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On G-quotient spaces

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Abstract. The G-convergence of sequences is a general method of convergence of sequences in topological spaces. The operations related to G-methods in topological spaces have been studied. In this paper, we discuss G-quotient spaces, which are a class of generalized quotient spaces, and obtain the relationship between G-quotient topology and the largest topology of the range that makes the mapping be G-continuous. These results can be applied to statistical convergence and ideal convergence in topological spaces.

1. Introduction

Various types of convergence of sequences in topological spaces can be defined, which mostly came from research with application backgrounds, and have already achieved rich results. The following method of G-convergence is general [5].

Let X be a set, $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x} \in s(X)$ if and only if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. A *method* on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$ [11, Definition 1.1]. A sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be *G-convergent* to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$. Therefore, the usual convergence of sequences [8], statistical convergence [7], and ideal convergence [9] are special cases of G-convergence.

The essential connection between G-methods and topological spaces is established by the concept of G-open sets. From this, we can define G-continuity and spaces determined by G-convergence, and discuss the mutual relationship between spaces and mappings with the help of relevant continuity [3, 11]. The G-method has become an effective and general method for studying convergence and continuity in general topology [1].

The operation of topological spaces is the fundamental form of studying topology [8]. The subspaces and product spaces defined by G-methods have obtained some basic results [11, 13, 14, 16, 17, 19]. Although the quotient mappings defined by G-methods have also been discussed in several papers, there has been little research on related quotient topologies [4, 11, 12].

A G-open set is a type of generalized open sets [11]. We can try to discuss G-quotient topologies from the perspective of generalized quotient spaces. On the other hand, the quotient topology is the largest topology of the range that makes the mapping be surjective and continuous [8], thus the following problem is natural.

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Problem 1.1. *[10, Question 1.4] How to characterize the largest topology of the range that makes the mapping preserve G-convergence or be G-continuous?*

It is easy to see that suppose that G is a method of sets X and Y, then a mapping $f : X \to Y$ is G-continuous (resp., preserves G-convergence) if and only if the discrete topology is the largest topology of Y that makes f be G-continuous (resp., preserve G-convergence). The answer to Problem 1.1 is so simple, because the concept of a method on a set is independent of the one of a topology on the set. In order to better analogize to quotient topology, we need to add some relations between methods and topologies when Problem 1.1 is discussed. If G is the usual convergence, statistical convergence, or ideal convergence in a topological space, then every open set in this space is a G-open set. This paper requires that every open subset of a topological space is always G-open in the space.

The above problem has the following partial answer for ideal convergence.

Theorem 1.2. ([10, Theorem 5.8]) *Let (X, τ) be an $\mathcal{I}$-topological space, $f : X \to Y$ be a surjective mapping and $(Y, \tau_{f,\mathcal{I}})$ be an $\mathcal{I}$-sequential space. Then $\tau_{f,\mathcal{I}}$ is the largest topology of the set Y that makes $f : (X, \tau) \to Y$ be $\mathcal{I}$-continuous.*

In this paper, we give some examples showing that the topology of G-quotient spaces is not the topology required by Problem 1.1, and obtain a necessary and sufficient condition to Problem 1.1 for G-continuity in a positive answer, which is a generalization of Theorem 1.2.

2. Generalized quotient spaces

A class $\mu \subset \exp X$ is called a *generalized topology* [6] on X when $\emptyset \in \mu$ and the union of every family of members of μ is again a member of μ. A pair (X, μ), where X is a non-empty set and μ is a generalized topology on it, is said to be a *generalized topological space* [6]. Every element of μ is called a *generalized open set* in X. Let us say that a function $f : (X, \mu_X) \to (Y, \mu_Y)$ between generalized topological spaces is *(μ_X, μ_Y)-continuous* [6] if $f^{-1}(V) \in \mu_X$ whenever $V \in \mu_Y$. Evidently, every topology is a generalized topology. However, a continuous mapping is not always generalized continuous [6].

