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On a modification of finite-dimensional Niemytzki spaces

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Abstract. In this paper we extend the construction of the Niemytzki plane to dimension $n \geq 3$. Further, we consider a poset of topologies on the closed n-dimensional Euclidean half-space similar to one from [1] which is related to the Niemytzki plane topology. Then we explore properties of topologies from the poset.

1. Introduction

The Niemytzki plane (cf. [9, Example 82]) is a classical example of a topological space (like the square of the Sorgenfrey line (cf. [9, Example 84])) which is Tychonoff but not normal. Besides that the Niemytzki plane is a separable, first-countable, perfect, realcompact, Cech-complete space which is neither countably paracompact nor weakly paracompact (cf. [5]). Recently it was proved that the Niemytzki plane is even κ-metrizable [2].

In this paper we extend the construction of the Niemytzki plane to dimension $n \geq 3$. Further, we consider a poset of topologies on the closed n-dimensional Euclidean half-space similar to one from [1] which is related to the Niemytzki plane topology. Then we explore properties of topologies from the poset.

For standard notions we refer to [5].

2. Finite-dimensional Niemytzki spaces and their properties

We generalize to dimension $n \geq 3$ the construction of Niemytzki plane (cf. [9, Example 82]).

Construction 2.1. Consider subsets $P_n = \{\overline{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, x_n > 0\}$ and $L_n = \{\overline{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, x_n = 0\}$ of $\mathbb{R}^n$.

We generate a topology τ_N on $X_n = P_n \cup L_n$ as follows.

If $\overline{a} \in P_n$, then a local base of τ_N at $\overline{a}$ consists of sets $B(\overline{a}, \epsilon) = \{\overline{x} \in R^n : |\overline{x} - \overline{a}| < \epsilon\}$, where $|\overline{x} - \overline{a}| = \sqrt{\Sigma_{i=1}^n (x_i - a_i)^2}$ and $0 < \epsilon < a_n$.

If $\overline{a} \in L_n$, then a local base of τ_N at $\overline{a}$ consists of sets $\tilde{B}(\overline{a}, \epsilon) = \{\overline{a}\} \cup B(\overline{a(\epsilon)}, \epsilon)$, where $\overline{a(\epsilon)} = (a_1, \ldots, a_{n-1}, \epsilon)$ and $0 < \epsilon$.

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Let us list some obvious properties of the finite-dimensional Niemytzki spaces.

Note that the space (X_n, τ_N) is first-countable, the restriction of the topology τ_N onto P_n (resp. L_n) coincides with the Euclidean topology on P_n (resp. with the discrete topology on L_n), the subset P_n (resp. L_n) is open and dense (resp. closed and nowhere dense) in the space (X_n, τ_N). In particular, the space (X_n, τ_N) is separable and perfect.

Denote the Euclidean topology on X_n by τ_E. It is evident that $\tau_E \subseteq \tau_N$. So the space (X_n, τ_N) is completely Hausdorff.

It is easy to see that for any $\overline{a} \in L_n$ the boundary $\mathrm{Bd}_N \tilde{B}(\overline{a}, \epsilon)$ of the set $\tilde{B}(\overline{a}, \epsilon)$ in the space (X_n, τ_N) is equal to $\mathrm{Bd}_E B(\overline{a(\epsilon)}, \epsilon) \setminus \{\overline{a}\}$, where $\mathrm{Bd}_E B(\overline{a(\epsilon)}, \epsilon)$ is the boundary of the set $B(\overline{a(\epsilon)}, \epsilon)$ in the space (X_n, τ_E).

Moreover, $B(\overline{a(\epsilon)}, \epsilon)$ is the disjoint union of the sets $\mathrm{Bd}_N \tilde{B}(\overline{a}, t\epsilon)$, where $0 < t < 1$. Note that for each $\overline{x} \in B(\overline{a(\epsilon)}, \epsilon)$ the point $\overline{x}$ belongs to the only set $\mathrm{Bd}_N \tilde{B}(\overline{a}, t(\overline{x})\epsilon)$ with $t(\overline{x}) = (\Sigma_{i=1}^{n-1}(x_i - a_i)^2 + x_n^2)/(2\epsilon x_n)$.

