Filomat 39:3 (2025), 809–818
https://doi.org/10.2298/FIL2503809Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some remarks on ϕ-Dedekind rings and ϕ-Prüfer rings

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Abstract. In this paper, we introduce and study the concepts of nonnil-injective modules and nonnil-FP-injective modules. Specifically, we show that a ϕ-ring R is an integral domain if and only if every nonnil-injective (resp., nonnil-FP-injective) R-module is injective (resp., FP-injective). Furthermore, we provide new characterizations of ϕ-von Neumann regular rings, nonnil-Noetherian rings, and nonnil-coherent rings. Lastly, we characterize ϕ-Dedekind rings and ϕ-Prüfer rings in terms of ϕ-flat modules, nonnil-injective modules, and nonnil-FP-injective modules.

1. Introduction

Recall from [5] that a commutative ring R is called an NP-*ring* if its nilpotent radical $\mathrm{Nil}(R)$ is a prime ideal, and a ZN-*ring* if $\mathrm{Z}(R) = \mathrm{Nil}(R)$, where $\mathrm{Z}(R)$ denotes the set of all zero-divisors of R. A prime ideal P of R is termed a *divided prime* if $P \subsetneq (x)$ for every $x \in R \setminus P$. Let

$$\mathcal{H} = \{R \mid R \text{ is a commutative ring and } \mathrm{Nil}(R) \text{ is a divided prime ideal of } R\}.$$

A ring R is referred to as a ϕ-*ring* if $R \in \mathcal{H}$. Moreover, a ZN ϕ-ring is called a *strong* ϕ-*ring*. Denote by $\mathrm{T}(R)$ the localization of R at the set of all regular elements. For a ϕ-ring R, there is a ring homomorphism $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ defined by $\phi(a/b) = a/b$. The image of ϕ restricted to R is denoted by $\phi(R)$.

In 2001, Badawi [6] investigated ϕ-*chained rings* (abbreviated as ϕ-CRs), which are ϕ-rings R such that for every $x, y \in R \setminus \mathrm{Nil}(R)$, either x divides y or y divides x. In 2004, Anderson and Badawi [1] extended the notion of Prüfer domains to ϕ-*Prüfer rings*, which are ϕ-rings R such that every finitely generated nonnil ideal is ϕ-invertible. The authors in [1] provided several characterizations of ϕ-Prüfer rings, stating that a ϕ-ring R is ϕ-Prüfer if and only if $R_{\mathfrak{m}}$ is a ϕ-chained ring for every maximal ideal $\mathfrak{m}$ of R, if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is Prüfer.

Later, in 2005, the authors in [2] generalized the concept of Dedekind domains to the context of rings in the class $\mathcal{H}$. A ϕ-ring is called a ϕ-*Dedekind ring* if every nonnil ideal is ϕ-invertible. They also proved that a ϕ-ring R is ϕ-Dedekind if and only if R is nonnil-Noetherian and $R_{\mathfrak{m}}$ is a discrete ϕ-chained ring for every maximal ideal $\mathfrak{m}$ of R, if and only if R is nonnil-Noetherian, ϕ-integrally closed, and has Krull

2020 *Mathematics Subject Classification*. Primary 13A15; Secondary 13F05.
Keywords. nonnil-injective module; nonnil-FP-injective module; ϕ-Dedekind ring; ϕ-Prüfer ring.
Received: 09 April 2024; Revised: 04 October 2024; Accepted: 13 October 2024
Communicated by Dijana Mosić
The first author was supported by the National Natural Science Foundation of China (No. 12061001). The second author was supported by the National Natural Science Foundation of China (No. 12201361).
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dimension ≤ 1, if and only if $R/\mathrm{Nil}(R)$ is a Dedekind domain. Generalizations of Noetherian domains, coherent domains, Bézout domains, and Krull domains to the context of rings in $\mathcal{H}$ were also introduced and studied (see [1, 2, 4, 7, 8]).

The module-theoretic study of rings in $\mathcal{H}$ began over a decade ago. In 2006, Yang [19] introduced nonnil-injective modules by replacing the ideals in Baer's criterion for injective modules with nonnil ideals. He showed that a ϕ-ring R is nonnil-Noetherian if and only if any direct sum of nonnil-injective modules is nonnil-injective. In 2013, Zhao et al. [23] introduced and studied the concept of *ϕ-von Neumann rings*, which can be characterized as follows: a ϕ-ring R is ϕ-von Neumann if and only if its Krull dimension is 0, if and only if every R-module is ϕ-flat, if and only if $R/\mathrm{Nil}(R)$ is a von Neumann regular ring. In 2018, Zhao [22] provided a homological characterization of ϕ-Prüfer rings: a strong ϕ-ring R is ϕ-Prüfer if and only if every submodule of a ϕ-flat module is ϕ-flat, if and only if every nonnil ideal of R is ϕ-flat.

The main motivation of this paper is to provide characterizations of ϕ-Dedekind rings and ϕ-Prüfer rings in terms of new versions of injective modules and FP-injective modules. We first introduce and study the notions of nonnil-injective modules and nonnil-FP-injective modules, and show that a ϕ-ring R is an integral domain if and only if every nonnil-injective R-module is injective, and if and only if every nonnil-FP-injective R-module is FP-injective (see Theorem 2.6). Additionally, new characterizations of ϕ-von Neumann regular rings, nonnil-Noetherian rings, and nonnil-coherent rings in terms of ϕ-flat modules, nonnil-injective modules, and nonnil-FP-injective modules are given (see Theorem 2.7, Proposition 2.8, and Proposition 2.9 respectively).

