holds, where A and B are two complex matrices. Since then, the reverse order law for the generalized

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inverses has been widely investigated, see for example, [7–17]. Later, Mosić and Djordjević [13, 18] considered the hybrid reverse order law $(ab)^{\#} = b^{\dagger}a^{\dagger}$ in rings with involution.

There are many conclusions about characterizations of EP element. For example, Hartwig [2] proved that if $a \in R^{\dagger}$, then $a \in R^{\mathrm{EP}}$ if and only if $aa^{\dagger} = a^{\dagger}a$. Patrício and Puystjens [3] characterized EP element by ideals, they proved that if $a \in R^{\dagger}$, then $a \in R^{\mathrm{EP}}$ if and only if $aR = a^{*}R$. Xu et al. [4] characterized EP element by three equations, that is to say, $a \in R^{\mathrm{EP}}$ if and only if there exists $x \in R$ such that

$$xa^2 = a, \quad ax^2 = x, \quad (xa)^* = xa.$$

In these equivalent characterizations of EP element, most of them are related to a^*, $a^{\dagger}$ and $a^{\#}$. So we first investigate the Moore-Penrose inverse of the product of two elements in the set $\{a, a^{\#}, (a^{\dagger})^*, a^{\dagger}, a^*, (a^{\#})^*\}$ under the condition $a \in R^{\dagger} \cap R^{\#}$, which will improve some conclusions in [19]. And then we give several characterizations such that a is an EP element by hybrid reverse order laws of the Moore-Penrose inverse and group inverse of two elements from the set $\{a, a^{\#}, (a^{\dagger})^*, a^{\dagger}, a^*, (a^{\#})^*\}$. In addition, we give some new characterizations of Hermitian element in the third section.

The following notations will be used in this paper:

$\tau_a = \{a, a^{\#}, (a^{\dagger})^*\}$,
$\gamma_a = \{a^{\dagger}, a^*, (a^{\#})^*\}$,
$\chi_a = \tau_a \cup \gamma_a = \{a, a^{\#}, (a^{\dagger})^*, a^{\dagger}, a^*, (a^{\#})^*\}$.

2. Characterizations of EP element by hybrid reverse order laws

In this section, we give expressions for the Moore-Penrose inverse and group inverse of the product of two elements in χ_a. And then we characterize EP element by hybrid reverse order laws.

Lemma 2.1. *[19] Let $a \in R^{\dagger} \cap R^{\#}$, then*
(1) $a^{\dagger} \in R^{\#}$ with $(a^{\dagger})^{\#} = (aa^{\#})^* a (aa^{\#})^*$;
(2) $a^{\#} \in R^{\dagger}$ with $(a^{\#})^{\dagger} = a^{\dagger}a^3a^{\dagger}$.

Lemma 2.2. *Let $a \in R^{\dagger} \cap R^{\#}$ and $w \in \chi_a$. Then $w \in R^{\dagger} \cap R^{\#}$, and we have the following results:*

(1) $ww^{\dagger} = \begin{cases} aa^{\dagger}, & w \in \tau_a, \\ a^{\dagger}a, & w \in \gamma_a. \end{cases}$

(2) $w^{\dagger}w = \begin{cases} a^{\dagger}a, & w \in \tau_a, \\ aa^{\dagger}, & w \in \gamma_a. \end{cases}$

(3) $ww^{\#} = w^{\#}w = \begin{cases} aa^{\#}, & w \in \tau_a, \\ (aa^{\#})^*, & w \in \gamma_a. \end{cases}$

Proof. If $a \in R^{\dagger} \cap R^{\#}$, according to Lemma 2.1 and the definitions of Moore-Penrose inverse and group inverse, it is easy to verify that

$$(a^{\#})^{\#} = a, \quad (a^{\#})^{\dagger} = a^{\dagger}a^3a^{\dagger}.$$

$$((a^{\dagger})^*)^{\#} = ((a^{\dagger})^{\#})^* = aa^{\#}a^*aa^{\#}, \quad ((a^{\dagger})^*)^{\dagger} = a^*.$$

Thus $w \in R^{\dagger} \cap R^{\#}$, where $w \in \tau_a$.

As we all know, $w \in R^{\dagger} \cap R^{\#}$ if and only if $w^* \in R^{\dagger} \cap R^{\#}$. When $w \in \gamma_a$, $w^* \in \tau_a$, thus $w^* \in R^{\dagger} \cap R^{\#}$, hence $w \in R^{\dagger} \cap R^{\#}$.

So we have proved above that if $a \in R^{\dagger} \cap R^{\#}$ and $w \in \chi_a$, then $w \in R^{\dagger} \cap R^{\#}$. Next we are going to prove conclusions (1)-(3) in this paper.