Proposition 2.2. *The space (X_n, τ_N) is Tychonoff.*

Proof. Since $\tau_E \subseteq \tau_N$, it is enough to show that for any $\overline{a} \in L_n$ and any $\tilde{B}(\overline{a}, \epsilon)$ there exists a continuous function $f : (X_n, \tau_N) \to [0, 1]$ such that $f(\overline{a}) = 0$ and $f|_{\mathrm{Bd}_N \tilde{B}(\overline{a}, \epsilon)} = 1$. In fact, set $f(\overline{x}) = 1$ for any $\overline{x} \in X_n \setminus \tilde{B}(\overline{a}, \epsilon)$, $f(\overline{a}) = 0$ and $f(\overline{x}) = t(\overline{x})$ for any $\overline{x} \in B(\overline{a(\epsilon)}, \epsilon)$. □

It is easy to see that the set $X'_{m,n} = \{\overline{x} \in X_n : x_1 = \ldots x_{n-m} = 0\}$, where $2 \le m < n$, is a closed subset of (X_n, τ_N), and the subspace $X'_{m,n}$ of (X_n, τ_N) is homeomorphic to the space (X_m, τ_N). Since the space (X_2, τ_N) is the Niemytzki plane, the space (X_n, τ_N) is in particular neither normal, countably paracompact nor weakly paracompact.

3. Topologies between the Euclidean and finite-dimensional Niemytzki

In [1] Abuzaid, Alqahtani and Kalantan suggested by the use of technique from [7] a poset $\mathcal{T}$ of topologies on the set X_2 such that the minimimal topology is τ_E and the maximal topology is τ_N. We will extend the construction to the sets $X_n, n \geq 3$.

Construction 3.1. Let A be a subset of L_n. We generate a topology $\tau(A)$ on $X_n = P_n \cup L_n$ as follows.

If $\overline{a} \in P_n$, then a local base of $\tau(A)$ at $\overline{a}$ consists of sets $B(\overline{a}, \epsilon)$, where $0 < \epsilon < a_n$.

If $\overline{a} \in A$, then a local base of $\tau(A)$ at $\overline{a}$ consists of sets $B(\overline{a}, \epsilon) \cap X_n$, where $0 < \epsilon$.

If $\overline{a} \in L_n \setminus A$, then a local base of $\tau(A)$ at $\overline{a}$ consists of sets $\tilde{B}(\overline{a}, \epsilon)$, where $0 < \epsilon$.

It is evident that $\tau_E = \tau(L_n) \subseteq \tau(A) \subseteq \tau(\emptyset) = \tau_N$. Note also that for any $A, B \subseteq L_n$ we have $A \subseteq B$ if and only if $\tau(A) \supseteq \tau(B)$.

Note that the space $(X_n, \tau(A))$ is first-countable and separable, the restriction of the topology $\tau(A)$ onto $P_n \cup A$ (resp. $L_n \setminus A$) coincides with the Euclidean topology on $P_n \cup A$ (resp. with the discrete topology on $L_n \setminus A$), the subset P_n (resp. L_n) is open and dense (resp. closed, even a zero set, and nowhere dense) in the space $(X_n, \tau(A))$.

Similarly to (X_n, τ_N), one can prove that the space $(X_n, \tau(A))$ is Tychonoff.

Proposition 3.2. (for $n = 2$ see [1, Theorem 2.3]) *The following are equivalent.*

(1) *The space $(X_n, \tau(A))$ is hereditarily Lindelöf.*

(2) $|L_n \setminus A| \leq \aleph_0$.

(3) *The space $(X_n, \tau(A))$ is second-countable.*

(4) *The space $(X_n, \tau(A))$ is metrizable.*

Proof. (1) => (2). Since the space $(L_n \setminus A, \tau(A)|_{L_n \setminus A})$ is discrete, $|L_n \setminus A| \leq \aleph_0$.

(2) => (3). Let $L_n \setminus A = \{\overline{b_1}, \overline{b_2}, \dots\}$, $\mathcal{B}$ be a countable base for the space (X_n, τ_E) and $\mathcal{B}_i$ be a countable local base of the space $(X_n, \tau(A))$ at the point $\overline{b_i}$, $i = 1, 2, \dots$ Then the family $\mathcal{B} \cup \cup_{i=1}^{\infty} \mathcal{B}_i$ is a countable base for the space $(X_n, \tau(A))$.