We further prove that a strong ϕ-ring R is a ϕ-Dedekind ring if and only if every divisible module is nonnil-injective, if and only if every h-divisible module is nonnil-injective, and if and only if every nonnil ideal of R is projective (see Theorem 3.8). Additionally, we show that a strong ϕ-ring R is ϕ-Prüfer if and only if every divisible module is nonnil-FP-injective, if and only if every finitely generated nonnil ideal of R is projective, if and only if every ideal of R is ϕ-flat, and if and only if every R-module has an epimorphic ϕ-flat envelope (see Theorem 3.13).

2. Nonnil-injective modules and nonnil-FP-injective modules

Throughout this paper, R denotes an NP-ring with identity, and all modules are unitary. We say that an ideal I of R is *nonnil* if there exists a non-nilpotent element in I. Denote by $\mathrm{NN}(R)$ the set of all nonnil ideals of R. It is easy to verify that $\mathrm{NN}(R)$ forms a multiplicative system of ideals, i.e., $R \in \mathrm{NN}(R)$ and $IJ \in \mathrm{NN}(R)$ for any $I, J \in \mathrm{NN}(R)$.

Let M be an R-module. Define

$$\phi\text{-tor}(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in \mathrm{NN}(R)\}.$$

An R-module M is said to be *ϕ-torsion* (resp., *ϕ-torsion free*) provided that $\phi\text{-tor}(M) = M$ (resp., $\phi\text{-tor}(M) = 0$). Clearly, the class of ϕ-torsion modules is closed under submodules, quotients, direct sums, and direct limits. Thus, an NP-ring R is ϕ-torsion free if and only if every flat module is ϕ-torsion free, and if and only if R is a ZN-ring (see [22, Proposition 2.2]). The classes of ϕ-torsion modules and ϕ-torsion free modules constitute a hereditary torsion theory of finite type. Recall that an ideal I of R is regular if there exists a regular element (i.e., a non-zero-divisor) in I.

Lemma 2.1. *Let R be a ϕ-ring, and let I be an ideal of R. Then the following assertions are equivalent:*

1. *I is a nonnil ideal of R;*
2. *$I/\mathrm{Nil}(R)$ is a nonzero ideal of $R/\mathrm{Nil}(R)$;*
3. *$\phi(I)$ is a regular ideal of $\phi(R)$.*

Proof. (1) $\Leftrightarrow$ (2): Obvious.

(1) $\Rightarrow$ (3): Let s be a non-nilpotent element in I. Then $\frac{s}{1} \in \phi(I)$ is regular in $\phi(R)$. Indeed, suppose $\frac{s}{1}\frac{t}{1} = 0$ in $\phi(R)$. Then there exists a non-nilpotent element $u \in R$ such that $ust = 0$. Since R is a ϕ-ring, us is non-nilpotent. Thus, $\frac{t}{1} = 0$ in $\phi(R)$.

(3) $\Rightarrow$ (1): Let $\frac{s}{1}$ be a regular element in $\phi(I)$ with $s \in I$. Then s is non-nilpotent. Indeed, if $s^n = 0$ in R, then $(\frac{s}{1})^n = \frac{s^n}{1} = 0$ in $\phi(R)$, which implies that $\frac{s}{1}$ is not regular in $\phi(R)$. □

Recall that an R-module M is *injective* (resp., *FP-injective*) if $\mathrm{Ext}_R^1(N, M) = 0$ for any (resp., finitely presented) R-module N. We now investigate the notions of nonnil-injective modules and nonnil-FP-injective modules using ϕ-torsion modules.

Definition 2.2. Let R be an NP-ring, and let M be an R-module.

1. M is called *nonnil-injective* if $\mathrm{Ext}_R^1(T, M) = 0$ for any ϕ-torsion module T.
2. M is called *nonnil-FP-injective* if $\mathrm{Ext}_R^1(T, M) = 0$ for any finitely presented ϕ-torsion module T.

Certainly, an R-module M is nonnil-injective if and only if $\mathrm{Ext}_R^1(R/I, M) = 0$ for any nonnil ideal I of R (see [24, Theorem 1.7]). The class of nonnil-injective modules is closed under direct summands, direct products, and extensions, while the class of nonnil-FP-injective modules is closed under pure submodules, direct sums, direct products, and extensions.

Recall from [23] that an R-module M is *ϕ-flat* if $\mathrm{Tor}_1^R(T, M) = 0$ for any ϕ-torsion module T. It is well-known that an R-module M is ϕ-flat if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for any (finitely generated) nonnil ideal I of R (see [23, Theorem 3.2]).

Proposition 2.3. *Let R be an* NP*-ring. The following assertions are equivalent:*

1. *M is ϕ-flat;*
2. $\mathrm{Hom}_R(M, E)$ *is nonnil-injective for any injective module E;*
3. $\mathrm{Hom}_R(M, E)$ *is nonnil-FP-injective for any injective module E;*
4. *If E is an injective cogenerator, then* $\mathrm{Hom}_R(M, E)$ *is nonnil-injective;*
5. *If E is an injective cogenerator, then* $\mathrm{Hom}_R(M, E)$ *is nonnil-FP-injective.*

Proof. (1) ⇒ (2): Let T be a ϕ-torsion R-module and E an injective R-module. Since M is ϕ-flat, we have $\mathrm{Ext}_R^1(T, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(T, M), E) = 0$. Thus, $\mathrm{Hom}_R(M, E)$ is nonnil-injective.

(2) ⇒ (3) ⇒ (5): Trivial.

(2) ⇒ (4) ⇒ (5): Trivial.