Given a generalized topological space (X, μ_X), and $\pi : X \to Y$ a surjective mapping, it is easy to see that

$$\mu_\pi = \{V \subset Y : \pi^{-1}(V) \in \mu_X\}$$

is a generalized topology on Y. The family μ_π is called the *generalized quotient topology* [2] induced on Y by π, the pair (Y, μ_π) is called the *generalized quotient space* [2] of X, and the mapping π is called a *generalized quotient mapping* [2].

It is easy to see that the generalized quotient topology μ_π is the largest generalized topology μ_Y on Y that makes the mapping π be (μ_X, μ_Y)-continuous [2]. Recently, generalized topological spaces with a hereditary class were studied [15].

Let $G : c_G(X) \to X$ be a method on a set X and $A \subset X$. The set A is called a *G-closed subset* of X if, whenever $\mathbf{x} \in s(A) \cap c_G(X)$, then $G(\mathbf{x}) \in A$ [11, Definition 2.1]; A is called a *G-open subset* of X if $X \setminus A$ is G-closed in X [11, Definition 3.1]. Open sets in a topological space are not always G-open [11, Example 2.13]. The family of all G-open subsets of X is a generalized topology on X [11, Proposition 3.2]. **For a method G, we require that the open sets are always G-open sets in this paper.**

Definition 2.1. Let G be a method of sets X and Y. Given a mapping $f : X \to Y$, f is called *G-continuous* if $f^{-1}(U)$ is a G-open subset of X whenever U is G-open in Y; f is called *preserving G-convergence* if $f(\mathbf{x}) \in c_G(Y)$ and $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for each $\mathbf{x} \in c_G(X)$.

The definition of G-continuity is given in the form of continuity in topological spaces or generalized continuity in generalized topological spaces. It is well-known that every mapping preserving G-convergence is G-continuous [11, Theorem 7.3], but the converse statement is not true, see Example 2.3. For a G-method, G-continuity is initially defined by preserving G-convergence [5, 11, 18].

Definition 2.2. Suppose that G is a method of sets X and Y, and a mapping $f : X \to Y$ is surjective. f is called a *G-quotient mapping* [4, Definition 3.1] if for each $U \subset Y$, the set $f^{-1}(U)$ is G-open in X if and only if U is G-open in Y, where the space Y is called a *G-quotient space* induced by the mapping f and the method G.

Example 2.3. There exist a method G of sets X and Y, and a G-quotient mapping $f : X \to Y$ which does not preserve G-convergence.

Let X be the set $\mathbb{Z}$ of all integers. Put

$$c_G(X) = \{\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X) : \text{ there exist } m \in \mathbb{N} \text{ and } a_{\boldsymbol{x}} \in X \text{ such that } x_n = x_{n-1} + a_{\boldsymbol{x}} \text{ for each } n > m\}.$$

Define $G : c_G(X) \to X$ by $G(\boldsymbol{x}) = a_{\boldsymbol{x}}$ for each $\boldsymbol{x} \in c_G(X)$. Then G is a method on X. It is easy to see that if A is a G-closed subset in X then $A = \emptyset$ or $0 \in A$.

Put $Y = \{0, 1\}$ with the method G. Then

$$c_G(Y) = \{\{y_n\}_{n\in\mathbb{N}} \in s(Y) : \text{ there exists } m \in \mathbb{N} \text{ such that } y_n = y_{n+1} \text{ for each } n > m\},$$

and $G(\boldsymbol{y}) = 0$ for each $\boldsymbol{y} \in c_G(Y)$. Then the family of all G-open subsets of Y is $\{\emptyset, \{1\}, Y\}$, which is denoted by $\tau_{Y,G}$.

A mapping $f : X \to Y$ is defined as follows: $f(x) = 0$ if and only if $x = 2k$, $k \in \mathbb{Z}$. Then $f^{-1}(0) = \{2k : k \in \mathbb{Z}\}$ is not G-open in X, and $f^{-1}(1) = \{2k - 1 : k \in \mathbb{Z}\}$ is G-open in X.

(1) $f : X \to Y$ is a G-quotient mapping.

It is easy to check that a subset U of Y is G-open in Y if and only if $f^{-1}(U)$ is G-open in X. This implies that f is G-quotient.

(2) $f : X \to Y$ does not preserve G-convergence.

