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# The complete comaximal decomposition in residuated lattices

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**Abstract.** In this paper, we introduce the concept of pseudo-irreducible ideals in residuated lattices and obtain its relationship with important concepts such as prime ideals and maximal ideals in residuated lattices. We then use this concept to define and study complete comaximal decomposition in residuated lattices. Specifically, we characterize residuated lattices in which proper ideals can be expressed as the intersection of pairwise comaximal of finitely many pseudo-irreducible ideals.

## 1. Introduction

The concept of residuated lattices was first introduced by M. Ward and R.P. Dilworth ([19]) in 1939 as a generalization of the ideal of rings. Residuated lattices have interesting algebraic properties and include some important classes of algebras such as MV-algebras, MTL-algebras, BL-algebras.
Residuated lattices play an important role in fuzzy logic theory. They provide an algebraic framework for fuzzy logic and reasoning. From a logical point of view, ideals correspond to sets of provable formulae. The notion of ideal has been introduced in many algebraic structures such as lattices, MV-algebras, Bl-algebras and residuated lattices. In BL-algebras, MTL-algebras or residuated lattices the focus has been on filters or deductive systems. However, in rings, MV-algebras and lattice implication algebras, the ideal is in the center position. By definition, ideals in MV-algebras are kernels of homomorphisms ([4]). In residuated lattices the notion of an ideal was introduced as a natural generalization of that of ideal in MV-algebras and the relation between filters and ideals was discussed (see [1], [15]). An ideal is a dual of a filter in some special logical algebras such as lattice implication algebras but in nonregular residuated lattices the dual of filters is quite differently.
In 2013, C. Lele et al. ([14]) constructed some examples to show that, unlike in MV-algebras, ideals and filters are dual but behave differently in BL-algebras. In recent years, many researchers have studied ideals in residuated lattices.
The concept of De Morgan residuated lattice was introduced by Holden in 2018 ([8]). The variety of De Morgan residuated lattices includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices. He study prime ideals and some special cases of prime ideals in De Morgan residuated.

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In the literature of BL-algebras and residuated lattices, there has been a growing interest in prime and maximal filters, stable topology on the set of prime and maximal filters, and pure filters. For instance, in 2003, Leustean ([16]) explored the prime and maximal spectra of BL-algebras and and considered the reticulation of them from a filter theory perspective. Haveshki et al. ([7]) developed a topology on BL-algebras induced by uniformity, grounded in filter theory. Also, in 2009, Eslami and Haghani ([5]) examined stable topology and $F$-topology on the set of all prime filters in a BL-algebra, demonstrating that the set of all prime filters endowed with stable topology forms a compact space that is not $T_0$. They further defined and analyzed pure filters in a BL-algebra through the lens of stable topology. In 2021, Bușneag and Piciu ([2]) explored another variant of topology on the set of all prime filters of a residuated lattice, known as the stable topology, which is coarser than the spectral topology. They introduced the concepts of pure $i$-filters within a residuated lattice and the notion of normal residuated lattices, studying their properties in depth. Also, Holdon and Borumand Saeid ([9]) investigated the regularity within residuated lattices from a filter theory perspective, providing valuable insights into this area.

In 2021, the notion of minimal prime ideals was introduced in residuated lattices and related properties were investigated. Also, new equivalent characterizations and properties for prime and maximal ideals were obtained and the relation between these ideals and minimal prime ideals was discussed for De Morgan residuated lattices ([18]). Then, Holdon and Borumand Saeid ([12]) explored various connections between obstinate ideals and other types of ideals in a residuated lattice, such as Boolean, primary, prime, implicative, maximal, and $\odot$-prime ideals. Also in 2022, the notion of pure ideals was introduced and investigated in residuated lattices, and using these ideals the related spectral topologies were studied ([17]). In this paper, we introduce the notion of pseudo-irreducible ideals of residuated lattices and investigate some related results, we show that every prime ideal is a pseudo-irreducible ideal but the converse is not true. We characterize the pseudo-irreducible ideals in De Morgan residuated lattices. Then, we introduce the notion of radical of an ideal residuated lattice and obtain the relation between pseudo irreduciblity of an ideal and its radical. Finally, we introduce the concept of a (complete) comaximal decomposition of an ideal of a residuated lattice. We prove that if a complete comaximal decomposition exists, then it is unique. If every proper ideal of $L$ has the complete comaximal decomposition, then we say it has the complete comaximal decomposition property (for short, it has the CCD property). We find the necessary and sufficient conditions for a residuated lattice to have CCD property. Moreover, we prove that an MTL-algebra has CCD property if and only if every closed subset of its spectrum has finitely many connected components.

In [2], the authors investigate the relationship between ideals and filters. Since filters correspond one-to-one with congruence relations on a residuated lattice, and we can define a congruence relation on a De Morgan residuated lattice using an ideal such that any two distinct ideals correspond to different congruences, we can consider ideals as special cases of filters in the context of quotient De Morgan residuated lattices.

Moreover, the quotient of a De Morgan residuated lattice by an ideal is an involution residuated lattice. By focusing on ideals instead of filters, we can highlight specific properties of a residuated lattice. For instance, when working with ideals, we can characterize De Morgan residuated lattices with Noetherian Max-spectrum; however, the results obtained when working with filters are entirely different.

This distinction motivates us to define pseudo-irreducible ideals instead of pseudo-irreducible filters when stating certain specialized results in this paper.

## 2. Preliminaries

In this section, we review some definitions and results which will be used throughout this paper.

**Definition 2.1.** *([6]) A residuated lattice is an algebra* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying the following axioms:*
*(RL1)* $(L, \wedge, \vee)$ *is a bounded lattice (the partial order is denoted by* $\leq$*).*
*(RL2)* $(L, \odot, 1)$ *is a commutative monoid.*
*(RL3) For every* $x, y, z \in L$, $x \odot z \leq y$ *if and only if* $z \leq x \rightarrow y$ *(residuation).*

If $L$ is a residuated lattice, then for $x, y \in L$ we define

$x^* := x \to 0$ and $x \oplus y := (x^* \odot y^*)^* = x^* \to y^{**} = y^* \to x^{**}$.

The operation $\oplus$ will be called *strong addition*. For $x \in L$, we define $0x := 0$ and $nx := (n-1)x \oplus x$ for $n \geqslant 1$. For more information see [8] and [18]. In the following proposition, we collect some main properties of residuated lattices.

**Proposition 2.2.** *([6]) Let $L$ be a residuated lattice and $x, y, z \in L$. Then we have the following statements:*
*(1) $x \leq y$ if and only if $x \to y = 1$.*
*(2) If $x \leq y$, then $y^* \leq x^*$.*
*(3) $x \odot x^* = 0$.*
*(4) $x \to (y \to z) = y \to (x \to z) = (x \odot y) \to z$.*
*(5) $(x \vee y)^* = x^* \wedge y^*$.*
*(6) $x \oplus 0 = x^{**}$, $x \oplus 1 = 1$, $x \oplus x^* = 1$.*
*(7) $x \oplus y = y \oplus x$.*
*(8) $x, y \leq x \oplus y$.*
*(9) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.*
*(10)If $x \leq y$, then $x \oplus z \leq y \oplus z$.*
*(11) $x \leq x^{**}$, $x^{***} = x^*$.*
*(12) $x \wedge (y \oplus z) \leq (x^{**} \wedge y^{**}) \oplus (x^{**} \wedge z^{**})$.*

**Definition 2.3.** *([8], [18]) A residuated lattice $L$ is called*
*(1) De Morgan residuated lattice, if $(x \wedge y)^* = x^* \vee y^*$ for all $x, y \in L$.*
*(2)Involution residuated lattice, if $x^{**} = x$ for all $x \in L$.*
*(3) MTL-algebra, if $(x \to y) \vee (y \to x) = 1$ for all $x, y \in L$.*
*(4) Stonean, if $x^* \vee x^{**} = 1$ for all $x \in L$.*

Examples of De Morgan residuated lattices are Boolean algebras, MV-algebras, BL-algebras and MTL-algebras.