(5) ⇒ (1): Let I be a finitely generated nonnil ideal of R and E an injective cogenerator. Since $\mathrm{Hom}_R(M, E)$ is nonnil-FP-injective, we have $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, M), E) \cong \mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(M, E)) = 0$. Since E is an injective cogenerator, it follows that $\mathrm{Tor}_1^R(R/I, M) = 0$. Therefore, M is ϕ-flat. □

Proposition 2.4. *Let R be a ϕ-ring, and let E be an* $R/\mathrm{Nil}(R)$*-module. Then E is injective over* $R/\mathrm{Nil}(R)$ *if and only if E is nonnil-injective over R.*

Proof. Let I be a nonnil ideal of R. Set $\overline{R} = R/\mathrm{Nil}(R)$ and $\overline{I} = I/\mathrm{Nil}(R)$. Let E be an $\overline{R}$-module. The short exact sequence $0 \to I \to R \to R/I \to 0$ induces the long exact sequence of R-modules:

$$0 \to \mathrm{Hom}_R(R/I, E) \to \mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(I, E) \to \mathrm{Ext}_R^1(R/I, E) \to 0. \qquad (a)$$

The short exact sequence $0 \to \overline{I} \to \overline{R} \to R/I \to 0$ induces the long exact sequence of $\overline{R}$-modules:

$$0 \to \mathrm{Hom}_{\overline{R}}(R/I, E) \to \mathrm{Hom}_{\overline{R}}(\overline{R}, E) \to \mathrm{Hom}_{\overline{R}}(\overline{I}, E) \to \mathrm{Ext}_{\overline{R}}^1(R/I, E) \to 0. \qquad (b)$$

By [21, Lemma 1.6], $I\mathrm{Nil}(R) = \mathrm{Nil}(R)$. Thus, $I \otimes_R \overline{R} \cong I/I\mathrm{Nil}(R) \cong \overline{I}$. Consequently, we have

$$\mathrm{Hom}_{\overline{R}}(\overline{I}, E) \cong \mathrm{Hom}_{\overline{R}}(I \otimes_R \overline{R}, E) \cong \mathrm{Hom}_R(I, \mathrm{Hom}_{\overline{R}}(\overline{R}, E)) \cong \mathrm{Hom}_R(I, E)$$

by the Adjoint Isomorphism Theorem (see [18, Theorem 2.2.16]). Combining (a) and (b), we conclude that E is injective over $R/\mathrm{Nil}(R)$ if and only if E is nonnil-injective over R (see Lemma 2.1 and [1, Lemma 2.4]). □

Proposition 2.5. *Let R be a ϕ-ring, and let M be an FP-injective* $R/\mathrm{Nil}(R)$*-module. Then M is nonnil-FP-injective over R.*

Proof. Let T be a finitely presented ϕ-torsion module over R. Then there exists a short exact sequence $0 \to K \to F \to T \to 0$, where F is a finitely generated free R-module and K is a finitely generated R-module. Set $\overline{R} = R/\mathrm{Nil}(R)$. By tensoring $\overline{R}$ over R, we obtain the following long exact sequence over $\overline{R}$:

$$\mathrm{Tor}_1^R(T, \overline{R}) \to K \otimes_R \overline{R} \to F \otimes_R \overline{R} \to T \otimes_R \overline{R} \to 0.$$

By [21, Proposition 1.7], $\overline{R}$ is ϕ-flat over R, thus $\mathrm{Tor}_1^R(T, \overline{R}) = 0$. It follows that $T \otimes_R \overline{R}$ is a finitely presented $\overline{R}$-module. We now have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F, M) & \longrightarrow & \mathrm{Hom}_R(K, M) & \longrightarrow & \mathrm{Ext}_R^1(T, M) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow f & & \\
\mathrm{Hom}_{\overline{R}}(F \otimes_R \overline{R}, M) & \longrightarrow & \mathrm{Hom}_{\overline{R}}(K \otimes_R \overline{R}, M) & \longrightarrow & \mathrm{Ext}_{\overline{R}}^1(T \otimes_R \overline{R}, M) & \longrightarrow & 0.
\end{array}$$

By the Adjoint Isomorphism Theorem, the first two vertical homomorphisms are isomorphisms. By the five Lemma, it follows that f is also an isomorphism. Since M is FP-injective over $\overline{R}$, we have $\mathrm{Ext}_{\overline{R}}^1(T \otimes_R \overline{R}, M) = 0$. Thus, $\mathrm{Ext}_R^1(T, M) = 0$, and hence M is nonnil-FP-injective over R. □

Obviously, any FP-injective module is nonnil-FP-injective, and any injective module is nonnil-injective. However, the converses characterize integral domains.

Theorem 2.6. *Let R be a ϕ-ring. Then the following assertions are equivalent:*

1. *R is an integral domain;*
2. *Any nonnil-injective module is injective;*
3. *Any nonnil-FP-injective module is FP-injective.*

Proof. (1) ⇒ (2) and (1) ⇒ (3): Trivial.

(2) ⇒ (1): By [9, Theorem 3.1.6], $\mathrm{Hom}_{\mathbb{Z}}(R/\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is an injective $R/\mathrm{Nil}(R)$-module. Thus, by Proposition 2.4, $\mathrm{Hom}_{\mathbb{Z}}(R/\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is a nonnil-injective R-module, and therefore an injective R-module. By [9, Theorem 3.2.10], $R/\mathrm{Nil}(R)$ is a flat R-module. Let K be a finitely generated nilpotent ideal. Then $K \subseteq \mathrm{Nil}(R) \subseteq \mathrm{Rad}(R)$. Thus, $K/K\mathrm{Nil}(R) = \frac{K \cap \mathrm{Nil}(R)}{K\mathrm{Nil}(R)} = \mathrm{Tor}_1^R(R/K, R/\mathrm{Nil}(R)) = 0$. It follows from Nakayama's Lemma that $K = 0$. Therefore, $\mathrm{Nil}(R) = 0$, and thus R is an integral domain.