Put $\boldsymbol{x} = \{n\}_{n\in\mathbb{N}} \in s(X)$. Then $\boldsymbol{x} \in c_G(X)$, $G(\boldsymbol{x}) = 1$, and $f(\boldsymbol{x}) = \{1, 0, 1, 0, \cdots\} \notin c_G(Y)$. Thus f does not preserve G-convergence.

(3) $\tau_{Y,G}$ is the largest topology of the set Y that makes $f : X \to Y$ be G-continuous.

Let $\tau_{X,G}$ be the topology of X generated by all G-open subsets of X as a subbase. The set Y is endowed with the topology $\tau_{Y,G}$. It follows from (1) that $f : (X, \tau_{X,G}) \to (Y, \tau_{Y,G})$ is G-continuous. Since $f^{-1}(\{0\})$ is not G-open in X, the family $\tau_{Y,G}$ is the largest topology of the set Y that makes $f : (X, \tau_{X,G}) \to Y$ be G-continuous.

Why is the discrete topology not the largest topology of the set Y that makes $f : X \to Y$ be G-continuous? Since we require each open set of Y to be a G-open set, if the discrete topology is the topology, then every subset of Y must be G-open, which is a contradiction.

(4) Changing the topologies of the spaces X and Y.

Regardless of whether the sets X and Y are endowed with topologies or not, under the given method G on X and Y, the mapping $f : X \to Y$ is always a G-quotient mapping which does not preserve G-convergence. If the sets X and Y are given appropriate topologies, since we require that the open sets are always G-open, the topologies τ_X, τ_Y of the sets X and Y satisfy $\tau_X \subset \tau_{X,G}$, $\tau_Y \subset \tau_{Y,G}$, respectively.

Let $f : X \to Y$ be a G-quotient mapping for a method G of topological spaces X and Y. Example 2.3 shows that the topology of Y may not necessarily be the largest topology that makes f be G-continuous? Example 2.4 provides another example about it.

Example 2.4. There exists a G-quotient mapping $f : X \to Y$ such that f preserves G-convergence and the topology of Y is not the largest topology of Y that makes f preserve G-convergence or be G-continuous.

Let X be a topological space with topology τ and Y a set as $Y = X$. Then define a G-method on X and therefore on Y with $G(\boldsymbol{x}) = x_1$ for any sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X)$. Then all subsets of X and Y are G-closed, thus they are G-open. Hence for any topology σ on Y a surjective mapping $f : (X, \tau) \to (Y, \sigma)$ is a G-quotient mapping and preserves G-convergence, but the greatest topology on Y such that f is G-continuous is the discrete topology on Y which might differ σ. It is obvious that all open subsets of X and Y are G-open.

Therefore, in order to provide a positive answer to Problem 1.1, we need to attach certain conditions to the topology of G-quotient spaces.

3. The main results

In this section, we seek a partial answer to Problem 1.1 for G-continuity. This problem involves the topology generated by all G-open sets in topological spaces.

Suppose that G is a method of sets X and Y. Let $f : X \to Y$ be a surjective mapping. Put

$$\mu_G = \{U \subset X : U \text{ is } G\text{-open in } X\},$$
$$\mu_{G,f} = \{V \subset Y : f^{-1}(V) \text{ is } G\text{-open in } X\}.$$

Then μ_G, $\mu_{G,f}$ are generalized topologies on X, Y, respectively. The family $\mu_{G,f}$ is the largest generalized topology on Y that makes $f : (X, \mu_G) \to (Y, \mu_{G,f})$ be G-continuous. Problem 1.1 is to characterize topology rather than generalized topology.

Let $f : X \to (Y, \tau_Y)$ be G-continuous, where τ_Y is a topology on Y. If $V \in \tau_Y$, then V is G-open in Y, thus $f^{-1}(V) \in \mu_{G,f}$; i.e., $\tau_Y \subset \mu_{G,f}$. Therefore, we can consider the topology of Y generated by $\mu_{G,f}$.

Definition 3.1. Let G be a method of a set X.

(1) The topology of the set X generated by all G-open subsets as a subbase is called a *G-open topology* induced by the method G, which is denoted by τ_G.