(1). When $w \in \tau_a$:

(i) If $w = a$, then $ww^{\dagger} = aa^{\dagger}$.

(ii) If $w = a^\#$, then

$$\begin{aligned} ww^\dagger &= a^\#(a^\#)^\dagger = a^\# a^\dagger a^3 a^\dagger = (a^\#)^2 a a^\dagger a^3 a^\dagger \\ &= (a^\#)^2 a^3 a^\dagger = a a^\dagger. \end{aligned}$$

(ii) If $w = (a^\dagger)^*$, then $ww^\dagger = (a^\dagger)^*((a^\dagger)^*)^\dagger = (a^\dagger)^* a^* = aa^\dagger$.

When $w \in \gamma_a$:
(i) If $w = a^\dagger$, then $ww^\dagger = a^\dagger (a^\dagger)^\dagger = a^\dagger a$.
(ii) If $w = a^*$, then $ww^\dagger = a^*(a^*)^\dagger = a^*(a^\dagger)^* = a^\dagger a$.
(ii) If $w = (a^\#)^*$, then

$$\begin{aligned} ww^\dagger &= (a^\#)^*((a^\#)^*)^\dagger = (a^\#)^*((a^\#)^\dagger)^* = (a^\#)^*(a^\dagger a^3 a^\dagger)^* = (a^\dagger a^3 a^\dagger a^\#)^* \\ &= (a^\dagger a^3 a^\dagger a a^\# a^\#)^* = (a^\dagger a^3 a^\# a^\#)^* = (a^\dagger a)^* = a^\dagger a. \end{aligned}$$

As a consequence, $ww^\dagger = \begin{cases} aa^\dagger, & w \in \tau_a, \\ a^\dagger a, & w \in \gamma_a. \end{cases}$.

(2). If $w \in \chi_a$, then $w^* \in \chi_a$. According to (1), we obtain

$$w^\dagger w = w^*(w^*)^\dagger = \begin{cases} aa^\dagger, & w^* \in \tau_a \\ a^\dagger a, & w^* \in \gamma_a \end{cases} = \begin{cases} aa^\dagger, & w \in \gamma_a, \\ a^\dagger a, & w \in \tau_a. \end{cases}$$

(3). When $w \in \tau_a$:
(i) If $w = a$, then $ww^\# = aa^\#$.
(ii) If $w = a^\#$, then $ww^\# = a^\#(a^\#)^\# = a^\# a = aa^\#$.
(ii) If $w = (a^\dagger)^*$, then

$$\begin{aligned} ww^\# &= (a^\dagger)^*((a^\dagger)^*)^\# = (a^\dagger)^*((a^\dagger)^\#)^* = (a^\dagger)^*((aa^\#)^* a (aa^\#)^*)^* \\ &= (a^\dagger a a^\dagger)^* a a^\# a^* a a^\# = (a^\dagger)^* a^\dagger a a a^\# a^* a a^\# = (a^\dagger)^* a^\dagger a a^* a a^\# \\ &= (a^\dagger)^* a^* a a^\# = (aa^\dagger)^* a a^\# = a a^\dagger a a^\# = aa^\#. \end{aligned}$$

When $w \in \gamma_a$, $w^* \in \tau_a$. Thus $ww^\# = (w^*(w^*)^\#)^* = (aa^\#)^*$.

Therefore, $ww^\# = w^\# w = \begin{cases} aa^\#, & w \in \tau_a, \\ (aa^\#)^*, & w \in \gamma_a. \end{cases}$ □

From Lemma 2.2, we have the following two results, which will be applied repeatedly in the following paper.

Corollary 2.3. *Let $a \in R^\dagger \cap R^\#$ and $w \in \chi_a$. Then $a \in R^{\mathrm{EP}}$ if and only if $w \in R^{\mathrm{EP}}$.*

Proof. $w \in R^{\mathrm{EP}} \Longleftrightarrow ww^\dagger = w^\dagger w \xLeftrightarrow{\text{Lemma 2.2}} \begin{cases} aa^\dagger = a^\dagger a, & w \in \tau_a, \\ a^\dagger a = aa^\dagger, & w \in \gamma_a. \end{cases} \Longleftrightarrow a \in R^{\mathrm{EP}}$.
□

Corollary 2.4. *Let $a \in R^\dagger \cap R^\#$ and $x, y \in \chi_a$. Then*

(1) $xx^\dagger = \begin{cases} yy^\dagger, & x, y \in \tau_a \text{ or } x, y \in \gamma_a, \\ y^\dagger y, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a. \end{cases}$

(2) $x^\dagger x = \begin{cases} y^\dagger y, & x, y \in \tau_a \text{ or } x, y \in \gamma_a, \\ yy^\dagger, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a. \end{cases}$

(3) $xx^\# = \begin{cases} yy^\#, & x, y \in \tau_a \text{ or } x, y \in \gamma_a, \\ (yy^\#)^*, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a. \end{cases}$

Li and Wei [19, Theorem 2.1] gave expressions for the Moore-Penrose inverse and the group inverse of product of two elements in χ_a. Now we are going to show you more concise formulae and the proof.