(4) => (1). Since the space $(X_n, \tau(A))$ is separable, it is second-countable and hence hereditarily Lindelöf. □

Let $\overline{a} \in L_n \setminus A$. It is easy to see that any sequence of points $\{\overline{x_i}\}_{i=1}^{\infty}$ in $\mathrm{Bd}\,_E B(\overline{a(\epsilon)}, \epsilon) \setminus \{\overline{a}\}$ converging to $\overline{a}$ in the Euclidean topology τ_E is discrete in the space $(X_n, \tau(A))$. This implies the following proposition.

Proposition 3.3. (for $n = 2$ see [1, Theorem 2.6]) *The space $(X_n, \tau(A))$ is locally compact if and only if $A = L_n$ i. e. $\tau(A) = \tau_E$.*

4. Some other properties of the spaces $(X_n, \tau(A))$

The following proposition is evident.

Proposition 4.1. *Let A be any subset of L_n. Then $(X_n, \tau(A))$ is perfect (resp. Lindelöf or σ-compact) if and only if $(L_n, \tau(A)|_{L_n})$ is the same.*

Let us observe that the topology $\tau(A)|_{L_n}$ can be considered as a modification of the Euclidean topology on the set L_n in the sense of Bing [3] and Hanner [6], see [5, Example 5.1.22] for the general construction.

Note also that the space $(L_n, (\tau_E)|_{L_n})$ is homeomorphic to the Euclidean space $\mathbb{R}^{n-1}$, $(\tau_E)|_{L_n} \subseteq \tau(A)|_{L_n}$, $(\tau_E)|_A = \tau(A)|_A$, the set A is closed in the space $(L_n, \tau(A)|_{L_n})$ and the subspace $L_n \setminus A$ of $(L_n, \tau(A)|_{L_n})$ is discrete.

So from [5, Problem 5.5.2 (b)] we get that the space $(L_n, \tau(A)|_{L_n})$ is hereditarily collectionwise normal.

By [5, Problem 5.5.2 (c)] we easily obtain

Lemma 4.2. *The space $(L_n, \tau(A)|_{L_n})$ is perfect if and only if A is a G_δ-set in $(L_n, (\tau_E)|_{L_n})$.*

Lemma 4.3. *The space $(L_n, \tau(A)|_{L_n})$ is Lindelöf if and only if $L_n \setminus A$ does not contain a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$.*

Proof. => Let us note that any closed subset Y of $(L_n, (\tau_E)|_{L_n})$ such that $Y \subset L_n \setminus A$ is a closed discrete subset of $(L_n, \tau(A)|_{L_n})$. Hence, $|Y| \leq \aleph_0$.

<= Let α be an open cover of the space $(L_n, \tau(A)|_{L_n})$. Since $\tau(A)|_A = (\tau_E)|_A$, there exists a countable subfamily α_1 of α such that $A \subseteq \cup\alpha_1$. Moreover, by the definition of $\tau(A)$ there is an open set O of (L_n, τ_E) such that $A \subseteq O \subseteq \cup\alpha_1$. Let us note that $B = L_n \setminus O \subseteq L_n \setminus A$ is a closed subset of (L_n, τ_E). By assumption B is countable. So there exists a countable subfamily α_2 of α such that $B \subseteq \cup\alpha_2$. Let us observe that the subfamily $\alpha_1 \cup \alpha_2$ of α is countable and it covers L_n. □

Lemma 4.4. *The space $(L_n, \tau(A)|_{L_n})$ is σ-compact if and only if A is a F_σ-set in $(L_n, (\tau_E)|_{L_n})$ and $|L_n \setminus A| \leq \aleph_0$.*

Proof. => Since A is closed in $(L_n, \tau(A)|_{L_n})$, A is σ-compact in $(L_n, \tau(A)|_{L_n})$ and hence in $(L_n, \tau(E)|_{L_n})$. So A is a F_σ-set in $(L_n, (\tau_E)|_{L_n})$. Further, $L_n \setminus A$ is a G_δ-set in $(L_n, (\tau_E)|_{L_n})$ which is homeomorphic to $\mathbb{R}^{n-1}$. If $L_n \setminus A$ is uncountable, then there is an uncountable compact subset Y of $(L_n, (\tau_E)|_{L_n})$ such that $Y \subset L_n \setminus A$. Let us note that Y is a closed discrete uncountable subset of $(L_n, \tau(A)|_{L_n})$. Since the space $(L_n, \tau(A)|_{L_n})$ is σ-compact it is impossible. So $|L_n \setminus A| \leq \aleph_0$.