(2) The topological space (X, τ_G) is called a *G-open topological space*, which is denoted by X_G.

(3) X is called a *G-topological space* [11, Definition 6.1] if the family of all G-open subsets of X is closed under finite intersections.

A topological space (X, τ) with a method G is called a *G-sequential space* if $\tau_G \subset \tau$ [11, Definition 5.1]. It follows from $\tau \subset \tau_G$ that X is G-sequential if and only if $\tau = \tau_G$. The following result is a partial answer to Problem 1.1, which shows some conditions for answering Problem 1.1 positively.

Theorem 3.2. *The following are equivalent for a topological space (X, τ) and a method G of X.*

(1) *Spaces X and X_G have the same G-open subsets.*

(2) *The identity $id : X \to X_G$ is a G-quotient mapping.*

(3) *The identity $id : X \to X_G$ is a G-continuous mapping.*

(4) *X is a G-topological space and X_G is a G-sequential space.*

(5) *τ_G is the largest topology μ of the set X that makes $id : (X, \tau) \to (X, \mu)$ be G-continuous.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (5). Suppose that $\mathrm{id} : X \to X_G$ is G-continuous. If $\mathrm{id} : (X, \tau) \to (X, \mu)$ is G-continuous and $U \in \mu$, then U is G-open in the space (X, μ). Since $\mathrm{id} : (X, \tau) \to (X, \mu)$ is G-continuous, the set U is G-open in (X, τ), i.e., $U \in \tau_G$, and thus $\mu \subset \tau_G$. This implies that (5) is true.

(5) ⇒ (4). We can assume that $\mathrm{id} : (X, \tau) \to X_G$ is G-continuous. If A and B are G-open subsets in (X, τ), then A and B are open in the space X_G, thus $A \cap B$ is also open in X_G, therefore $A \cap B$ is G-open in X_G. Since $\mathrm{id} : (X, \tau) \to X_G$ is G-continuous, the set $A \cap B$ is G-open in the space (X, τ). Hence, the family of G-open subsets of (X, τ) is closed under finite intersections, and it implies that (X, τ) is a G-topological space.

If V is a G-open subset of the space X_G, it follows from $\mathrm{id} : (X, \tau) \to X_G$ being G-continuous that V is G-open in (X, τ), thus $V \in \tau_G$, i.e., V is open in X_G. This shows that X_G is a G-sequential space.

(4) ⇒ (1). Suppose that X is a G-topological space and X_G is a G-sequential space. Obviously, every G-open subset of X is G-open in X_G. If U is a G-open subset of X_G, it follows from X_G being G-sequential that U is open in X_G. Thus, U can be represented as a finite intersection of G-open subsets in X. Since X is a G-topological space, the set U is G-open in X. Therefore, the spaces X and X_G have the same G-open subsets. □

The following result is a generalization of [12, Theorem 3.2], which was proved under the assumption G being a subsequential method. A method $G : c_G(X) \to X$ is called *subsequential* [11, Definition 1.1] if, whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, there exists a subsequence $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $\boldsymbol{x}'$ is the usual convergent sequence in the space X with $\lim \boldsymbol{x}' = l$. If G is a subsequential method on X, then every open set of X is G-open [11, Lemma 2.11].

Corollary 3.3. *Let G be a method of a set X. If τ, μ are the topologies of X with $\tau_G \subset \mu$, then the spaces (X, τ) and (X, μ) have the same G-open subsets if and only if $\mu = \tau_G$, (X, τ) is a G-topological space and (X, μ) is a G-sequential space.*

Proof. The sufficiency is true by Theorem 3.2. Next, we will show the necessary.

Suppose that the spaces (X, τ) and (X, μ) have the same G-open subsets. Since $\tau_G \subset \mu$ and $\mu \subset \mu_G = \tau_G$, we have that $\mu = \tau_G$. Thus the spaces X and X_G have the same G-open subsets. By Theorem 3.2, the space (X, τ) is a G-topological space and the space (X, μ) is a G-sequential space. □

Next, we will further consider extending the identity in Theorem 3.2 to a mapping.