Theorem 2.5. *Let $a \in R^\dagger \cap R^\#$. If $x, y \in \chi_a$, then $xy \in R^\dagger \cap R^\#$. And in this case,* *(1) $(xy)^\dagger = y^\dagger x^\# xx^\dagger$.*
In particular, $(xy)^\dagger = y^\dagger x^\# xx^\dagger = y^\dagger x^\dagger$ when $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$.

(2) $(xy)^\# = \begin{cases} y^\# x^\#, & x, y \in \tau_a \text{ or } x, y \in \gamma_a, \\ y^\dagger x^\dagger, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a. \end{cases}$

Proof. (1). Since $a \in R^\dagger \cap R^\#$ and $x, y \in \chi_a$, from Lemma 2.2, we know that $x, y \in R^\dagger \cap R^\#$.
According to Corollary 2.4,

$$(xy)(y^\dagger x^\# xx^\dagger) = \begin{cases} xxx^\dagger x^\# xx^\dagger, & x, y \in \tau_a \text{ or } x, y \in \gamma_a \\ xx^\dagger xx^\# xx^\dagger, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a \end{cases} = xx^\dagger \tag{1}$$

is symmetric. And

$$(y^\dagger x^\# xx^\dagger)(xy) = y^\dagger x^\# xy = \begin{cases} y^\dagger y^\# yy, & x, y \in \tau_a \text{ or } x, y \in \gamma_a \\ y^\dagger (y^\# y)^* y, & x \in \tau_a, y \in \gamma_a \text{ or } x \in \gamma_a, y \in \tau_a \end{cases} = y^\dagger y$$

is also symmetric. Thus, $y^\dagger x^\# xx^\dagger$ is a $\{3, 4\}$-inverse of xy. Furthermore,

$$(xy)(y^\dagger x^\# xx^\dagger)(xy) \overset{(1)}{=\!=} xx^\dagger xy = xy$$

and

$$(y^\dagger x^\# xx^\dagger)(xy)(y^\dagger x^\# xx^\dagger) \overset{(1)}{=\!=} (y^\dagger x^\# xx^\dagger)xx^\dagger = y^\dagger x^\# xx^\dagger$$

show that $y^\dagger x^\# xx^\dagger$ is a $\{1, 2\}$-inverse of xy. Hence, $(xy)^\dagger = y^\dagger x^\# xx^\dagger$, where $x, y \in \chi_a$.
In particular, when $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$,

$$(xy)^\dagger = y^\dagger x^\# xx^\dagger = y^\dagger (yy^\#)^* x^\dagger = y^\dagger (yy^\# yy^\dagger)^* x^\dagger = y^\dagger (yy^\dagger)^* x^\dagger = y^\dagger x^\dagger.$$

(2). From the above proof, we know that $y^\dagger x^\# xx^\dagger$ is a $\{1, 2\}$-inverse of xy, $(xy)(y^\dagger x^\# xx^\dagger) = xx^\dagger$ and $(y^\dagger x^\# xx^\dagger)(xy) = y^\dagger y$.
According to Corollary 2.4, when $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$, $xx^\dagger = y^\dagger y$. Thus $(xy)(y^\dagger x^\# xx^\dagger) = (y^\dagger x^\# xx^\dagger)(xy)$. So, $(xy)^\# = y^\dagger x^\# xx^\dagger = y^\dagger x^\dagger$.
When $x, y \in \tau_a$ or $x, y \in \gamma_a$, $xx^\# = yy^\#$. Since $(xy)(y^\# x^\#) = xxx^\# x^\# = xx^\#$ and $(y^\# x^\#)(xy) = y^\# y^\# yy = yy^\#$, we obtain $(xy)(y^\# x^\#) = (y^\# x^\#)(xy)$. Moreover,

$$(xy)(y^\# x^\#)(xy) = xx^\# xy = xy, \quad (y^\# x^\#)(xy)(y^\# x^\#) = y^\# x^\# xx^\# = y^\# x^\#,$$

thus $(xy)^\# = y^\# x^\#$. □

Let $a \in R^\dagger \cap R^\#$, from Theorem 2.5 we know that $(xy)^\dagger = y^\dagger x^\dagger = (xy)^\#$ when $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$, thus we obtain the following result.