<= It is trivial. □

Proposition 4.1 and Corollories 4.2-4.4 imply

Theorem 4.5. *Let A be any subset of L_n. Then $(X_n, \tau(A))$ is perfect (resp. Lindelöf or σ-compact) if and only if A is a G_δ-set in $(L_n, (\tau_E)|_{L_n})$ (resp. $L_n \setminus A$ does not contain a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$ or A is a F_σ-set in $(L_n, (\tau_E)|_{L_n})$ and $|L_n \setminus A| \leq \aleph_0$).*

Corollary 4.6. *Let $B \subseteq L_n \setminus A$ be a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$, for example, B is homeomorphic to the Cantor set. Then the space $(X_n, \tau(A))$ is not Lindelöf.*

Remark 4.7. Since any Cantor set in $\mathbb{R}^n$ is nowhere dense in $\mathbb{R}^n$, Corollary 4.6 (the case $n = 2$) evidently disagrees with [1, Theorem 2.8 and Theorem 2.9]. Indeed, the fourth sentence of the proof [1, Theorem 2.8] is not correct. Roughly speaking it implies that any open dense subset of the real line must coincide with the real line.

Corollary 4.8. *If $(X_n, \tau(A))$ is σ-compact, then $(X_n, \tau(A))$ is second-countable.*

Corollary 4.9. *Let M be any countable dense subset of $(L_n, (\tau_E)|_{L_n})$. Then*

(1) *$(X_n, \tau(M))$ is neither perfect nor Lindelöf,*

(1) *$(X_n, \tau(L_n \setminus M))$ is second-countable but it is not σ-compact.*

Corollary 4.10. *Let A be any uncountable compact subset of $(L_n, (\tau_E)|_{L_n})$. Then $(X_n, \tau(A))$ (as well as $(X_n, \tau(L_n \setminus A))$) is perfect but it is not Lindelöf.*

A subset A of the Euclidean space $\mathbb{R}^n$ we will call *a Bernstein set* (cf. [8, p. 24] for the case $n = 1$) if both A and $\mathbb{R}^n \setminus A$ intersect every uncountable compact subspace F of $\mathbb{R}^n$. It is easy to see that if A is a Bernstein set of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus A$ is also a Bernstein set of $\mathbb{R}^n$. Moreover, the Bernstein sets are of the cardinality continuum and they do not contain uncountable compacta. It implies, in particular, that any Bernstein set in $\mathbb{R}^n$ is neither a G_δ-set nor an F_σ-set in $\mathbb{R}^n$.

Corollary 4.11. *Let A be a Bernstein set of the space $(L_n, (\tau_E)|_{L_n})$. Then the space $(X_n, \tau(A))$ is Lindelöf but it is not perfect.*

Recall (cf. [4, p. 65]) that a subset Y of a space X is called

(1) *C^*-embedded in X* if every bounded continuous function on Y can be extended to a bounded continuous function on X,

(2) *z-embedded* in X if every zero set of Y is the trace on Y of some zero set of X.

Let us recall that any closed subset Y of a normal space X is C^*-embedded in X, and if a subset Y of a space X is C^*-embedded in X, then Y is z-embedded in X.

Lemma 4.12. *Let Y be a discrete subspace of cardinality continuum of a separable space X. Then Y is not z-embedded in X.*

Proof. Let X be a separable space, Y be its discrete subspace of cardinality $\mathfrak{c}$, $\mathcal{F}$ be the family of all continuous functions on the space X and Z_X be the family of all zero sets on X. Since X is separable, the cardinality of $\mathcal{F}$ is at most $\mathfrak{c}$ and hence the cardinality of Z_X is also at most $\mathfrak{c}$. Let Z_Y be the family of all zero sets on Y. It is easy to see that the cardinality of Z_Y is at least $2^{\mathfrak{c}} > \mathfrak{c}$. So Y is not z-embedded in X. □

The following corollary is evident.