**Definition 2.4.** *([8], [18]) A nonempty subset $I$ of a residuated lattice $L$ is called an ideal of $L$ if the following conditions hold:*
*(I1) If $x \leq y$ and $y \in I$, then $x \in I$.*
*(I2) If $x, y \in I$, then $x \oplus y \in I$.*

We denote by $I(L)$ the set of all ideals of $L$. Every ideal is a lattice ideal in the lattice $(L, \wedge, \vee, 0, 1)$, but the converse is not true. Moreover, the intersection of any set of ideals becomes an ideal. An ideal $I$ is called *proper* if $I \neq L$. For a nonempty subset $S$ of $L$, we set $(S] := \bigcap\{I \in I(L) \mid S \subseteq I\}$ that is called *the ideal of $L$ generated by $S$* and for $x \in L$ we set $(x]$ the ideal of $L$ generated by $\{x\}$. An ideal is called *principal* if it is of the form $(x]$ for some $x \in L$. Also, for $I \in I(L)$ and $x \in L$, we set $I(x) := (I \cup \{x\}]$. The lattice $(I(L), \subseteq)$ is distributive, complete and algebraic where the compact elements are the principal ideals of $L$. Also, for $I, J \in I(L)$ we have $I \vee J = (I \cup J]$ and for a family $\{I_i\}_{i \in A}$ of ideals of $L$ we have $\wedge_{i \in A} I_i = \cap_{i \in A} I_i$, see [11] for more information.

**Proposition 2.5.** *([8], [18]) Let $S$ be a nonempty subset of a residuated lattice $L$, $x, y \in L$ and $I, J \in I(L)$. Then*
*(1) $(S] = \{x \in L \mid x \leq s_1 \oplus \cdots \oplus s_n$ for some $n \geqslant 1$ and $s_1, ..., s_n \in S\}$.*
*(2) $(x] = \{z \in L \mid z \leq nx$ for some $n \geqslant 1\}$.*
*(3) $I(x) = \{z \in L \mid z \leq i \oplus nx$ for some $i \in I$ and $n \geqslant 1\}$.*
*(4) $I(x \wedge y) \subseteq I(x) \cap I(y) = I(x^{**} \wedge y^*)$ and $I(x) = I(x^{**})$.*
*(5) If $L$ is a De Morgan residuated lattice, then $I(x \wedge y) = I(x) \cap I(y)$ and $(x] \cap (y] = (x \wedge y]$.*
*(6) $(x] \vee (y] = (x \oplus y]$, $(x] \cap (y] = (x^{**} \wedge y^*]$ and $(x] = (x^{**}]$.*
*(7) $I \vee J = (I \cup J] = \{x \in L \mid x \leq i \oplus j$ for some $i \in I$ and $j \in J\}$.*
*(8) $x \in I$ if and only if $x^{**} \in I$.*
*(9) If $x \leq y$, then $I(x) \subseteq I(y)$.*
*(10) $I(x) \vee I(y) = I(x \vee y) = I(x \oplus y)$.*

If $I$ is an ideal of a residuated lattice $L$, then the binary relation $\theta_I$ on $L$ defined by $(x, y) \in \theta_I$ if and only if $x^* \odot y \in I$ and $x \odot y^* \in I$ is an equivalence relation on $L$.

We recall that $\theta_I$ is a congruence on the reduct $(L, \vee, \odot, \rightarrow, 0, 1)$ of the residuated lattice $L$. For $x \in L$, we denote by $\frac{x}{I}$ the class of $x$ concerning to $\theta_I$ and the quotient set $\frac{L}{\theta_I}$ by $\frac{L}{I}$.

**Proposition 2.6.** *([8]) Let I be an ideal of a De Morgan residuated lattice L. Then $\theta_I$ is a congruence on L and $(\frac{L}{I}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an involution residuated lattice by natural actions induced from L.*

For a nonempty subset $X$ of a De Morgan residuated lattice $L$ we set $X/I := \{\frac{x}{I} \mid x \in X\}$. Clearly, for $x \in L$; $\frac{x}{I} = \frac{1}{I}$ if and only if $x^* \in I$, and $\frac{x}{I} = \frac{0}{I}$ if and only if $x \in I$.

While the concept of prime ideals has been defined in various ways in different papers, we will be using the definition provided in [8, Page 451] in this paper. Recall a proper ideal $P \in I(L)$ is called *prime* if for $x, y \in L$, $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. We denote by Spec$(L)$ the set of all prime ideals of $L$. Also, a proper ideal $P \in I(L)$ is called *$\cap$-prime* if $P$ is a prime element in $(I(L), \subseteq)$, that is, for $I, J \in I(L)$ if $I \cap J \subseteq P$ then we have either $I \subseteq P$ or $J \subseteq P$.

**Theorem 2.7.** *([8]) Let P be a proper ideal if a De Morgan residuated lattice L. Then P is prime iff P is $\cap$-prime.*

**Proposition 2.8.** *([8]) Let L be a De Morgan residuated lattice, $I \in I(L)$ and $a \notin I$. Then we have the following:*
*(1) There is a prime ideal P such that $I \subseteq P$ and $a \notin P$.*
*(2) I is the intersection of all prime ideals which contain I.*

**Proposition 2.9.** *([8], [18]) Let P be a proper ideal of an MTL-algebra L. Then the following are equivalent:*

1. *P is a prime ideal.*
2. *For every $x, y \in L$, we have either $x \odot y^* \in P$ or $x^* \odot y \in P$.*
3. *$\frac{L}{P}$ is a chain.*

A proper ideal $M \in I(L)$ is called *maximal* if $M$ is not strictly contained in a proper ideal of $L$. We denote by MaxI$(L)$ the set of all maximal ideals of $L$. Clearly, every proper ideal is contained in a maximal ideal.

**Proposition 2.10.** *([18]) Let M be a proper ideal of a residuated lattice L. Then the following are equivalent:*
*(1) $M \in$ MaxI$(L)$.*
*(2) For any $x \notin M$, there exist $d \in M$, $n \geqslant 1$ such that $d \oplus (nx) = 1$.*
*(3) For any $x \in L$, $x \notin M$ if and only if $(nx)^* \in M$ for some $n \geqslant 1$.*

**Proposition 2.11.** *([8], [18]) Every maximal ideal of a De Morgan residuated lattice is prime.*

For every subset $X$ of a residuated lattice $L$ we set $V(X) := \{P \in \text{Spec}(L) \mid X \subseteq P\}$, and for each $x \in L$ we denote $V(x)$ by $V((x])$. The family $\{V(X)\}_{X\subseteq L}$ satisfies the axioms for closed sets for a topology over Spec$(L)$.

Also, we denote Spec$(L) \setminus V(X)$ by $D(X)$ (for each $x \in L$ we denote $D(x)$ by $D((x])$), that is, $D(X) = \{P \in \text{Spec}(L) \mid X \nsubseteq P\}$. Thus, the family $\{D(X)\}_{X\subseteq L}$ satisfies the axioms for open sets for a topology over Spec$(L)$. By Proposition 2.11, every maximal ideal of a De Morgan residuated lattice is prime. Thus, we can consider MaxI$(L)$ as a subspace of Spec$(L)$ in a De Morgan residuated lattice. For a De Morgan residuated lattice $L$ and $X \subseteq L$ we set $V_{Max}(X) := V(X) \cap \text{MaxI}(L)$ and $D_{Max}(X) := D(X) \cap \text{MaxI}(L)$. Then the family $\{V_{Max}(X)\}_{X\subseteq L}$ ($\{D_{Max}(X)\}_{X\subseteq L}$) satisfies the axioms for closed (open) sets for a topology over MaxI$(L)$, for more information see [11] and [18, Propositions 37, 38 and 41].

Let $L$ be a residuated lattice. For $I, J \in I(L)$, we set $I \rightarrow J := \{x \in L \mid (x] \cap I \subseteq J\}$.