Corollary 2.6. *Let $a \in R^\dagger \cap R^\#$. If $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$, then $xy \in R^{\mathrm{EP}}$.*

If $w \in \tau_a$ (resp. $w \in \gamma_a$), then $w^* \in \gamma_a$ (resp. $w \in \tau_a$). Thus, from Corollary 2.6 and Theorem 2.5 we have the following conclusion.

Corollary 2.7. *Let $a \in R^\dagger \cap R^\#$ and $w \in \chi_a$. Then $ww^*, w^*w \in R^{\mathrm{EP}}$. And in this case,*

$$(ww^*)^\dagger = (ww^*)^\# = (w^*)^\dagger w^\dagger,$$

$$(w^*w)^\dagger = (w^*w)^\# = w^\dagger (w^*)^\dagger.$$

Many references have considered the characterizations of EP element, we characterize $a \in R^{\mathrm{EP}}$ by hybrid reverse order laws in the following theorem, which is an interesting conclusion.

Theorem 2.8. *Let $a \in R^{\dagger} \cap R^{\#}$ and $x, y \in \chi_a$. Then the following conditions are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $(xy)^{\dagger} = y^{\dagger}x^{\#}$;
(3) $(xy)^{\dagger} = y^{\#}x^{\dagger}$;
(4) $(xy)^{\dagger} = y^{\#}x^{\#}$;
(5) $(xy)^{\#} = y^{\dagger}x^{\#}$;
(6) $(xy)^{\#} = y^{\#}x^{\dagger}$.

Proof. (1) ⇒ (2)(3)(4)(5)(6) by Corollary 2.3 and Theorem 2.5.

Conversely, according to Corollary 2.4 and Theorem 2.5, we show that the condition (1) can be derived from any one of conditions (2)-(6). If we want to prove that $a \in R^{\mathrm{EP}}$, we just need to prove that $x \in R^{\mathrm{EP}}$ or $y \in R^{\mathrm{EP}}$ by Corollary 2.3.

(2) ⇒ (1). Since $(xy)^{\dagger} = y^{\dagger}x^{\#}xx^{\dagger}$, the condition $(xy)^{\dagger} = y^{\dagger}x^{\#}$ can be equivalently replaced by $y^{\dagger}x^{\#}xx^{\dagger} = y^{\dagger}x^{\#}$. Pre-multiplying the equality by xy, we have

$$xyy^{\dagger}x^{\#}xx^{\dagger} = xyy^{\dagger}x^{\#},$$

i.e.,

$$\begin{cases} xxx^{\dagger}x^{\#}xx^{\dagger} = xxx^{\dagger}x^{\#}, & x, y \in \tau_a \ \ or \ \ x, y \in \gamma_a, \\ xx^{\dagger}xx^{\#}xx^{\dagger} = xx^{\dagger}xx^{\#}, & x \in \tau_a, y \in \gamma_a \ \ or \ \ x \in \gamma_a, y \in \tau_a. \end{cases}$$

Both of these two equations can be simplified to $xx^{\dagger} = xx^{\#}$, thus $x \in R^{\mathrm{EP}}$.

(3) ⇒ (1). The condition $(xy)^{\dagger} = y^{\#}x^{\dagger}$ is equivalent to $y^{\dagger}x^{\#}xx^{\dagger} = y^{\#}x^{\dagger}$. Post-multiplying the equality by xy, we have

$$y^{\dagger}x^{\#}xx^{\dagger}xy = y^{\#}x^{\dagger}xy,$$

i.e.,

$$\begin{cases} y^{\dagger}y^{\#}yy^{\dagger}yy = y^{\#}y^{\dagger}yy, & x, y \in \tau_a \ \ or \ \ x, y \in \gamma_a, \\ y^{\dagger}(yy^{\#})^{*}yy^{\dagger}y = y^{\#}yy^{\dagger}y, & x \in \tau_a, y \in \gamma_a \ \ or \ \ x \in \gamma_a, y \in \tau_a. \end{cases}$$

The equality $y^{\dagger}y^{\#}yy^{\dagger}yy = y^{\#}y^{\dagger}yy$ can be simplified to $y^{\dagger}y = y^{\#}y$, thus $y \in R^{\mathrm{EP}}$. For the equality $y^{\dagger}(yy^{\#})^{*}yy^{\dagger}y = y^{\#}yy^{\dagger}y$, the left-hand side $y^{\dagger}(yy^{\#})^{*}yy^{\dagger}y = y^{\dagger}yy^{\dagger}(yy^{\#})^{*}y = y^{\dagger}(yy^{\#}yy^{\dagger})^{*}y = y^{\dagger}(yy^{\dagger})^{*}y = y^{\dagger}y$ and the right-hand side $y^{\#}yy^{\dagger}y = y^{\#}y$, thus $y^{\dagger}y = y^{\#}y$, which leads to $y \in R^{\mathrm{EP}}$.