(3) ⇒ (1): Similar to (2) ⇒ (1). □

Recall from [23] that a ϕ-ring R is said to be ϕ-von Neumann if the Krull dimension of R is 0. It is well known that a ϕ-ring R is ϕ-von Neumann if and only if $R/\mathrm{Nil}(R)$ is a von Neumann ring, and if and only if any R-module is ϕ-flat (see [23, Theorem 4.1]).

Theorem 2.7. *Let R be a ϕ-ring. Then the following assertions are equivalent:*

1. *R is a ϕ-von Neumann regular ring;*
2. *$R/\mathrm{Nil}(R)$ is a field;*
3. *Any non-nilpotent element in R is invertible;*
4. *Any R-module is ϕ-flat;*
5. *Any R-module is nonnil-FP-injective;*
6. *Any R-module is nonnil-injective.*

Proof. (1) ⇔ (4): See [23, Theorem 4.1].

(1) ⇒ (2): Since $\mathrm{Nil}(R)$ is a prime ideal of R, $R/\mathrm{Nil}(R)$ is a 0-dimensional domain, and thus a field by [12, Theorem 3.1].

(2) ⇒ (3): Let a be a non-nilpotent element in R. Since $R/\mathrm{Nil}(R)$ is a field, there exists $b \in R$ such that $1 - ab \in \mathrm{Nil}(R)$. That is, $(1 - ab)^n = 0$ for some n. It is easy to verify that a is invertible.

(3) ⇒ (2) ⇒ (1): Trivial.

(3) ⇒ (5): It follows from (3) that the only nonnil ideal of R is R itself. Let T be a finitely presented ϕ-torsion module. Then $T = \phi\text{-tor}(T) = \{x \in T \mid Ix = 0 \text{ for some nonnil ideal } I \text{ of } R\} = 0$. Hence, $\mathrm{Ext}^1_R(T, M) = 0$. Consequently, M is nonnil-FP-injective.

(5) ⇒ (1): Let I be a finitely generated nonnil ideal of R. Since, for any R-module M, $\mathrm{Ext}^1_R(R/I, M) = 0$ by (5), R/I is projective. Thus, I is an idempotent ideal of R. By [10, Proposition 1.10], I is generated by an idempotent $e \in R$. Therefore, R is a ϕ-von Neumann regular ring by [23, Theorem 4.1].

(3) ⇒ (6) and (6) ⇒ (5): Obvious. □

Recall from [7] that a ϕ-ring R is called *nonnil-Noetherian* if any nonnil ideal of R is finitely generated.

Proposition 2.8. *Let R be a ϕ-ring. Then R is nonnil-Noetherian if and only if any nonnil-FP-injective module is nonnil-injective.*

Proof. Suppose R is a nonnil-Noetherian ring. Let I be a nonnil ideal of R and M a nonnil-FP-injective module. Then I is finitely generated, and thus R/I is a finitely presented ϕ-torsion module. It follows that $\mathrm{Ext}^1_R(R/I, M) = 0$. Consequently, M is nonnil-injective by [24, Theorem 1.7].

Conversely, since the class of nonnil-FP-injective modules is closed under direct sums, R is a nonnil-Noetherian ring by [19, Theorem 1.9]. □

Recall from [4] that a ϕ-ring R is called *nonnil-coherent* if any finitely generated nonnil ideal of R is finitely presented. A ϕ-ring R is nonnil-coherent if and only if any direct product of ϕ-flat modules is ϕ-flat, and if and only if R^I is ϕ-flat for any indexing set I (see [4, Theorem 2.4]). Now we give a new characterization of nonnil-coherent rings using the preenveloping properties of ϕ-flat modules.

Proposition 2.9. *Let R be a ϕ-ring. Then R is nonnil-coherent if and only if the class of ϕ-flat modules is preenveloping.*

Proof. Suppose R is a nonnil-coherent ring. By [4, Theorem 2.4], the class of ϕ-flat modules is closed under direct products. Note that any pure submodule of a ϕ-flat module is ϕ-flat. Thus, the class of ϕ-flat modules is preenveloping by [9, Lemma 5.3.12, Corollary 6.2.2].

Conversely, let $\{F_i\}_{i \in I}$ be a family of ϕ-flat modules. Let $\prod_{i \in I} F_i \to F$ be a ϕ-flat preenvelope. Then there is a factorization $\prod_{i \in I} F_i \to F \to F_i$ for each $i \in I$. Consequently, the natural composition $\prod_{i \in I} F_i \to F \to \prod_{i \in I} F_i$ is the identity. Thus, $\prod_{i \in I} F_i$ is a direct summand of F, and hence $\prod_{i \in I} F_i$ is ϕ-flat. It follows from [4, Theorem 2.4] that R is nonnil-coherent. □

The following corollary follows from Theorem 3.8 and [9, Corollary 6.3.5].

Corollary 2.10. *Let R be a nonnil-coherent ring. If the class of ϕ-flat modules is closed under inverse limits, then the class of ϕ-flat modules is enveloping.*

3. ϕ-Dedekind rings and ϕ-Prüfer rings

Recall that an R-module E is said to be *divisible* if $sM = M$ for any regular element $s \in R$, and an R-module M is said to be *h-divisible* provided that M is a quotient of an injective module. Evidently, any injective module is h-divisible and any h-divisible module is divisible. It is well known that an integral domain R is a Dedekind domain if and only if any h-divisible module is injective, and if and only if any divisible module is injective (see [18, Theorem 5.2.15], for example).

Definition 3.1. Let R be an NP-ring. An R-module E is called *nonnil-divisible* provided that for any $m \in E$ and any non-nilpotent element $a \in R$, there exists $x \in E$ such that $ax = m$.

Lemma 3.2. *Let R be an* NP*-ring and E an R-module. Consider the following statements:*

1. *E is nonnil-divisible;*
2. *E is divisible;*
3. $\mathrm{Ext}_R^1(R/\langle a\rangle, E) = 0$ *for any* $a \notin \mathrm{Nil}(R)$.