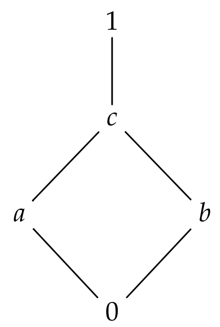
**Lemma 2.12.** *([11], [18]) Let L be a residuated lattice and $I, J \in I(L)$. Then we have the following:*
*(1) $I \rightarrow J \in I(L)$.*
*(2) $I \cap J \subseteq K$ if and only if $I \subseteq J \rightarrow K$, that is, $J \rightarrow K = sup\{I \in I(L) \mid I \cap J \subseteq K\}$.*
*(3) $I \rightarrow I = L$, $L \rightarrow I = I$.*
*(4) $(I \vee J) \rightarrow K = (I \rightarrow K) \cap (I \rightarrow J)$.*
*(5) $K \rightarrow (I \cap J) = (K \rightarrow I) \cap (K \rightarrow J)$.*

## 3. Pseudo-irreducible ideals

In this section, we introduce the notion of pseudo-irreducible ideals of residuated lattices and investigate some of their properties.

**Definition 3.1.** *A proper ideal $I$ of a residuated lattice $L$ is called pseudo-irreducible, whenever for $J, K \in I(L)$, if $I = J \cap K$ and $J \vee K = L$, then either $J = L$ or $K = L$.*

**Example 3.2.** *Let $A = \{0, a, b, c, 1\}$ such that $0 < a, b < c < 1$ and $a, b$ are incomparable.*




*Consider the operations $\odot$ and $\to$ given by the following tables:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | $a$ | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ | $b$ |
| $c$ | 0 | $a$ | $b$ | $c$ | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | $b$ | 1 | 1 |
| $b$ | $a$ | $a$ | 1 | 1 | 1 |
| $c$ | 0 | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

*Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice ([8]). We have $I(L) = \{\{0\}, \{0, a\}, \{0, b\}, L\}$.*

1. *It is easy to see that $\{0, a\}$ and $\{0, b\}$ are pseudo-irreducible ideals of $L$.*
2. *$\{0\}$ is not a pseudo-irreducible ideal because $\{0\} = \{0, a\} \cap \{0, b\}$ and $\{0, a\} \vee \{0, b\} = L$, but $\{0, a\} \neq L$ and $\{0, b\} \neq L$.*
3. *It is easy to verify that $\{0, a\}$ and $\{0, b\}$ are prime ideals of $L$ and since $a^{**} \wedge b^{**} = 0$ with $a, b \neq 0$, then $\{0\}$ is not a prime ideal of $L$. Thus, $\mathrm{Spec}(L) = \mathrm{MaxI}(L) = \{\{0, a\}, \{0, b\}\}$ and the topological spaces $\mathrm{Spec}(L)$ and $\mathrm{MaxI}(L)$ are discrete in this case.*

**Lemma 3.3.** *Every maximal ideal of a residuated lattice is $\cap$-prime.*

*Proof.* Let $M$ be a maximal ideal of a residuated lattice $L$ and $I$ and $J$ be two ideals of $L$ with $I \vee J \subseteq M$. If $I \nsubseteq M$ and $J \nsubseteq M$, then $I \vee M = L$ and $J \vee M = L$, and so $M = (I \cap J) \vee M = (I \vee M) \cap (J \vee M) = L$, a contradiction. Hence, we have either $I \subseteq M$ or $J \subseteq M$, actually, $M$ is a prime element in $(I(L), \subseteq)$, that is, it is $\cap$-prime. □

**Proposition 3.4.** *Let $L$ be a residuated lattice. Then we have the following statements.*

1. *Every prime ideal is a pseudo-irreducible ideal.*
2. *Every $\cap$-prime ideal is a pseudo-irreducible ideal.*
3. *Every maximal ideal is a pseudo-irreducible ideal.*

*Proof.* (1) Let $I$ be a prime ideal of $L$ such that $I = J \cap K$ and $J \vee K = L$ for some $J, K \in I(L)$. Thus there exist $a \in J$ and $b \in K$ such that $a \oplus b = 1$. Hence, $a \wedge b \in J \cap K = I$. Now since $I$ is prime we have either $a \in I$ or $b \in I$. Assume that $a \in I$. Hence, $a \in K$, and so $1 = a \oplus b \in K$. Thus, $K = L$, and so $I$ is a pseudo-irreducible ideal..

(2) Let $I$ be a $\cap$-prime ideal of $L$ such that $I = J \cap K$ and $J \vee K = L$ for some $J, K \in I(L)$. Then either $J \subseteq I$ or $K \subseteq I$. So we have either $J \subseteq K$ or $K \subseteq J$. By assumption, we have $L = J \vee K = (J \cup K]$. Hence, we have either $L = (J] = J$ or $L = (K] = K$, and so $I$ is a pseudo-irreducible ideal.

(3) The proof is straightforward by (1) and Lemma 3.3. □

We now consider residuated lattices whose proper ideals are pseudo-irreducible.

**Proposition 3.5.** *A residuated lattice $L$ is local (that is, has only one maximal ideal) if and only if every proper ideal of $L$ is a pseudo-irreducible ideal.*

*Proof.* ⇒). Let $M$ be the unique maximal ideal of $L$ and $I$ be a proper ideal of $L$. Assume that there exist two ideals $J$ and $K$ of $L$ such that $I = J \cap K$ and $J \vee K = L$. Then we have either $J \nsubseteq M$ or $K \nsubseteq M$. Hence, we have either $J = L$ or $K = L$. Therefore, $I$ is a pseudo-irreducible ideal of $L$.

⇐). Let every proper ideal of $L$ be a pseudo-irreducible ideal and $M$ and $N$ be two distinct maximal ideals. Set $I := M \cap N$. Now since $M \vee N = L$ and $I \neq L$, we have either $M = L$ or $N = L$, which is a contradiction. □

The following example shows that pseudo-irreducible ideals of a residuated lattice is not prime in general.

**Example 3.6.** *Let $C$ be the MV-algebra described in [3, p. 474]. Consider the subalgebra $A := \{(x, y) \in C \times C \mid ord(z) = ord(y) = \infty$ or $ord(z), ord(y) \lneq \infty\}$ of $C \times C$. Then $A$ is a local MV-algebra that has two minimal prime ideals, see [1, p. 341] for more information. Since $A$ has two minimal prime ideals, the zero ideal is not prime, but by Proposition 3.5 since $A$ is local, every proper ideal is pseudo-irreducible. Hence, the zero ideal is a pseudo-irreducible ideal that is not prime. Thus, in MV-algebra and so in BL-algebras, MTL-algebras and De Morgan residuated lattice the concept of pseudo-irreducible ideal and (maximal) prime ideal is not equal in general.*

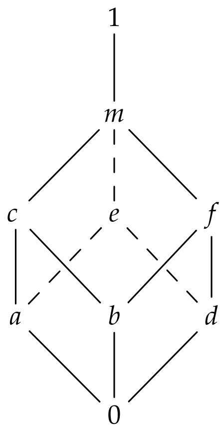
**Remark 3.7.** *In the above example,* $\mid \mathrm{MaxI}(C \times C) \mid = 1$ *and* $\mathrm{Spec}(C \times C) \neq \mathrm{MaxI}(C \times C)$. *Clearly, the topolgy on* $\mathrm{Spec}(C \times C)$ *is not discrete.*

**Theorem 3.8.** *Let $I$ be an ideal of a residuated lattice $L$ such that for each $x, y \in L$, $x \wedge y \in I$ and $x \oplus y = 1$ imply either $x \in I$ or $y \in I$. Then $I$ is pseudo-irreducible.*

*Proof.* Suppose that $J, K \in I(L)$ such that $I = J \cap K$ and $J \vee K = L$. Since $1 \in L = J \cap K$, we have $1 \leq j \oplus k$ for some $j \in J$ and $k \in K$. So $j \wedge k \in J \cap K = I$. By assumption, we have either $j \in I$ or $k \in I$. Without loss of generality, suppose that $j \in I$. Since $I \subseteq K$, we have $j \in K$. We obtain that $j \oplus k \in K$. Therefore, $1 \in K$ and so $K = L$. □

The converse of Theorem 3.8 is not true in general.