(4) ⇒ (1). The assumption $(xy)^{\dagger} = y^{\#}x^{\#}$ is equivalent to $y^{\dagger}x^{\#}xx^{\dagger} = y^{\#}x^{\#}$, pre-multiplying the equality by y, we obtain

$$yy^{\dagger}x^{\#}xx^{\dagger} = yy^{\#}x^{\#}. \tag{2}$$

When $x, y \in \tau_a$ *or* $x, y \in \gamma_a$, Eq.(2) implies that $xx^{\dagger}x^{\#}xx^{\dagger} = xx^{\#}x^{\#}$, pre-multiplying the equality by x, we have $xxx^{\dagger}x^{\#}xx^{\dagger} = xxx^{\#}x^{\#}$. simplifying this formula yields $xx^{\dagger} = xx^{\#}$. Hence $x \in R^{\mathrm{EP}}$.

When $x \in \tau_a, y \in \gamma_a$ *or* $x \in \gamma_a, y \in \tau_a$, Eq. (2) follows that $x^{\dagger}xx^{\#}xx^{\dagger} = (xx^{\#})^{*}x^{\#}$, this is further reduced to $x^{\dagger} = (xx^{\#})^{*}x^{\#}$, thus

$$xx^{\dagger}x^{\dagger} = xx^{\dagger}(xx^{\#})^{*}x^{\#} = (xx^{\#}xx^{\dagger})^{*}x^{\#} = xx^{\dagger}x^{\#} = x^{\#}.$$

Furthermore,

$$xx^{\#} = x(xx^{\dagger}x^{\dagger}) = (xxx^{\dagger}x^{\dagger})xx^{\dagger} = xx^{\#}xx^{\dagger} = xx^{\dagger},$$

so $x \in R^{\mathrm{EP}}$.

Since $(xy)^{\dagger} = y^{\dagger}x^{\dagger} = (xy)^{\#}$ when $x \in \tau_a, y \in \gamma_a$ *or* $x \in \gamma_a, y \in \tau_a$ by Theorem 2.5, in this case, conditions (5) and (6) are equivalent to conditions (2) and (3), respectively. Therefore, we only need to show that the condition (1) can be derived from any one of conditions (5) and (6) when $x, y \in \tau_a$ *or* $x, y \in \gamma_a$.

(5) $\Rightarrow$ (1). When $x, y \in \tau_a$ or $x, y \in \gamma_a$, $(xy)^\# = y^\# x^\#$, thus the condition $(xy)^\# = y^\dagger x^\#$ is equivalent to $y^\# x^\# = y^\dagger x^\#$. Post-multiplying the equality by xy, we have $y^\# x^\# xy = y^\dagger x^\# xy$, i.e., $y^\# y^\# yy = y^\dagger y^\# yy$, this yields $y^\# y = y^\dagger y$, which shows that $y \in R^{\mathrm{EP}}$.

(6) $\Rightarrow$ (1). When $x, y \in \tau_a$ or $x, y \in \gamma_a$, the hypothesis $(xy)^\# = y^\# x^\dagger$ is equivalent to $y^\# x^\# = y^\# x^\dagger$. Pre-multiplying the equality by xy, we have $xyy^\# x^\# = xyy^\# x^\dagger$, i.e., $xxx^\# x^\# = xxx^\# x^\dagger$, this yields $xx^\# = xx^\dagger$, which shows that $x \in R^{\mathrm{EP}}$. □

After reading Theorem 2.8, you may ask: whether $(xy)^\# = y^\dagger x^\dagger$ and $(xy)^\# = y^\# x^\#$ are also equivalent to $a \in R^{\mathrm{EP}}$? The following two results will tell you the answer.

Corollary 2.9. *Let $a \in R^\dagger \cap R^\#$ and $x, y \in \tau_a$ or $x, y \in \gamma_a$. Then the following conditions are equivalent:*
(1) $a \in R^{\mathrm{EP}}$;
(2) $xy \in R^{\mathrm{EP}}$;
(3) $(xy)^\# = y^\dagger x^\dagger$.

Proof. When $x, y \in \tau_a$ or $x, y \in \gamma_a$, $(xy)^\dagger = y^\dagger x^\# xx^\dagger$ and $(xy)^\# = y^\# x^\#$ according to Theorem 2.5.