Definition 3.4. Suppose that X and Y are sets, $f : X \to Y$ is a surjective mapping and G is a method on X. The family $\tau_{f,G} = \{U \subset Y : f^{-1}(U) \in \tau_G\}$ is a topology of Y, and is called a *G-open topology* induced by the method G and the mapping f.

The following is the main result in this paper, in which the assumption of G-topological spaces is also a necessary condition under identity mappings by Theorem 3.2.

Theorem 3.5. *Let G be a method on sets X and Y. Suppose that (X, τ) is a G-topological space and $f : X \to Y$ is a surjective mapping, then the following are equivalent.*

(1) *$(Y, \tau_{f,G})$ is a G-sequential space.*
(2) *$f : (X, \tau) \to (Y, \tau_{f,G})$ is a G-quotient mapping.*
(3) *$f : (X, \tau) \to (Y, \tau_{f,G})$ is a G-continuous mapping.*
(4) *$\tau_{f,G}$ is the largest topology of Y that makes $f : (X, \tau) \to Y$ be G-continuous.*

Proof. (1) ⇒ (2). Suppose that $(Y, \tau_{f,G})$ is a G-sequential space. Let U be a G-open subset of the space Y. It follows from Y being G-sequential that the set U is open in Y, i.e., $U \in \tau_{f,G}$, thus $f^{-1}(U) \in \tau_G$. Since X is a G-topological space, we have that $f^{-1}(U)$ is G-open in the space X. Hence, f is G-continuous. On the other hand, let U be a subset of the space Y such that $f^{-1}(U)$ is G-open in X. Then $U \in \tau_{f,G}$, and thus U is G-open in the space Y. Therefore, f is a G-quotient mapping.

(2) ⇒ (3). It is obvious.

(3) ⇒ (4). Suppose that $f : (X, \tau) \to (Y, \tau_{f,G})$ is G-continuous. Let ν be a topology of the set Y such that $f : (X, \tau) \to (Y, \nu)$ is G-continuous. If $V \in \nu$, then V is G-open in Y, thus $f^{-1}(V)$ is G-open in X, and so $V \in \tau_{f,G}$. Hence, $\nu \subset \tau_{f,G}$. On the other hand, the mapping $f : (X, \tau) \to (Y, \tau_{f,G})$ is G-continuous. This implies that $\tau_{f,G}$ is the largest topology of Y that makes $f : (X, \tau) \to Y$ be G-continuous.

(4) ⇒ (1). Suppose that $\tau_{f,G}$ is the largest topology of Y that makes $f : (X, \tau) \to Y$ be G-continuous. Then the mapping $f : (X, \tau) \to (Y, \tau_{f,G})$ is G-continuous. If V is a G-open subset of the space $(Y, \tau_{f,G})$, then $f^{-1}(V)$ is a G-open subset of the space (X, τ), thus $V \in \tau_{f,G}$, i.e., V is open in $(Y, \tau_{f,G})$. Therefore, the space $(Y, \tau_{f,G})$ is G-sequential. □

The following result is a generalization of [10, Theorem 5.8].

Corollary 3.6. *Let G be a method on sets X and Y. Suppose that (X, τ) is a G-topological space and $f : X \to Y$ is a surjective mapping. If $(Y, \tau_{f,G})$ is a G-sequential space, then the following are equivalent for a topology μ of the set Y.*

(1) *μ is the largest topology that makes $f : (X, \tau) \to (Y, \mu)$ be G-continuous.*
(2) *$\mu = \tau_{f,G}$.*
(3) *$f : (X, \tau) \to (Y, \mu)$ is a G-quotient mapping and (Y, μ) is a G-sequential space.*

Proof. Since $(Y, \tau_{f,G})$ be a G-sequential space, it follows from Theorem 3.5 that (1) ⇔ (2) ⇒ (3). Next, we will show that (3) ⇒ (1), which does not need to assume that $(Y, \tau_{f,G})$ is G-sequential. Suppose that $f : (X, \tau) \to (Y, \mu)$ is a G-quotient mapping and (Y, μ) is a G-sequential space. Let $f : (X, \tau) \to (Y, \nu)$ be G-continuous, where ν is a topology of Y. If $V \in \nu$, then V is G-open in the space (Y, ν), and thus $f^{-1}(V)$ is G-open in the space (X, τ). Since $f : (X, \tau) \to (Y, \mu)$ is a G-quotient mapping, we have that the set V is G-open in (Y, μ). And since (Y, μ) is a G-sequential space, we obtain that $V \in \mu$. This implies that $\nu \subset \mu$.