Then we have (1) ⇒ (2) *and* (1) ⇒ (3)*. Moreover, if R is a* ZN*-ring, all statements are equivalent.*

Proof. (1) ⇒ (2) and (2) ⇒ (1) for ZN-rings: Trivial.

(1) ⇒ (3): Let a be a non-nilpotent element (then regular) in R and $f : \langle a\rangle \to E$ be an R-homomorphism. Then there exists an element $x \in E$ such that $f(a) = ax$ since E is nonnil-divisible. Set $g(r) = rx$ for any $r \in R$. Then g is an extension of f to R. Thus $\mathrm{Ext}_R^1(R/\langle a\rangle, E) = 0$.

(3) ⇒ (1) for ZN-rings: Let a be a non-nilpotent element in R and m an element in E. Set $f(ra) = rm$. Then f is a well-defined R-homomorphism from $\langle a\rangle$ to E. Since $\mathrm{Ext}_R^1(R/\langle a\rangle, E) = 0$, there exists an R-homomorphism $g : R \to E$ such that $g|_{\langle a\rangle} = f$. Let $x = g(1)$. Then $m = f(a) = g(a) = ag(1) = ax$. Thus, E is nonnil-divisible. □

The following result is an easy corollary of Lemma 3.2.

Corollary 3.3. *Let R be a* ZN*-ring, and let E be a nonnil-FP-injective R-module. Then E is a nonnil-divisible R-module.*

Lemma 3.4. *Let R be an* NP*-ring, and let E be a nonnil-divisible R-module. Then $E_\mathfrak{p}$ is a nonnil-divisible $R_\mathfrak{p}$-module for any prime ideal $\mathfrak{p}$ of R.*

Proof. Suppose E is a nonnil-divisible R-module. Let $\frac{m}{s}$ be an element in $E_\mathfrak{p}$ and $\frac{r}{t}$ a non-nilpotent element in $R_\mathfrak{p}$. Then s, t, and r are non-nilpotent elements in R. Thus, there exists $y \in E$ such that $tm = sry$ in R. Therefore, $\frac{m}{s} = \frac{r}{t}\frac{y}{1}$. It follows that $E_\mathfrak{p}$ is a nonnil-divisible $R_\mathfrak{p}$-module. □

Recall from [1] that a ϕ-ring R is called a *ϕ-chained ring* if for every $x \in R_{\mathrm{Nil}(R)} - \phi(R)$, we have $x^{-1} \in \phi(R)$, equivalently, if for any $a, b \in R - \mathrm{Nil}(R)$, either $a \mid b$ or $b \mid a$ in R. Moreover, a ϕ-ring R is said to be a *discrete ϕ-chained ring* if R is a ϕ-chained ring with at most one nonnil prime ideal and every nonnil ideal of R is principal (see [2]).

Proposition 3.5. *Let R be a discrete ϕ-chained ring, and let E be a nonnil-divisible R-module. Then E is a nonnil-injective R-module.*

Proof. Let I be a nonnil ideal of R. Since R is a discrete ϕ-chained ring, I is generated by a non-nilpotent element $a \in R$. Let $f : I \to E$ be an R-homomorphism. Then there exists $x \in E$ such that $f(a) = ax$ as E is divisible. Define $g : R \to E$ by $g(r) = rx$. Then g is an extension of f to R. Hence, E is a nonnil-injective R-module. □

Recall that a regular ideal I of R is called *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in \mathrm{T}(R) \mid Ix \subseteq R\}$. It follows from [12, Lemma 18.1] and [11, Lemma 5.3] that a regular ideal is invertible if and only if it is finitely generated and locally principal, and if and only if it is projective. Recall from [1] that a nonnil ideal I of a ϕ-ring R is said to be *ϕ-invertible* provided that $\phi(I)$ is an invertible ideal of $\phi(R)$.

Proposition 3.6. *Let R be a ϕ-ring, and let I be a nonnil ideal of R. If I is projective over R, then I is ϕ-invertible.*

Proof. Since I is a projective R-ideal, I is a direct summand of a free R-module $R^{(\kappa)}$. Then $\phi(I)$ is a direct summand of a free $\phi(R)$-module $\phi(R)^{(\kappa)}$. Thus, $\phi(I)$ is a projective $\phi(R)$-ideal. Since I is a nonnil ideal of R, $\phi(I)$ is a regular ideal of $\phi(R)$ by Lemma 2.1. By [11, Lemma 5.3], $\phi(I)$ is an invertible ideal of $\phi(R)$. Thus, I is ϕ-invertible. □

Recall that an integral domain R is a Dedekind domain if any nonzero ideal is invertible. Utilizing the concept of ϕ-invertibility, the authors in [2] introduced ϕ-Dedekind rings, which generalize Dedekind domains to the context of rings that are in the class $\mathcal{H}$.

Definition 3.7. A ϕ-ring R is called *ϕ-Dedekind* provided that any nonnil ideal of R is ϕ-invertible.

Theorem 3.8. *Let R be a ϕ-ring. Then the following statements are equivalent for R:*

1. *R is a ϕ-Dedekind ring and a strong ϕ-ring;*
2. *Any divisible module is nonnil-injective;*
3. *Any h-divisible module is nonnil-injective;*
4. *Any nonnil ideal of R is projective.*

Proof. (1) ⇒ (2): Let E be a divisible module, and let I be a nonnil ideal of R. By [2, Theorem 2.10], R is nonnil-Noetherian. Hence, I is finitely generated, and thus R/I is finitely presented. Let $\mathfrak{m}$ be a maximal ideal of R. Then $E_{\mathfrak{m}}$ is a divisible module over $R_{\mathfrak{m}}$ by Lemma 3.2 and Lemma 3.4. By [2, Theorem 2.10] again, $R_{\mathfrak{m}}$ is a discrete ϕ-chained ring, so $E_{\mathfrak{m}}$ is a nonnil-injective $R_{\mathfrak{m}}$-module by Proposition 3.5. By [18, Theorem 3.9.11], $\mathrm{Ext}^1_R(R/I,E)_{\mathfrak{m}} = \mathrm{Ext}^1_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/I_{\mathfrak{m}},E_{\mathfrak{m}}) = 0$. Thus, $\mathrm{Ext}^1_R(R/I,E) = 0$, and E is nonnil-injective.