(1) $\Leftrightarrow$ (2). Suppose that $a \in R^{\mathrm{EP}}$, then $x, y \in R^{\mathrm{EP}}$ by Corollary 2.3. Therefore, $(xy)^\dagger = y^\dagger x^\# xx^\dagger = y^\# x^\# xx^\# = y^\# x^\# = (xy)^\#$, thus $xy \in R^{\mathrm{EP}}$.

Conversely, if $xy \in R^{\mathrm{EP}}$, then $(xy)^\dagger = (xy)^\# = y^\# x^\#$, so $a \in R^{\mathrm{EP}}$ by Theorem 2.8.

(1) $\Leftrightarrow$ (3). Assume that $a \in R^{\mathrm{EP}}$, then $x, y \in R^{\mathrm{EP}}$ by Corollary 2.3. Hence $(xy)^\# = y^\# x^\# = y^\dagger x^\dagger$.

Conversely, the condition $(xy)^\# = y^\dagger x^\dagger$ is equivalent to $y^\# x^\# = y^\dagger x^\dagger$, pre-multiplying the equality by y, we get $yy^\# x^\# = yy^\dagger x^\dagger$, i.e., $xx^\# x^\# = xx^\dagger x^\dagger$, which yields $x^\# = xx^\dagger x^\dagger$. Moreover,

$$xx^\# = x(xx^\dagger x^\dagger) = (xxx^\dagger x^\dagger)xx^\dagger = xx^\# xx^\dagger = xx^\dagger,$$

so $x \in R^{\mathrm{EP}}$. Hence $a \in R^{\mathrm{EP}}$ by Corollary 2.3. □

Corollary 2.10. *Let $a \in R^\dagger \cap R^\#$. If $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$, then $a \in R^{\mathrm{EP}}$ if and only if $(xy)^\# = y^\# x^\#$.*

Proof. According to Corollary 2.6, $xy \in R^{\mathrm{EP}}$ When $x \in \tau_a, y \in \gamma_a$ or $x \in \gamma_a, y \in \tau_a$, thus $(xy)^\# = (xy)^\dagger$. Therefore, $(xy)^\# = y^\# x^\#$ if and only if $(xy)^\dagger = y^\# x^\#$, which is equivalent to the condition $a \in R^{\mathrm{EP}}$ by Theorem 2.8. □

3. Characterizations of Hermitian Element

An element $a \in R$ is called Hermitian (or symmetric) if $a = a^*$. In this section, we characterize Hermitian element by hybrid reverse order laws.

Lemma 3.1. *[4, Theorem 3.9] Suppose that $a \in R^\#$. Then $a \in R^{\mathrm{EP}}$ if and only if one of the following equivalent conditions holds:*
(1) $aR \subseteq a^*R$;
(2) $a^*R \subseteq aR$;
(3) $Ra \subseteq Ra^*$;
(4) $Ra^* \subseteq Ra$.

Lemma 3.2. *[5, Theorem 1.4.2] Suppose that $a \in R^\dagger \cap R^\#$. Then $a \in R^{\mathrm{Her}}$ if and only if one of the following equivalent conditions holds:*
(1) $aa^\# = a^* a^\dagger$;
(2) $aa^\# = a^\dagger a^*$;
(3) $aa^* a^\dagger = a$.

From the definition of Hermitian element, we immediately get the following lemma.

Lemma 3.3. *Let $a \in R^\dagger \cap R^\#$ and $w \in \chi_a$. Then $a \in R^{\mathrm{Her}}$ if and only if $w \in R^{\mathrm{Her}}$.*

Therorem 2.8 give the characterizations of EP element by hybrid reverse order laws. Inspired by it, we made some changes to hybrid reverse order laws in Therorem 2.8 and found the following interesting results.

Theorem 3.4. *Let $a \in R^{\dagger} \cap R^{\#}$ and $x, y \in \chi_a$. Then the following conditions are equivalent:*
(1) $a \in R^{\mathrm{Her}}$;
(2) $(xy)^{\dagger} = (y^*)^{\dagger}x^{\#}$;
(3) $(xy)^{\dagger} = y^{\#}(x^*)^{\dagger}$;
(4) $(xy)^{\dagger} = (y^*)^{\dagger}x^{\dagger}$;
(5) $(xy)^{\dagger} = y^{\dagger}(x^*)^{\dagger}$.

Proof. (1) ⇒ (2)(3)(4)(5). Let $a \in R^{\dagger} \cap R^{\#}$. If $a \in R^{\mathrm{Her}}$, then $a \in R^{\mathrm{EP}}$ by Lemma 3.1. In this case, $x, y \in R^{\mathrm{Her}}$, so $x = x^*$ and $y = y^*$, therefore, conditions (2) and (3) are valid according to Theorem 2.8.