It follows from that $f : (X, \tau) \to (Y, \mu)$ is G-continuous that the family μ is the largest topology that makes $f : (X, \tau) \to (Y, \mu)$ be G-continuous. □

The above corollary is also a generalization of Theorem 1.2. Let us recall the related concepts of ideal convergence. An *ideal* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ which is closed under the operations of taking finite unions and subsets of its elements. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ provided for any neighborhood U of x, we have the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ [9]. Since ideal convergence is a special kind of G-convergence, the concepts and results of G-convergence are all applicable to ideal convergence [10, 11, 20], in which every open subset is G-open, i.e., $\mathcal{I}$-open.

The usual convergence of sequences in topological spaces can be extended to statistical convergence [7], and statistical convergence can be extended to ideal convergence on $\mathbb{N}$ [9]. Thus, the results in this paper can be applied to statistical convergence and ideal convergence in topological spaces.

The following example shows "G-quotient" in part (3) of Corollary 3.6 cannot be weakened to "G-contunuous".

Example 3.7. There exists a non-G-quotient mapping preserving G-convergence $f : X \to Y$ such that

(1) X is a G-topological space;

(2) Y is a G-sequential space;

(3) the topology of Y is not the largest topology of Y that makes f preserve G-convergence or be G-continuous.

Let $X = \{0\} \cup \bigcup_{i\in\mathbb{N}} X_i$, where each $X_i = \{1/i\} \cup \{1/i + 1/k : k \in \mathbb{N}, k \geq i^2\}$. The set X is endowed with the discrete topology τ. Let G be the usual convergence method. Then every subset of X is G-open. Thus X is a G-topological space.

Let Y be the set X endowed with the following topology μ [8].

(a) Each point of the form $1/i + 1/j$ is isolated.

(b) Each neighborhood of each point of the form $1/i$ contains a set of the form $\{1/i\} \cup \{1/i + 1/k : k \geq j\}$, where each $j \geq i^2$.

(c) Each neighborhood of the point 0 contains a set obtained from Y by removing a finite number of Y_i's and a finite number of points of the form $1/i + 1/j$ in all the remaining Y_i's.

Then Y is a G-sequential space [8]. Let $f : X \to Y$ be the identity. Then f preserves G-convergence, thus it is G-continuous. If f is G-quotient, then every subset of Y is G-open, thus every subset of Y is open, because Y is G-sequential. Hence, Y is discrete, which is a contradiction. This implies that the mapping f is not G-quotient.

It is easy to see that $\tau_{f,G}$ is the discrete topology of the set Y. The topology μ of Y is not the largest topology of Y that makes f preserve G-convergence or be G-continuous.

4. Conclusions

In this paper, we discuss the topology of G-quotient spaces, and study the following problem [10, a part of Question 1.4]: how to characterize the largest topology of the range that makes the mapping be G-continuous?

In general, the discrete topology is the largest topology of a G-quotient space that makes the mapping be G-continuous. Suppose that every open subset of a topological space is always G-open in the space, we introduce the G-open topologies τ_G and $\tau_{f,G}$, and obtain the following result: Let G be a method on sets X and Y. Suppose that (X, τ) is a G-topological space and $f : X \to Y$ is a surjective mapping, then $f : (X, \tau) \to (Y, \tau_{f,G})$ is a G-quotient mapping if and only if $\tau_{f,G}$ is the largest topology of Y that makes $f : (X, \tau) \to Y$ be G-continuous.

We will continue to study the following question [10, another part of Question 1.4]: how to characterize the largest topology of the range that makes the mapping preserve G-convergence?

Finally, in this paper we assume that G is a method of sets X and Y. As a more detailed discussion, we can further assume that G_1 and G_2 are methods of sets X and Y, respectively, and study some properties of (G_1, G_2)-quotient spaces [4]. Here, we will not continue to discuss this topic.

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