**Example 3.9.** *Let $L = \{0, a, b, c, d, e, f, m, 1\}$ with $0 < a < c < m < 1, 0 < a < e < m < 1, 0 < b < c < m < 1, 0 < b < f < m < 1, 0 < d < f < m < 1, 0 < d < e < m < 1$ and elements $\{a, b\}$, $\{a, f\}$, $\{a, d\}$, $\{b, d\}$, $\{b, e\}$, $\{d, c\}$, $\{c, e\}$, $\{c, f\}$, and $\{e, f\}$ are pairwise incomparable.*




*Consider the operations $\odot$ and $\to$ given by the following tables (see [13, Page. 252]):*

| $\odot$ | 0 | a | b | c | d | e | f | m | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | a |
| b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | b |
| c | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | c |
| d | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | d |
| e | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | e |
| f | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | f |
| m | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | m |
| 1 | 0 | a | b | c | d | e | f | m | 1 |

| $\to$ | 0 | a | b | c | d | e | f | m | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | m | 1 | m | 1 | m | 1 | m | 1 | 1 |
| b | m | m | 1 | 1 | m | m | 1 | 1 | 1 |
| c | m | m | m | 1 | m | m | m | 1 | 1 |
| d | m | m | m | m | 1 | 1 | 1 | 1 | 1 |
| e | m | m | m | m | m | 1 | m | 1 | 1 |
| f | m | m | m | m | m | m | 1 | 1 | 1 |
| m | m | m | m | m | m | m | m | 1 | 1 |
| 1 | a | b | c | d | e | f | m | 1 | 1 |

*Consider the ideal $I := \{0\}$. Clearly $I$ is the only proper ideal and so is the only maximal ideal. Hence, $I$ is a pseudo-irreducible ideal by Proposition 3.4, but we have $a \wedge b \in I$, $a \oplus b = 1$, $a \notin I$ and $b \notin I$, see [11, Example 3] for more information.*

We recall that a residuated lattice is indecomposable if $L \cong L_1 \times L_2$ implies either $L_1$ or $L_2$ is trivial, where $L_1$ and $L_2$ are two residuated lattices and $L_1 \times L_2$ is their direct product. A nontrivial residuated lattice $L$ is indecomposable if and only if $B(L) = \{0, 1\}$.

Now we want to characterize pseudo-irreducible ideals in De Morgam lattices.

**Theorem 3.10.** *The following statements are equivalent for an ideal $I$ of a De Morgan residuated lattice $L$:*

1. *For $x, y \in L$, $x \wedge y \in I$ and $x \oplus y = 1$ imply either $x \in I$ or $y \in I$.*
2. *$I$ is pseudo-irreducible.*
3. *$\frac{L}{I}$ is an indecomposable residuated lattice.*

4. *$V(I)$ is connected as a subspace of* Spec($L$).

*Proof.* (1) ⇒ (2). By Theorem 3.8.

(2) ⇒ (1). Suppose that $x \wedge y \in I$ and $x \oplus y = 1$ for some $x, y \in L$. By Proposition 2.5, we have $I = I(x \wedge y) = I(x) \cap I(y)$. Using Proposition 2.5, we obtain $I(x) \vee I(y) = I(x \vee y) = I(x \oplus y) = I(1) = L$. By assumption, we have either $I(x) = L$ or $I(y) = L$. Suppose that $I(x) = L$. Then $y \in L = I(x)$. Since $y \in I(y)$, then $y \in I(x) \cap I(y) = I$. Similarly, if $I(y) = L$, then we have $x \in I$.

(1) ⇒ (3). We want to prove that $B(\frac{L}{I}) = \{\frac{0}{I}, \frac{1}{I}\}$, equivalently, $\frac{L}{I}$ is indecomposable. Let $\frac{x}{I} \in B(\frac{L}{I})$ be arbitrary. Then $\frac{x}{I} \wedge \frac{x^*}{I} = \frac{x}{I} \wedge (\frac{x}{I})^* = \frac{0}{I}$ and $\frac{x}{I} \vee \frac{x^*}{I} = \frac{1}{I}$. Form $\frac{x}{I} \vee \frac{x^*}{I} = \frac{1}{I}$ we have $\frac{x \vee x^*}{I} = \frac{1}{I}$, and so $(x \vee x^*)^* = x^* \wedge x^{**} \in I$ by Proposition 2.2. Also, we have $x^* \oplus x^{**} = (x^{**} \odot x^{***})^* = (x^{**} \odot x^*)^* = 0^* = 1$. Thus, we obtain that $x^* \in I$ or $x^{**} \in I$. If $x^* \in I$, then $\frac{x}{I} = \frac{1}{I}$. Now if $x^{**} \in I$, then $\frac{x^{**}}{I} = \frac{0}{I}$. By Proposition 2.6 since $\frac{L}{I}$ is an involution residuated lattice, then $\frac{x}{I} = \frac{x^{**}}{I} = \frac{0}{I}$. Hence $B(\frac{L}{I}) = \{\frac{0}{I}, \frac{1}{I}\}$.

(3) ⇒ (2). Suppose that $\frac{L}{I}$ is indecomposable but $I$ is not a pseudo-irreducible ideal. Then there exist $J, K \in I(L)$ such that $I = J \cap K$, $J \vee K = L$, $J \neq L$ and $K \neq L$. Define $\varphi : \frac{L}{I} \to \frac{L}{J} \times \frac{L}{K}$ by $\varphi(\frac{x}{I}) = (\frac{x}{J}, \frac{x}{K})$. It is easy to see that $\varphi$ is a homomorphism. Now we prove that $\varphi$ is onto. Since $1 \in L = J \vee K$, there exist $a \in J$ and $b \in K$ such that $a \oplus b = 1$ by Proposition 2.5. We have $\frac{1}{J} = \frac{a \oplus b}{J} = \frac{a}{J} \oplus \frac{b}{J} = \frac{0}{J} \oplus \frac{b}{J} = \frac{b^{**}}{J} = \frac{b}{J}$ by Proposition 2.6. Similarly, we can show that $\frac{a}{K} = \frac{1}{K}$. Now, let $(\frac{x}{J}, \frac{y}{K}) \in \frac{L}{J} \times \frac{L}{K}$ be arbitrary. It is straightforward to prove that $\varphi(\frac{(a \oplus x) \wedge (b \oplus y)}{I}) = (\frac{x}{J}, \frac{y}{K})$, and hence $\varphi$ is onto. Now suppose that $\varphi(\frac{x}{I}) = \varphi(\frac{y}{I})$. It is easy to see that $x^* \odot y \in J \cap K = I$ and $x \odot y^* \in J \cap K = I$. Hence $\frac{x}{I} = \frac{y}{I}$, that is, $\varphi$ is one to one.

Since $J \neq L$ and $K \neq L$, we have $\frac{L}{J}$ and $\frac{L}{K}$ are nontrivial residuated lattices. Therefore, $\frac{L}{I}$ is decomposable, which is a contradiction.

(3) ⇔ (4). Prime ideals of $\frac{L}{I}$ are exactly of the form $\frac{P}{I} := \{\frac{x}{I} \mid x \in P\}$, where $P$ is a prime ideal of $L$ containing $I$. Hence, $V(I)$ is homeomorphic to Spec($\frac{L}{I}$). Thus by [18, Theorems 44 and 45], $\frac{L}{I}$ is indecomposable if and only if $B(\frac{L}{I}) = \{\frac{0}{I}, \frac{1}{I}\}$ if and only if Spec($\frac{L}{I}$) is connected if and only if $V(I)$ is connected as a subspace of Spec($L$). □

**Remark 3.11.** *Let $I$ be a pseudo-irreducible ideal of a Stonean residuated lattice $L$. Then by [10, Theorem 5], $\frac{L}{I}$ is a Boolean residuated lattice. Also by Theorem 3.10, $\frac{L}{I}$ must be indecomposable. Thus, $\frac{L}{I} = \{\frac{0}{I}, \frac{1}{I}\}$ and so by [10, Theorems 4 and 6], $I$ is prime and maximal. Therefore by Proposition 3.4, the concepts maximal, prime and pseudo-irreducible are equivalent for a proper ideal of an Stonean residuated lattice $L$.*

We end this section with the following proposition that will be used in Section 5 for ideal decomposition.