Corollary 4.13. *No separable normal space contains a closed discrete subspace of cardinality continuum.*

Recall ([5, Exercise 5.2. C (b)]) that no separable countably paracompact space contains a closed discrete subspace of cardinality continuum.

Proposition 4.14. *Let $B \subseteq L_n \setminus A$ be a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$. Then the space $(X_n, \tau(A))$ is neither normal nor countably paracompact.*

Proof. Since $(\tau_E)|_{L_n} \subseteq \tau(A)|_{L_n}$, the set B is a closed discrete subset of $(L_n, \tau(A)|_{L_n})$ (and even $(X_n, \tau(A))$) of cardinality continuum. Since the space $(X_n, \tau(A))$ is separable, it is neither normal nor countably paracompact. □

Corollary 4.15. *If the space $(X_n, \tau(A))$ is normal (or countably paracompact), then $L_n \setminus A$ does not contain a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$.*

Theorem 4.16. *The following are equivalent.*

(1) *The space $(X_n, \tau(A))$ is Lindelöf.*

(2) *The space $(X_n, \tau(A))$ is paracompact.*

(3) *The space $(X_n, \tau(A))$ is countably paracompact.*

(4) *The space $(X_n, \tau(A))$ is normal.*

(5) *The set $L_n \setminus A$ does not contain a closed uncountable subset $(L_n, (\tau_E)|_{L_n})$.*

Proof. Let us only note that (3) (or (4)) => (5) (by Corollary 4.15). (5) => (1) (by Theorem 4.5). □

The following lemma is evident.

Lemma 4.17. *Let X be a space and $Z \subseteq Y \subseteq X$. If Z is z-embedded in Y and Y is z-embedded in X, then Z is z-embedded in X.*

Proposition 4.18. (1) *Let B be a subset of $L_n \setminus A$ of cardinality continuum. Then B is not z-embedded in $(X_n, \tau(A))$.*

(2) *Let $B \subseteq L_n \setminus A$ be a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$, for example, B is homeomorphic to the Cantor set. Then the subset L_n of $(X_n, \tau(A))$ is not z-embedded in $(X_n, \tau(A))$.*

Proof. (1) Observe that the subspace $(B, \tau(A)|_B)$ of the space $(X_n, \tau(A))$ is discrete and has the cardinality continuum. Since $(X_n, \tau(A))$ is separable, by Lemma 4.12 the set B is not z-embedded in $(X_n, \tau(A))$.

(2) Note that B is a closed discrete subset of the space $(L_n, \tau(A)|_{L_n})$ (and also $(X_n, \tau(A))$ of cardinality continuum. Hence by (1) the set B is not z-embedded in $(X_n, \tau(A))$. Recall that the space $(L_n, \tau(A)|_{L_n})$ is normal. So the set B is z-embedded in $(L_n, \tau(A)|_{L_n})$. If we assume that the closed subset L_n of $(X_n, \tau(A))$ is z-embedded in $(X_n, \tau(A))$ we will get a contradiction with Lemma 4.17. □

Corollary 4.19. *If the subset L_n of $(X_n, \tau(A))$ is z-embedded in $(X_n, \tau(A))$, then the set $L_n \setminus A$ does not contain a closed uncountable subset of $(L_n, (\tau_E)|_{L_n})$,*

Theorem 4.20. *The following are equivalent.*

(1) *The space $(X_n, \tau(A))$ is normal.*

(2) *The subset L_n of $(X_n, \tau(A))$ is C^*-embedded in $(X_n, \tau(A))$.*

(3) *The subset L_n of $(X_n, \tau(A))$ is z-embedded in $(X_n, \tau(A))$.*

Proof. (3) => (1) (by Corollary 4.19 and Theorem 4.16). □

Remark 4.21. For any subset A of L_n we have
$\mathrm{ind}\,(L_n, \tau(A)|_{L_n}) = \mathrm{Ind}\,(L_n, \tau(A)|_{L_n}) = \dim(L_n, \tau(A)|_{L_n}) = \dim A$, and
$\mathrm{ind}\,(X_n, \tau(A)) = \mathrm{Ind}\,(X_n, \tau(A)) = \dim(X_n, \tau(A)) = n$ for any A for which the space $(X_n, \tau(A))$ is normal.

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