(2) ⇒ (3): Trivial.

(3) ⇒ (4): Let N be an R-module, and let I be a nonnil ideal of R. There is a long exact sequence as follows:

$$0 = \mathrm{Ext}^1_R(R,N) \to \mathrm{Ext}^1_R(I,N) \to \mathrm{Ext}^2_R(R/I,N) \to \mathrm{Ext}^2_R(R,N) = 0.$$

Let $0 \to N \to E \to K \to 0$ be an exact sequence where E is the injective envelope of N. There exists a long exact sequence as follows:

$$0 = \mathrm{Ext}^1_R(R/I,E) \to \mathrm{Ext}^1_R(R/I,K) \to \mathrm{Ext}^2_R(R/I,N) \to \mathrm{Ext}^2_R(R/I,E) = 0.$$

Thus, $\mathrm{Ext}^1_R(I,N) \cong \mathrm{Ext}^2_R(R/I,N) \cong \mathrm{Ext}^1_R(R/I,K) = 0$ as K is nonnil-injective. It follows that I is a projective ideal of R.

(4) ⇒ (1): It follows from Proposition 3.6 that we only need to show that R is a strong ϕ-ring. Indeed, let a be a non-nilpotent element in R. Then $\langle a\rangle$ is a projective ideal of R. By [13, Corollary 2.6], R is a strong ϕ-ring. □

The next example shows that every divisible module is not necessarily nonnil-injective for ϕ-Dedekind rings. Thus, the condition that R is a strong ϕ-ring in Theorem 3.8 cannot be removed.

Example 3.9. Let D be a non-field Dedekind domain and K its quotient field. Let $R = D(+)K/D$ be the idealization construction. Then $\mathrm{Nil}(R) = 0(+)K/D$. Since $D \cong R/\mathrm{Nil}(R)$ is a Dedekind domain, R is a ϕ-Dedekind ring by [2, Theorem 2.5]. Denote by $U(R)$ and $U(D)$ the sets of unit elements of R and D, respectively. Since $\mathrm{Z}(R) = \{(r,m) \mid r \in \mathrm{Z}(D) \cup \mathrm{Z}(K/D)\} = [R - \mathrm{U}(D)](+)K/D = R - \mathrm{U}(R)$ by [3, Theorem 3.5, Theorem 3.7], R is a total ring of quotients. Thus, any R-module is divisible. However, since $\mathrm{Nil}(R)$ is not a maximal ideal of R, there exists an R-module M that is not nonnil-injective by Theorem 2.7.

Recall that an integral domain R is a Prüfer domain if any finitely generated nonzero ideal is invertible. The following definition generalizes Prüfer domains to the context of rings in the class $\mathcal{H}$ (see [1]).

Definition 3.10. A ϕ-ring R is called *ϕ-Prüfer* provided that any finitely generated nonnil ideal of R is ϕ-invertible.

Lemma 3.11. *Let R be an* NP*-ring, $\mathfrak{p}$ a prime ideal of R, and I an ideal of R. Then I is nonnil over R if and only if $I_{\mathfrak{p}}$ is nonnil over $R_{\mathfrak{p}}$.*

Proof. Let I be nonnil over R, and let x be a non-nilpotent element in I. We will show that the element $x/1$ in $I_{\mathfrak{p}}$ is non-nilpotent in $R_{\mathfrak{p}}$. If $(x/1)^n = x^n/1 = 0$ in $R_{\mathfrak{p}}$ for some positive integer n, there is an $s \in R - \mathfrak{p}$ such that $sx^n = 0$ in R. Since R is an NP-ring, $\mathrm{Nil}(R)$ is the minimal prime ideal of R. In the integral domain $R/\mathrm{Nil}(R)$, we have $\overline{sx^n} = \overline{0}$, thus $\overline{x^n} = \overline{0}$ since $s \notin \mathrm{Nil}(R)$. So $x \in \mathrm{Nil}(R)$, a contradiction.

Now, let x/s be a non-nilpotent element in $I_{\mathfrak{p}}$ where $x \in I$ and $s \in R - \mathfrak{p}$. Clearly, x is non-nilpotent in R, and thus I is nonnil over R. □

Proposition 3.12. *Let R be an* NP*-ring, $\mathfrak{p}$ a prime ideal of R, and M an R-module. Then M is ϕ-torsion over R if and only if $M_{\mathfrak{p}}$ is ϕ-torsion over $R_{\mathfrak{p}}$.*

Proof. Let M be an R-module and $x \in M$. If $M_{\mathfrak{p}}$ is ϕ-torsion over $R_{\mathfrak{p}}$, there exists a nonnil ideal $I_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ such that $I_{\mathfrak{p}}(x/1) = 0$ in $R_{\mathfrak{p}}$. Let I be the preimage of $I_{\mathfrak{p}}$ in R. Then I is nonnil by Lemma 3.11. Thus, there is a non-nilpotent element $t \in I$ such that $tkx = 0$ for some $k \notin \mathfrak{p}$. Let $s = tk$. Then we have $\langle s\rangle$ is nonnil and $\langle s\rangle x = 0$. Thus, M is ϕ-torsion over R.