**Proposition 3.12.** *Let $I$ and $J$ be pseudo-irreducible ideals of a residuated lattice $L$. Then $I \vee J \neq L$ if and only if $I \cap J$ is a pseudo-irreducible ideal.*

*Proof.* *⇒). Suppose that $K_1, K_2 \in I(L)$ such that $I \cap J = K_1 \cap K_2$ and $K_1 \vee K_2 = L$. Then $(I \vee K_1) \cap (I \vee K_2) = I \vee (K_1 \cap K_2) = I$ and $(I \vee K_1) \vee (I \vee K_2) = I \vee (K_1 \vee K_2) = L$. Since $I$ is a pseudo-irreducible ideal, we have either $I \vee K_1 = L$ or $I \vee K_2 = L$. Similarly, we can prove either $J \vee K_1 = L$ or $J \vee K_2 = L$. By assumption, $I \vee J \neq L$, so there exists a maximal ideal $M$ of $L$ such that $I \vee J \subseteq M$. Hence $I, J \subseteq M$. Now since $K_1 \cap K_2 \subseteq I \cap J \subseteq M$, we have either $K_1 \subseteq M$ or $K_2 \subseteq M$ by Lemma 3.3. Suppose that $K_1 \subseteq M$. Then $I, J, K_1 \subseteq M$. Thus, we have $I \vee K_2 = L$ and $J \vee K_2 = L$, and we have $L = (I \vee K_2) \cap (J \vee K_2) = (K_2 \cap K_2) \vee (I \cap J) \vee (I \cap K_2) \vee (J \cap K_2) \leq K_2$. Thus, $K_2 = L$.*

*⇐). It is clear since pseudo-irreducible ideals are proper.* □

## 4. Radical of an ideal and its pseudo-irreducibility

In this section, we recall the definition of the radical of an ideal of a residuated lattice and then we consider its pseudo-irreducibility. We begin with the following definition.

**Definition 4.1.** *Let $I$ be a proper ideal of a residuated lattice $L$. The intersection of all maximal ideals of $L$ which contain $I$ is called the radical of $I$, and it is denoted by* Rad($F$). *If $I = L$, then we put* Rad($I$) = $L$.

**Example 4.2.** *If $L$ is a local residuated lattice with the unique maximal ideal $M$ that is not zero, then for each proper ideal $I$ of $L$ we have,* $\mathrm{Rad}(I) = M$. *In particular,* $\mathrm{Rad}(\{0\}) = M \neq \{0\}$.

In the following proposition we state some properties of radical of ideals.

**Proposition 4.3.** *Let $I, J$ be ideals of a residuated lattice $L$. Then we have the following statements:*

1. $\mathrm{Rad}(I) \in I(L)$.
2. $I \subseteq \mathrm{Rad}(I)$.
3. *If* $I \in \mathrm{MaxI}(L)$, *then* $\mathrm{Rad}(I) = I$.
4. *If* $I \subseteq J$, *then* $\mathrm{Rad}(I) \subseteq \mathrm{Rad}(J)$.
5. $\mathrm{Rad}(I) = L$ *if and only if* $I = L$
6. $\mathrm{Rad}(I) \cap \mathrm{Rad}(J) = \mathrm{Rad}(I \cap J)$.
7. $I \vee \mathrm{Rad}(J) \subseteq \mathrm{Rad}(I \vee J)$.
8. $\mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I)$.
9. $\mathrm{Rad}(I \to J) \subseteq I \to \mathrm{Rad}(J)$.
10. $I \vee J = L$ *if and only if* $\mathrm{Rad}(I) \vee \mathrm{Rad}(J) = L$.
11. $\mathrm{Rad}(\mathrm{Rad}(I) \to \mathrm{Rad}(J)) = \mathrm{Rad}(I) \to \mathrm{Rad}(J)$.

*Proof.* *By Definition 4.1, the proof of (1)-(4) is clear.*

(5). *Let* $\mathrm{Rad}(I) = L$. *If* $I \neq L$, *then there exists* $M \in \mathrm{MaxI}(L)$ *such that* $I \subseteq M$. *Hence,* $\mathrm{Rad}(I) \subseteq M \subsetneq L$, *which is a contradiction. The converse is obvious.*

(6). *By (1), we have* $\mathrm{Rad}(I) \cap \mathrm{Rad}(J) \supseteq \mathrm{Rad}(I \cap J)$. *Now let $M$ be a maximal ideal of $L$ containing $I \cap J$. By Lemma 3.3 we have either $I \subseteq M$ or $J \subseteq M$. Thus* $\mathrm{Rad}(I) \cap \mathrm{Rad}(J) \subseteq \mathrm{Rad}(I \cap J)$, *and so* $\mathrm{Rad}(I) \cap \mathrm{Rad}(J) = \mathrm{Rad}(I \cap J)$.

(7). *By (1) and (4) we have* $I \subseteq I \vee J \subseteq \mathrm{Rad}(I \vee J)$ *and* $\mathrm{Rad}(J) \subseteq \mathrm{Rad}(I \vee J)$. *Hence,* $I \vee \mathrm{Rad}(J) \subseteq \mathrm{Rad}(I \vee J)$.

(8). *By part (1) and then part (4), we have* $\mathrm{Rad}(I) \subseteq \mathrm{Rad}(\mathrm{Rad}(I))$. *Conversely, suppose that* $x \in \mathrm{Rad}(\mathrm{Rad}(I))$. *Then $x \in M$ for all* $M \in \mathrm{MaxI}(L)$ *containing* $\mathrm{Rad}(I)$. *Now, let* $N \in \mathrm{MaxI}(L)$ *containing $I$ be arbitrary. Then* $\mathrm{Rad}(I) \subseteq \mathrm{Rad}(N) = N$. *Thus $x \in N$. Hence,* $x \in \mathrm{Rad}(I)$, *that is,* $\mathrm{Rad}(\mathrm{Rad}(I)) \subseteq \mathrm{Rad}(I)$. *Therefore* $\mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I)$.

(9). *Since* $I \cap (I \to J) \subseteq I \subseteq \mathrm{Rad}(I)$, *we have* $\mathrm{Rad}(I \cap (I \to J)) \subseteq \mathrm{Rad}(I) \subseteq \mathrm{Rad}(\mathrm{Rad}(I))$ *by part(4). So* $\mathrm{Rad}(I) \cap \mathrm{Rad}(I \to J) \subseteq \mathrm{Rad}(I)$ *by parts (6) and (8). We obtain* $I \cap \mathrm{Rad}(I \to J) \subseteq \mathrm{Rad}(I)$ *by part (1). Hence* $\mathrm{Rad}(I \to J) \subseteq I \to \mathrm{Rad}(J)$ *by Lemma 2.12.*

(10). *Let* $I \vee J = L$. *Using part (1), we get* $L = I \vee J \subseteq \mathrm{Rad}(I) \vee \mathrm{Rad}(J)$. *Conversely, suppose that* $\mathrm{Rad}(I) \vee \mathrm{Rad}(J) = L$ *but* $I \vee J \neq L$. *Then $I \vee J$ is a proper ideal of $L$. So there exists a maximal ideal $M$ such that $I \vee J \subseteq M$. Thus $I, J \subseteq M$. We obtain that* $L = \mathrm{Rad}(I) \vee \mathrm{Rad}(J) \subseteq M$, *that is, $M = L$ which is a contradiction.*

(11). *Using parts (2), (8) and (9), we obtain*
$\mathrm{Rad}(I) \to \mathrm{Rad}(J) \subseteq \mathrm{Rad}(\mathrm{Rad}(I) \to \mathrm{Rad}(J)) \subseteq \mathrm{Rad}(I) \to \mathrm{Rad}(\mathrm{Rad}(J)) = \mathrm{Rad}(I) \to \mathrm{Rad}(J)$. □

The following proposition characterizes the radical of an ideal in an MTL-algebra.