In addition, when $a \in R^{\mathrm{EP}} \cap R^{\mathrm{Her}}$, $x, y \in R^{\mathrm{EP}} \cap R^{\mathrm{Her}}$. Thus from Theorem 2.5, we have $(xy)^{\dagger} = y^{\dagger}x^{\#}xx^{\dagger} = y^{\dagger}x^{\dagger} = (y^*)^{\dagger}x^{\dagger} = y^{\dagger}(x^*)^{\dagger}$, which shows that conditions (4) and (5) are valid.

Conversely, if we want to prove that $a \in R^{\mathrm{Her}}$, we just need to prove that $x \in R^{\mathrm{Her}}$ or $y \in R^{\mathrm{Her}}$ by Lemma 3.3.

(2) ⇒ (1). The assumption $(xy)^{\dagger} = (y^*)^{\dagger}x^{\#}$ can be equivalently written as $y^{\dagger}x^{\#}xx^{\dagger} = (y^*)^{\dagger}x^{\#}$, post-multiplying the equality by x, we have $y^{\dagger}x^{\#}xx^{\dagger}x = (y^*)^{\dagger}x^{\#}x$, i.e.,

$$y^{\dagger}x^{\#}x = (y^*)^{\dagger}x^{\#}x. \tag{3}$$

When $x, y \in \tau_a$ *or* $x, y \in \gamma_a$, Eq.(3) is equivalent to $y^{\dagger}y^{\#}y = (y^*)^{\dagger}y^{\#}y$, post-multiplying the equality by y we get $y^{\dagger}y^{\#}yy = (y^*)^{\dagger}y^{\#}yy$, it is further reduced to

$$y^{\dagger}y = (y^*)^{\dagger}y, \tag{4}$$

taking an involution on it we get $y^{\dagger}y = y^*y^{\dagger}$. Pre-multiplying the equality by y, we have $y = yy^*y^{\dagger}$, thus $y \in R^{\mathrm{Her}}$ according to Lemma 3.2.

When $x \in \tau_a, y \in \gamma_a$ *or* $x \in \gamma_a, y \in \tau_a$, Eq.(3) is equivalent to

$$y^{\dagger}(yy^{\#})^* = (y^*)^{\dagger}(yy^{\#})^*.$$

Post-multiplying the equality by y^* we get

$$y^{\dagger}(yy^{\#})^*y^* = (y^*)^{\dagger}(yy^{\#})^*y^*,$$

which can be simplified to

$$y^{\dagger}y^* = yy^{\dagger}. \tag{5}$$

Hence

$$y = yy^{\dagger}y = y^{\dagger}y^*y = y^*(y^{\dagger})^*y^{\dagger}y^*y \in y^*R,$$

which yields that $y \in R^{\mathrm{EP}}$ by Lemma 3.1. Moreover, from Eq.(5) we obtain $yy^{\#} = yy^{\dagger} = y^{\dagger}y^*$, thus $y \in R^{\mathrm{Her}}$ according to Lemma 3.2.

(3) ⇒ (1). Since $x, y \in \chi_a$, $x^*, y^* \in \chi_a$. Suppose that $(xy)^{\dagger} = y^{\#}(x^*)^{\dagger}$, taking an involution on it, we get $(y^*x^*)^{\dagger} = ((x^*)^*)^{\dagger}(y^*)^{\#}$, from the proof of (2) ⇒ (1) we obtain that $x^* \in R^{\mathrm{Her}}$, thus $x \in R^{\mathrm{Her}}$.

(4) ⇒ (1) Suppose that $(xy)^{\dagger} = (y^*)^{\dagger}x^{\dagger}$, then $y^{\dagger}x^{\#}xx^{\dagger} = (y^*)^{\dagger}x^{\dagger}$. Post-multiplying the equality by xy we have

$$y^{\dagger}x^{\#}xx^{\dagger}xy = (y^*)^{\dagger}x^{\dagger}xy,$$

i.e.,

$$y^{\dagger}x^{\#}xy = (y^*)^{\dagger}x^{\dagger}xy.$$

According to Corollary 2.4, the above equality can be written as

$$\begin{cases} y^\dagger y^\# yy = (y^*)^\dagger y^\dagger yy, & x, y \in \tau_a \ \ or \ \ x, y \in \gamma_a, \\ y^\dagger (yy^\#)^* y = (y^*)^\dagger yy^\dagger y, & x \in \tau_a, y \in \gamma_a \ \ or \ \ x \in \gamma_a, y \in \tau_a. \end{cases}$$

Both of these two equations can be simplified to $y^\dagger y = (y^*)^\dagger y$, which is the same as Eq.(4), thus $y \in R^{\mathrm{Her}}$ by the proof of (2) ⇒ (1).