Conversely, suppose M is ϕ-torsion over R. Let x/s be an element in $M_{\mathfrak{p}}$. Then there exists a nonnil ideal I such that $Ix = 0$, and thus $I_{\mathfrak{p}}(x/s) = 0$, with $I_{\mathfrak{p}}$ nonnil over $R_{\mathfrak{p}}$ by Lemma 3.11. It follows that $M_{\mathfrak{p}}$ is ϕ-torsion over $R_{\mathfrak{p}}$. □

Theorem 3.13. *Let R be a ϕ-ring. Then the following statements are equivalent for R:*

1. *R is a ϕ-*Prüfer *ring and a strong ϕ-ring;*
2. *Any divisible module is nonnil-FP-injective;*
3. *Any h-divisible module is nonnil-FP-injective;*
4. *Any finitely generated nonnil ideal of R is projective;*
5. *Any (finitely generated) nonnil ideal of R is flat;*
6. *Any (finitely generated) ideal of R is ϕ-flat;*
7. *Any submodule of a ϕ-flat module is ϕ-flat;*
8. *Any R-module has an epimorphism ϕ-flat preenvelope;*
9. *Any R-module has an epimorphism ϕ-flat envelope.*

Proof. (1) ⇒ (2): Let T be a finitely presented ϕ-torsion module, and let $\mathfrak{m}$ be a maximal ideal of R. Then by Proposition 3.12, $T_{\mathfrak{m}}$ is a finitely presented ϕ-torsion $R_{\mathfrak{m}}$-module. By [1, Corollary 2.10], $R_{\mathfrak{m}}$ is a ϕ-chained ring. Since R is a strong ϕ-ring, $R_{\mathfrak{m}}$ is also a strong ϕ-ring. Thus, $T_{\mathfrak{m}} \cong \bigoplus_{i=1}^{n} R_{\mathfrak{m}}/R_{\mathfrak{m}}x_i$ for some regular elements $x_i \in R_{\mathfrak{m}}$ by [22, Theorem 4.1]. Let E be a divisible module. Then $E_{\mathfrak{m}}$ is a divisible module over $R_{\mathfrak{m}}$ by Lemma 3.2 and Lemma 3.4. Thus,

$$\mathrm{Ext}_R^1(T,E)_{\mathfrak{m}} = \mathrm{Ext}_{R_{\mathfrak{m}}}^1(T_{\mathfrak{m}},E_{\mathfrak{m}}) = \bigoplus_{i=1}^{n} \mathrm{Ext}_{R_{\mathfrak{m}}}^1(R_{\mathfrak{m}}/R_{\mathfrak{m}}x_i,E_{\mathfrak{m}}) = 0$$

by Lemma 3.2 and [18, Theorem 3.9.11]. It follows that $\mathrm{Ext}_R^1(T,E) = 0$. Therefore, E is nonnil-FP-injective.

(2) ⇒ (3): Trivial.

(3) ⇒ (4): Let N be an R-module, and let I be a finitely generated nonnil ideal of R. The short exact sequence $0 \to I \to R \to R/I \to 0$ induces a long exact sequence as follows:

$$0 = \mathrm{Ext}_R^1(R,N) \to \mathrm{Ext}_R^1(I,N) \to \mathrm{Ext}_R^2(R/I,N) \to \mathrm{Ext}_R^2(R,N) = 0.$$

Let $0 \to N \to E \to K \to 0$ be an exact sequence where E is the injective envelope of N. There exists a long exact sequence as follows:

$$0 = \mathrm{Ext}_R^1(R/I,E) \to \mathrm{Ext}_R^1(R/I,K) \to \mathrm{Ext}_R^2(R/I,N) \to \mathrm{Ext}_R^2(R/I,E) = 0.$$

Thus, $\mathrm{Ext}_R^1(I,N) \cong \mathrm{Ext}_R^2(R/I,N) \cong \mathrm{Ext}_R^1(R/I,K) = 0$ as K is nonnil-FP-injective. It follows that I is a projective ideal of R.

(4) ⇒ (1): It follows from Proposition 3.6 and Theorem 3.8.

(4) ⇒ (5): Let I be a nonnil ideal of R, and let a be a non-nilpotent element in I. Let $\{I_i\}_{i\in\Gamma}$ be a family of finitely generated subideals of I such that $\varinjlim I_i = I$. Set $I'_i = \langle I, a\rangle$. Then I'_i is a finitely generated nonnil ideal of R such that $\varinjlim I'_i = I$. Since each I'_i is projective by (4), I is a flat ideal of R.

(5) ⇔ (6): Let I be a (resp., finitely generated) nonnil ideal of R, and let J be a (resp., finitely generated) ideal of R. Then we have

$$\mathrm{Tor}_1^R(R/J, I) \cong \mathrm{Tor}_2^R(R/I, R/J) \cong \mathrm{Tor}_1^R(R/I, J).$$

Thus, I is flat if and only if J is ϕ-flat.

(5) ⇒ (1): It follows from [13, Corollary 2.6] that R is a strong ϕ-ring. Let K and L be non-zero (resp., finitely generated) ideals of $R/\mathrm{Nil}(R)$ (denoted by $\overline{R}$). Then $K = I/\mathrm{Nil}(R)$ and $L = J/\mathrm{Nil}(R)$ for some (resp., finitely generated) nonnil ideals I and J of R (see [1, Lemma 2.4]). By [21, Lemma 1.6], $J\mathrm{Nil}(R) = \mathrm{Nil}(R)$. Thus, $L = J/\mathrm{Nil}(R) \cong J \otimes_R \overline{R}$.