**Proposition 4.4.** *Let $I$ be a proper ideal of an MTL-algebra $L$. Then*

$$\mathrm{Rad}(I) = \{x \in L \mid x \odot (nx)^{**} \in I \text{ for all } n \in N\}.$$

*Proof.* *Suppose that* $x \in \mathrm{Rad}(I)$ *and there exists $k \in N$ such that* $x \odot (kx)^{**} \notin I$. *By Proposition 2.8, there exists a prime ideal $P$ such that $I \subseteq P$ and* $x \odot (kx)^{**} \notin P$. *Since $P$ is a prime ideal of an MTL-algebra, then* $x^* \odot (kx)^* \in P$ *by Proposition 2.9. Let $M$ be a maximal ideal with $P \subseteq M$. Thus,* $x^* \odot (kx)^* \in M$. *If we can prove that $x \notin M$, then* $x \notin \mathrm{Rad}(I)$ *and we have a contradiction. Hence assume that $x \in M$. Then $kx \in M$. Now by Proposition 2.2, we have*

$$x^* \to (kx \oplus ((kx)^* \odot x^*)) = x^* \to ((kx)^* \to ((kx)^* \odot x^*)^{**}) = ((kx)^* \odot x^*) \to (((kx)^* \odot x^*)^{**}) = 1.$$

*Thus,* $x^* \leq (kx \oplus ((kx)^* \odot x^*)) \in M$. *Hence,* $x^* \in M$, *and so* $1 = x \oplus x^* \in M$, *a contradiction.*

*Conversely, let* $x \odot (nx)^{**} \in I$ *for all $n \in N$ and* $x \notin \mathrm{Rad}(I)$. *Then there exists a maximal ideal $M$ such that $I \subseteq M$ and $x \notin M$. By Proposition 2.10, there exists $k \in N$ such that* $(kx)^* \in M$. *Since* $(kx)^{**} \odot x \in I \subseteq M$, *as above argument we have $x \in M$, which is a contradiction.* □

In general, Rad(_) does not commute with infinite intersection.

**Proposition 4.5.** *Let* $\{I_i\}_{i\in I}$ *be a family of ideals of an MTL-algebra L. Then* $\mathrm{Rad}(\bigcap_{i\in I} I_i) = \bigcap_{i\in I} \mathrm{Rad}(I_i)$.

*Proof.* *Suppose that* $x \in \bigcap_{i\in I} \mathrm{Rad}(I_i)$. *Then* $x \in \mathrm{Rad}(I_i)$ *for all* $i \in I$. *By Proposition 4.4, we get that* $x^{**} \odot (nx) \in I_i$ *for all* $n \in N$ *and for all* $i \in I$. *Thus* $x \odot (nx)^{**} \in \bigcap_{i\in I} I_i$ *for all* $n \in N$. *We obtain* $x \in \mathrm{Rad}(\bigcap_{i\in I} I_i)$, *that is,* $\mathrm{Rad}(\bigcap_{i\in I} I_i) \supseteq \bigcap_{i\in I} \mathrm{Rad}(I_i)$. *The reverse inclusion is true in general.* □

Recall that two ideals $I$ and $J$ of $L$ are *comaximal* if $I \vee J = L$, and a family $\{I_i\}_{i\in A}$ of ideals of $L$ is *pairwise comaximal* if for every $i \neq j$ in $A$, $I_i \vee I_j = L$.

**Proposition 4.6.** *For a residuated lattice L let* $I_1, I_2, ..., I_n$ *be n proper pairwise comaximal ideals and* $J_1, J_2, ..., J_n$ *be n ideals such that* $I_i \subseteq J_i$ *for* $1 \leq i \leq n$. *If* $\cap_{i=1}^n I_i = \cap_{i=1}^n J_i$, *then* $I_i = J_i$ *for all for* $1 \leq i \leq n$.

*Proof.* *Since* $I(L)$ *is distributive, we have* $I_1 \vee (\cap_{i=2}^n I_i) = \cap_{i=2}^n (I_1 \vee I_i) = \cap_{i=2}^n L = L$. *Then there exist* $a \in I_1$ *and* $b \in \cap_{i=2}^n I_i$ *such that* $1 = a \oplus b$. *Now, let* $x \in J_1$ *be arbitrary. Then* $x^{**} \wedge b^{**} \in J_1 \cap (\cap_{i=2}^n I_i) \subseteq \cap_{i=1}^n J_i = \cap_{i=1}^n I_i \subseteq I_1$. *Also,* $x^{**} \wedge a^{**} \in I_1$. *By Proposition 2.2, we have* $x = x \wedge 1 = x \wedge (a \oplus b) \leq (x^{**} \wedge a^{**}) \oplus (x^{**} \wedge b^{**}) \in I_1$ *and so* $a \in I_1$. *Therefore* $I_1 = J_1$. *Similarly, we can prove* $I_i = J_i$ *for* $i = 2, ..., n$. □

**Lemma 4.7.** *Let I be an ideal of a residuated lattice L such that* $\mathrm{Rad}(I) = I_1 \cap I_2$, *where* $I_1, I_2$ *are proper comaximal ideals of L. Then* $\mathrm{Rad}(I_i) = I_i$ *for* $i = 1, 2$.

*Proof.* *We have* $I_1 \cap I_2 = \mathrm{Rad}(I) = \mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I_1) \cap \mathrm{Rad}(I_2)$, $\mathrm{Rad}(I_1) \vee \mathrm{Rad}(I_1) = L$ *and* $I_i \subseteq \mathrm{Rad}(I_i)$ *for* $i = 1, 2$. *Using proposition 4.6, we get* $\mathrm{Rad}(I_i) = I_i$ *for* $i = 1, 2$. □

**Theorem 4.8.** *Let I be an ideal of an MTL-algebra L such that* $\mathrm{Rad}(I) = I_1 \cap I_2$ *where* $I_1, I_2$ *are proper comaximal ideals of L, then there exist proper comaximal ideals* $J_1$ *and* $J_2$ *such that* $I = J_1 \cap J_2$ *and* $\mathrm{Rad}(J_i) = \mathrm{Rad}(I_i) = I_i$ *for* $i = 1, 2$.

*Proof.* Since $I_1 \vee I_2 = L$, there exist $a \in I_1$ and $b \in I_2$ such that $1 = a \oplus b$ and we have $a \wedge b \in I_1 \cap I_2 = \mathrm{Rad}(I)$. Let $P$ be an arbitrary prime ideal of $L$ such that $I \subseteq P$. Then either $a^* \odot b \in P$ or $a \odot b^* \in P$ by Proposition 2.9. There exists a maximal ideal $M$ such that $P \subseteq M$. So $\mathrm{Rad}(I) \subseteq M$. Thus $a \wedge b \in M$. By proposition 2.11, $M$ is prime and so we have either $a \in M$ or $b \in M$, note that both cases do not occur together since $a \oplus b = 1$. Also, we have either $a \odot b^* \in M$ or $a^* \odot b \in M$ since $P \subseteq M$. We have two cases:

Case 1: If $a \in M$. Since $a \odot b^* \leq a$, we have $a \odot b^* \in M$. We will prove that $a^* \odot b \notin M$. Suppose that $a^* \odot b \in M$. Then by Proposition 2.2 we have

$$b \to (a \oplus (a^* \odot b)) = b \to (a^* \to (a^* \odot b)^{**}) = (a^* \odot b) \to ((a^* \odot b)^{**}) = 1.$$

Thus $b \leq (a \oplus (a^* \odot b)) \in M$ and so $b \in M$, which is a contradiction.

Case 2: If $a \notin M$. So $b \in M$ and then $a^* \odot b \in M$. Similar to Case 1, we can show that $a \odot b^* \notin M$.

We conclude that either ($a \odot b^* \in P$ and $a^* \odot b \notin P$) or ($a^* \odot b \in P$ and $a \odot b^* \notin P$). Put $A := \{P \in \mathrm{Spec}(L) \mid a \odot b^* \in P, I \subseteq P\}$, $B := \{P \in \mathrm{Spec}(L) \mid a^* \odot b \in P, I \subseteq P\}$, $J_1 := \bigcap A$ and $J_2 := \bigcap B$. By Proposition 2.8, it is clear that $I = (\bigcap A) \cap (\bigcap B) = J_1 \cap J_1$.