(5) ⇒ (1). Since $x, y \in \chi_a$, $x^*, y^* \in \chi_a$. Assume that $(xy)^\dagger = y^\dagger (x^*)^\dagger$, taking an involution on it, we get $(y^* x^*)^\dagger = ((x^*)^*)^\dagger (y^*)^\dagger$, from the proof in (4) ⇒ (1) we obtain that $x^* \in R^{\mathrm{Her}}$, so $x \in R^{\mathrm{Her}}$. □

Theorem 3.5. *Let $a \in R^\dagger \cap R^\#$ and $x, y \in \tau_a$ or $x, y \in \gamma_a$. Then the following conditions are equivalent:*
(1) $a \in R^{\mathrm{Her}}$;
(2) $(xy)^\# = (y^*)^\# x^\#$;
(3) $(xy)^\# = y^\# (x^*)^\#$;
(4) $(xy)^\# = (y^*)^\# x^\dagger$;
(5) $(xy)^\# = y^\dagger (x^*)^\#$.

Proof. Let $a \in R^\dagger \cap R^\#$, if $a \in R^{\mathrm{Her}}$, then $a \in R^{\mathrm{EP}}$ by Lemma 3.1. From Theorem 2.5 we know that $(xy)^\# = y^\# x^\#$ when $x, y \in \tau_a$ or $x, y \in \gamma_a$.

(1) ⇒ (2)(3)(4)(5). Obviously.

(2) ⇒ (1). Suppose that $(xy)^\# = (y^*)^\# x^\#$, then $y^\# x^\# = (y^*)^\# x^\#$. Post-multiplying the equality by x we have

$$y^\# x^\# x = (y^*)^\# x^\# x,$$

which is equivalent to

$$y^\# y^\# y = (y^*)^\# y^\# y$$

when $x, y \in \tau_a$ or $x, y \in \gamma_a$. The above equality can be further simplified as

$$y^\# = (y^\#)^* y^\# y, \tag{6}$$

Thus

$$y = y^\# y^2 = (y^\#)^* y^\# y y^2 = (y^\#)^* y^2 = y^* (y^\# y^\#)^* y^2 \in y^* R,$$

which yields $y \in R^{\mathrm{EP}}$. In this case, Eq.(6) can be written as $y^\dagger = (y^\dagger)^* y^\dagger y = (y^\dagger)^*$, thus $y = (y^\dagger)^\dagger = ((y^\dagger)^*)^\dagger = y^*$ shows that $y \in R^{\mathrm{Her}}$.

(3) ⇒ (1). Since $x, y \in \chi_a$, $x^*, y^* \in \chi_a$. Taking an involution on the hypothesis $(xy)^\# = y^\# (x^*)^\#$, we get $(y^* x^*)^\# = ((x^*)^*)^\# (y^*)^\#$, from the proof in (2) ⇒ (1) we obtain that $x^* \in R^{\mathrm{Her}}$, so $x \in R^{\mathrm{Her}}$.

(4) ⇒ (1). Assume that $(xy)^\# = (y^*)^\# x^\dagger$, then $y^\# x^\# = (y^*)^\# x^\dagger$. Post-multiplying the equality by x we have

$$y^\# x^\# x = (y^*)^\# x^\dagger x,$$

i.e.,

$$y^\# y^\# y = (y^*)^\# y^\dagger y,$$

where $x, y \in \tau_a$ or $x, y \in \gamma_a$. the above formula can be simplified as

$$y^\# = (y^*)^\# y^\dagger y. \tag{7}$$

Furthermore,

$$y = y^\# y^2 = (y^*)^\# y^\dagger y y^2 = (y^\#)^* y^\dagger y^3 = y^* (y^\# y^\#)^* y^\dagger y^3 \in y^* R,$$

so $a \in R^{\mathrm{EP}}$. Moreover, Eq.(7) can be written as $y^\dagger = (y^\dagger)^* y^\dagger y = (y^\dagger)^*$, Hence $y \in R^{\mathrm{Her}}$.

(5) ⇒ (1). Since $x, y \in \chi_a$, $x^*, y^* \in \chi_a$. Taking an involution on the assumption $(xy)^\# = y^\dagger (x^*)^\#$ we get $(y^* x^*)^\# = ((x^*)^*)^\# (y^*)^\dagger$. According to the proof of (4) ⇒ (1), we obtain that $x^* \in R^{\mathrm{Her}}$, thus $x \in R^{\mathrm{Her}}$. □

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