We claim that $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, L) = 0$. Indeed, by change of rings, the exact sequence of $\overline{R}$-modules:

$$0 \to \mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, J \otimes_R \overline{R}) \to K \otimes_{\overline{R}} J \otimes_R \overline{R} \to \overline{R} \otimes_{\overline{R}} J \otimes_R \overline{R} \to \overline{R}/K \otimes_{\overline{R}} J \otimes_R \overline{R} \to 0$$

is naturally isomorphic to

$$0 \to \mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, J \otimes_R \overline{R}) \to K \otimes_R J \to \overline{R} \otimes_R J \to \overline{R}/K \otimes_R J \to 0.$$

Thus, there is a commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}_1^R(R/I, J) & \longrightarrow & I \otimes_R J & \longrightarrow & R \otimes_R J & \longrightarrow & R/I \otimes_R J & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, J \otimes_R \overline{R}) & \longrightarrow & K \otimes_R J & \longrightarrow & \overline{R} \otimes_R J & \longrightarrow & \overline{R}/K \otimes_R J & \longrightarrow & 0.
\end{array}$$

Since g and h are epimorphisms, f is also an epimorphism by the five Lemma (see [18, Theorem 1.9.9]). By (5), J is flat, then $\mathrm{Tor}_1^R(R/I, J) = 0$. Thus, $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, L) \cong \mathrm{Tor}_1^{\overline{R}}(\overline{R}/K, J \otimes_R \overline{R}) = 0$. Consequently, $\overline{R} = R/\mathrm{Nil}(R)$ is a Prüfer domain. By [1, Corollary 2.10], R is a ϕ-Prüfer ring.

(5) ⇒ (7): Let M be a ϕ-flat module and N a submodule of M. Let I be a nonnil ideal of R. Then I is flat by (6). Thus, $\mathrm{fd}_R(R/I) \leq 1$. By considering the long exact sequence $\mathrm{Tor}_2^R(R/I, M/N) \to \mathrm{Tor}_1^R(R/I, N) \to \mathrm{Tor}_1^R(R/I, M)$, we have $\mathrm{Tor}_1^R(R/I, N) = 0$ since $\mathrm{Tor}_2^R(R/I, M/N) = \mathrm{Tor}_1^R(R/I, M) = 0$. Thus, N is ϕ-flat.

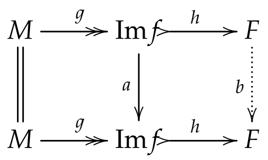
(7) ⇒ (6) and (9) ⇒ (8): Trivial.

(8) ⇒ (7): Let F be a ϕ-flat module, $i : K \rightarrowtail F$ a monomorphism, and $f : K \twoheadrightarrow F'$ an epimorphism ϕ-flat preenvelope. Then there exists a homomorphism $g : F' \to F$ such that $i = gf$. Thus, f is a monomorphism. Consequently, $K \cong F'$, and K is ϕ-flat.

(1) + (4) + (7) ⇒ (9): Let R be a ϕ-Prüfer ring and I a finitely generated nonnil ideal of R. By (4), I is projective and thus finitely presented. It follows that R is nonnil-coherent. Thus, the class of ϕ-flat modules is preenveloping by Proposition 2.9. Let $\{F_i \mid i \in I\}$ be a family of ϕ-flat modules. Then $\prod_{i\in I} F_i$ is ϕ-flat by [4, Theorem 2.4]. By (7), the class of ϕ-flat modules is closed under submodules. Thus, the class of ϕ-flat modules is closed under inverse limits. By Corollary 2.10, the class of ϕ-flat modules is enveloping.

We claim that the ϕ-flat envelope of any R-module M is an epimorphism. Indeed, suppose $f : M \to F$ is the ϕ-flat envelope of M. Let $f = h \circ g$ with $g : M \twoheadrightarrow \mathrm{Im} f$ an epimorphism and $h : \mathrm{Im} f \rightarrowtail F$ the embedding map. We will show that g is the ϕ-flat envelope of M. For any $f' : M \to F'$ with F' ϕ-flat, there exists $l : F \to F'$ such that $l \circ f = f'$. Then $g \circ h \circ l = f'$, and thus g is a ϕ-flat preenvelope of M as $\mathrm{Im} f$ is ϕ-flat by (7).

Suppose $a : \mathrm{Im}f \to \mathrm{Im}f$ such that $g = a \circ g$. Then a is an epimorphism. Consider the following commutative diagram:



Since $f = h \circ g$ is a ϕ-flat envelope, there exists $b : F \to F$ such that $b \circ f = b \circ h \circ g = h \circ a \circ g = h \circ g = f$. Since g is an epimorphism, $h \circ a = b \circ h$. Then a is a monomorphism, and thus a is an isomorphism. It follows that g is the ϕ-flat envelope of M. □

Remark 3.14. Actually, Zhao [22, Theorem 4.3] showed that if R is a strong ϕ-ring, then R is a ϕ-Prüfer ring if and only if each submodule of a ϕ-flat R-module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat, if and only if every finitely generated nonnil ideal of R is ϕ-flat. In Theorem 3.13, we provide simplified versions of [22, Theorem 4.3] and several new characterizations of ϕ-Prüfer rings using divisible modules, nonnil-FP-injective modules, and the epimorphic enveloping properties of ϕ-flat modules.

The following example demonstrates that not every divisible R-module is necessarily nonnil-FP-injective for ϕ-Prüfer rings. Therefore, the condition that R is a strong ϕ-ring in Theorem 3.13 cannot be omitted.

Example 3.15. Let D be a non-field Prüfer domain and K its quotient field. Let $R = D(+)K/D$ be the idealization construction. As in Example 3.9, we can show that R is a ϕ-Prüfer ring and a total ring of quotients. Hence, every R-module is divisible. However, since $\mathrm{Nil}(R)$ is not a maximal ideal of R, the Krull dimension of R is greater than 1. Therefore, by Theorem 2.7, there exists an R-module M that is not nonnil-FP-injective.

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