If $J_1 \vee J_2 \neq L$, then there exists a maximal ideal $M$ such that $J_1 \vee J_2 \subseteq M$. We obtain that $a \odot b^* \in M$ and $a^* \odot b \in M$, which is a contradiction. Thus $J_1 \vee J_2 = L$ and so $\mathrm{Rad}(J_1) \vee \mathrm{Rad}(J_2) = L$. It is clear that $J_i \subseteq I_i$. We get $\mathrm{Rad}(J_i) \subseteq \mathrm{Rad}(I_i)$. Also, we have $\mathrm{Rad}(J_1) \cap \mathrm{Rad}(J_2) = \mathrm{Rad}(I) = \mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I_1) \cap \mathrm{Rad}(I_2)$. Applying Proposition 4.6, we obtain $\mathrm{Rad}(J_i) = \mathrm{Rad}(I_i)$. Hence $\mathrm{Rad}(J_i) = I_i$ by Lemma 4.7. □

**Theorem 4.9.** *Let I be an ideal of an MTL-algebra L such that* $I = I_1 \cap \cdots \cap I_n$ *where* $I_1, I_2, ..., I_n$ *be n proper pairwise comaximal ideals of L. Then every ideal J of L with* $\mathrm{Rad}(I) = \mathrm{Rad}(J)$ *can be written uniquely as* $J = J_1 \cap \cdots \cap J_n$ *for some pairwise comaximal elements* $J_1, ..., J_n$ *such that* $\mathrm{Rad}(I_i) = \mathrm{Rad}(J_i)$ *for* $1 \leq i \leq n$.

*Proof.* By Proposition 4.3, we have $\mathrm{Rad}(I) = \mathrm{Rad}(I_1) \cap \cdots \cap \mathrm{Rad}(I_n) = \mathrm{Rad}(J)$. Using Theorem 4.8 and an inductive argument, we can prove that $J = J_1 \cap \cdots \cap J_n$ for some pairwise comaximal elements $J_1, ..., J_n$ such that $\mathrm{Rad}(I_i) = \mathrm{Rad}(J_i)$ for $1 \leq i \leq n$. We will prove the uniqueness. Let $J = J_1 \cap \cdots \cap J_n$ and $J = J'_1 \cap \cdots \cap J'_n$ be two decompositions with desired property. Put $K_i := J_i \cap J'_i$ for each $1 \leq i \leq n$. By Proposition 4.3, $\mathrm{Rad}(K_i) = \mathrm{Rad}(J_i) = \mathrm{Rad}(J'_i)$. Hence, $K'_i$s are pairwise comaximal, and so $J = K_1 \cap \cdots \cap K_n$ is another desired decomposition, and since $K_i \subseteq J_i \cap J'_i$ by Proposition 4.6, we have $J_i = K_i = J'_i$. □

**Proposition 4.10.** *Let $I, J$ be ideals of an MTL-algebra $L$ such that $I \subseteq J \subseteq \mathrm{Rad}(I)$. Then $I$ is a pseudo-irreducible ideal if and only if $J$ is a pseudo-irreducible ideal.*

*Proof.* Suppose that $I$ is a pseudo-irreducible ideal but $J$ is not a pseudo-irreducible ideal. Then there exist proper comaximal ideals $J_1, J_2$ such that $J = J_1 \cap J_2$. Since $I \subseteq J \subseteq \mathrm{Rad}(I)$, we have $\mathrm{Rad}(I) \subseteq \mathrm{Rad}(J) \subseteq \mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I)$ by Proposition 4.3. We obtain $\mathrm{Rad}(I) = \mathrm{Rad}(J) = \mathrm{Rad}(J_1) \cap \mathrm{Rad}(J_2)$, $\mathrm{Rad}(J_1) \vee \mathrm{Rad}(J_2) = L$ and $\mathrm{Rad}(J_i) \neq L$ by Proposition 4.3. Using Theorem 4.8, there exist proper ideals $I_1$ and $I_2$ such that $I = I_1 \cap I_2$ and $I_1 \vee I_2 = L$, which is a contradiction by pseudo-irreducibility of ideal $I$. Now if $J$ is pseudo-irreducible, then $I$ is also pseudo-irreducible by an easy argument as above and the fact that $\mathrm{Rad}(I) = \mathrm{Rad}(J)$ . □

## 5. Complete comaximal decomposition

In this section, we consider residuated lattices whose proper ideals can be written as an intersection of pairwise comaximal of finitely many pseudo-irreducible ideals. We start with the following definition.

**Definition 5.1.** *Suppose $I$ is a proper ideal of a residuated lattice $L$. We say that:*

1. *$I$ has a comaximal decomposition if $I$ can be written as $I = I_1 \cap \cdots \cap I_n$ of ideals of $L$ such that $I'_i$s are pairwise comaximal.*
2. *$I$ has a complete comaximal decomposition, whenever either $I$ is pseudo-irreducible or has a comaximal decomposition in which each of its factors is pseudo-irreducible.*

**Example 5.2.** *Suppose a residuated lattice $L$ has at least two maximal ideals $M$ and $N$. Put $I := M \cap N$. In this case, $M \cap N$ is a complete comaximal decomposition for $I$. Also, $I$ is not pseudo-irreducible because $M \vee N = L$, $M \neq L$ and $N \neq L$.*

In the following theorem, we show that a complete comaximal decomposition is unique if it exists. Hence, we can speak of *the* complete comaximal decomposition for an ideal $I$, if such a decomposition exists for $I$.

**Theorem 5.3** (Uniqueness decomposition)**.** *Suppose $I$ is an ideal of a residuated lattice $L$ that has a complete maximal decomposition. In this case, this compete comaximal decomposition is unique.*

*Proof.* Suppose $I = I_1 \cap \cdots \cap I_n$ and $I = J_1 \cap \cdots \cap J_m$ are two complete comaximal decompositions for $I$. For each $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ put $K_{ij} := I_i \vee J_j$. Clearly, $K'_{ij}$s are pairwise comaximal and for each $i \in \{1, ..., n\}$ since $I(L)$ is distributive, we have

$$I_i \subseteq K_{i1} \cap \cdots \cap K_{im} = (I_i \vee J_1) \cap \cdots \cap (I_i \vee J_m) = I_i \vee (J_1 \cap \cdots \cap J_m) = I_i \vee I = I_i.$$

As a result $I_i = K_{i1} \cap \cdots \cap K_{im}$ and since $I_i$ is pseudo-irreducible and $K'_{ij}$s are pairwise comaximal, we have there exists $j_i \in \{1, ..., m\}$ such that $I_i = K_{ij_i}$ and for every $j \in \{1, ..., m\} \setminus \{j_i\}$, $K_{ij} = L$. Similarly, for each $j \in \{1, ..., m\}$ there exists $i_j \in \{1, ..., n\}$ such that $J_j = K_{i_j j}$ and for every $i \in \{1, ..., n\} \setminus \{i_j\}$, $K_{ij} = L$. Hence, there is a one-to-one correspondence between two sets $\{I_1, ..., I_n\}$ and $\{J_1, ..., J_m\}$, and the corresponding ideals are equal. Therefore, regardless of the order of factors in decompositions of $I$, the number and factors are unique. □

**Remark 5.4.** *According to Proposition 3.12, if an ideal $I$ can be expressed as $I = I_1 \cap \cdots \cap I_n$, where $I'_i$s are pseudo-irreducible, then we can find the complete comaximal decomposition for $I$.*

**Definition 5.5.** *We say that a residuated lattice L has the complete comaximal decomposition property (for short, it has the CCD property) if every proper ideal of L has the complete comaximal decomposition.*

**Example 5.6.** *(1) By Proposition 3.5, every local (e.g., chain) residuated lattice has the CCD property.*
*(2) Let M be an infinite MV-algebra. Since $x \oplus y = x \vee y$ and $x \odot y = x \wedge y$ for each $x, y \in M$, every proper ideal of M is prime. Thus, for each $1 \neq x \in M$, the ideal $(x]$ is prime and so $|\operatorname{Spec}(M)| = \infty$ and $|\operatorname{MaxI}(M)| = 1$. Hence, by Proposition 3.5 M is a residuated lattice with infinitely many prime ideals that has the CCD property.*

Now we want to obtain equivalent conditions for a residuated lattice with the CCD property. For this recall that a topological space $X$ is called *Noetherian* if every ascending chain of open subsets of $X$ is stationary.

**Theorem 5.7.** *Let L be a residuated lattice. Then following are equivalent:*

1. *L has the CCD property.*
2. *For every subset $\{M_i\}_{i\in A}$ of $\operatorname{MaxI}(L)$, except for a finite number of elements of A, for each $j \in A$ we have $\bigcap_{j\neq i\in A} M_i \subseteq M_j$.*
3. *For every infinite subset $\{M_i\}_{i\in A}$ of $\operatorname{MaxI}(L)$ there exists $j \in A$ such that $\bigcap_{j\neq i\in A} M_i \subseteq M_j$.*
4. *$\operatorname{MaxI}(L)$ is a Noetherian topological space.*

*Proof.* (1) ⇒ (2). It is sufficient to prove the statement for infinite set index set $A$. Hence, suppose that $\{M_i\}_{i\in A}$ is an infinite subset of $\operatorname{MaxI}(L)$ and Put $I := \cap_{i\in I} M_i$. By assumption and Theorem 5.3, the pseudo-irreducible factors in the complete comaximal decomposition of $I$ are unique. Now suppose there are an infinite subset $B$ of $A$ such that for every $j \in B$ we have $\bigcap_{j\neq i\in A} M_i \nsubseteq M_j$. Suppose $j \in B$ is fixed. Then we have $(\bigcap_{j\neq i\in A} M_i) \vee M_j = 1$. Set $J := \bigcap_{j\neq i\in A} M_i$. By assumption $J$ has the complete comaximal decomposition, named, $J = J_1 \cap \cdots \cap J_n$. Now, since every maximal ideal is also pseudo-irreducible, $I = M \cap J_1 \cap \cdots \cap J_n$ is the complete comaximal decomposition for $I$. Therefore, for each $j \in B$, $M_j$ a pseudo-irreducible factor in the complete comaximal decomposition of $I$, and this is a contradiction.

(2) ⇒ (3) It is clear.

(3) ⇒ (4) On the contrary, assume that $\operatorname{MaxI}(L)$ is not a Noetherian topological space. In this case, there is a chain $D_{Max}(I_1) \subsetneq D_{Max}(I_2) \subsetneq \cdots \subsetneq D_{Max}(I_n) \subsetneq \cdots$ of open sets of $\operatorname{MaxI}(L)$. Thus, for every $2 \leqslant i$ there exists

$$M_i \in D_{Max}(I_i) \setminus D_{Max}(I_{i-1}).$$

Now let $2 \leqslant j$ be fixed and set $I := M_1 \cap \cdots \cap M_{j-1} \cap I_j$. If $I \subseteq M_j$, then by Lemma 3.3 either $I_j \leqslant M_j$ or $M_i \subseteq M_j$ for at least one $i \lneqq j$ both of which are contradiction. So $I \nsubseteq M_j$. Now if $i \in \{2, 3, ...\} \setminus \{j\}$, Then we consider two cases. If $i \lneqq j$, then $I \subseteq M_i$ and if $j \lneqq i$, then according to the choice $M_i$ and the definition of $I$ we have $I \subseteq M_i$. Hence, $I \subseteq \bigcap_{j\neq i\in A} M_i$. So $\bigcap_{j\neq i\in A} M_i \nsubseteq M_j$, which is a contradiction

(4) ⇒ (1) Suppose the statement (1) is not true. In this case, there is a proper ideal $I$ of $L$ that does not have the complete comaximal decomposition. So $I$ is not pseudo-irreducible and therefore there are ideals $I_1$ and $J_1$ of $L$ such that $I = I_1 \cap I_1$ and $I_1 \vee I_1 = L$. If $I_1$ and $J_1$ have the complete comaximal decomposition, then $I$ also have a complete comaximal decomposition according to Remark 5.4, which is a contradiction. So either $I_1$ or $J_1$ does not have the complete comaximal decomposition. Without loss of generality, we assume that $J_1$ does not have the complete comaximal decomposition. Therefore, there are proper ideals $I_2$ and $J_2$ of $L$ such that $J_1 = I_2 \cap J_2$ and $I_2 \vee J_2 = L$. By continuing the same process, we can obtain the following comaximal decompositions for $I$.

$$I = I_1 \cap J_1 = I_1 \cap I_2 \cap J_2 = I_1 \cap I_2 \cap I_3 \cap J_3 = \cdots .$$

Now for every $i \in \{1, 2, ...\}$, there exists a maximal ideal $M_i$ of $L$ such that $I_i \leqslant M_i$. According to the construction of $I_i$, we have

$$M_i \vee (\textstyle\bigcap_{j\neq i} M_i). \geqslant I_i \vee (I_1 \cap \cdots \cap I_{i-1} \cap J_i) = L.$$

Hence for every $i \in \{1, 2, ...\}$, we have $M_i \vee (\bigcap_{j\neq i} M_i) = L$. Consequently, for each $i \in \{1, 2, ...\}$, we have $M_i \in D_{Max}(\bigcap_{j\geqslant i+1} M_j) \setminus D_{Max}(\bigcap_{j\geqslant i} M_j)$. Therefore, we have the following non-stationary ascending chain of open subsets of MaxI($L$) that is a contradiction

$$D_{Max}(\bigcap_{j\geqslant 1} M_j) \subsetneq D_{Max}a(\bigcap_{j\geqslant 2} M_j) \subsetneq \cdots \subsetneq D_{Max}(\bigcap_{j\geqslant n} M_j) \subsetneq \cdots,$$

which is a contradiction. □

**Corollary 5.8.** *(1) Every finite residuated lattice has the CCD property.*
*(2) Every residuated lattice that has a finite number of maximal ideals also has the CCD property.*

**Theorem 5.9.** *Let $L$ be an MTL-algebra. Then equivalent conditions of Theorem 5.7 are equivalent to the fact that every closed subset of* Spec($L$) *has finitely many connected components.*

*Proof.* ⇒). Let $C$ be a closed subset of Spec($L$). Thus, there is an ideal $I$ of $L$ such that $C = V(I)$. Let $I = \cap_{i=1}^{n} I_i$ be the complete comaximal decomposition of $I$. Since $I_1, I_2, ..., I_n$ are pairwise comaximal, we have $V(I) = \cup_{i=1}^{n} V(I_i)$ is a disjoint union of connected closed subsets of Spec($L$) by Theorem 3.10. Now let $C$ be a connected component of $V(I)$. Thus $C = \cup_{i=1}^{n}(C \cap V(I_i))$. Now since $C \cap V(I_1), C \cap V(I_2), ..., C \cap V(I_n)$ are $n$ pairwise disjoint closed subsets of $V(I)$ which cover $C$, there is $1 \leq i \leq n$ such that $C = C \cap V(I_i)$ and $C \cap V(I_j) = \emptyset$ for each $j \in \{1, 2, ..., n\} \setminus \{i\}$. Thus $C \subseteq V(I_i)$. By Theorem 3.10, $V(I_i)$ is connected. Hence, $C = V(I_i)$ and so $C$ has finitely many connected components.

⇐). Let $I$ be a proper ideal of $L$ and let $C_1, C_2, ..., C_m$ be all connected components of $V(I)$. Hence by Theorem 3.10 for each $i = 1, ..., n$, there exists a pseudo-irreducible ideal $I_i$ such that $C_i = V(I_i)$ and since $C_i's$ are disjoint, we have $I_i's$ are pairwise comaximal. Now by Proposition 2.8, we have

$$I = \bigcap V(I) = \bigcap_{i=1}^{n} V(I_i) = I_1 \cap \cdots \cap I_n,$$

and we are done. □

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