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The hyperbolic mate of an oval

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Abstract. The hyperbolic mate C_h of an Euclidean oval C is introduced and studied. This new curve in the Lorentz plane resembles a spacelike curve. Here, we concentrate on the curvature of C_h and provide various examples.

Introduction

The enormous influence of convexity in practically every area of mathematics is widely known. We highlight the concept of *convex curve* by limiting the discussion to geometry, specifically Euclidean plane geometry. As an illustration, the recent book [1] includes a whole chapter, namely chapter 6, devoted to this topic.

This brief note attempts to correlate a second curve, C_h, which is thought to be *spacelike* in the Lorentzian plane geometry, to a given particular convex curve C, referred to as *oval*. The support function defining C serves as the foundation for the full analysis of this pair of curves. More specifically, we concentrate on the curvature, which is the only differential invariant for both settings.

These are the contents. The differential (and integral) geometry of the ovals is reviewed in the first part. Our new idea of *hyperbolic mate* of the given oval C is presented in the next section. It is important to note that, apart from the pair (C, C_h), there exists another curve P that is naturally connected to the support function p of G. In fact, we study three curves. We focus on a few cases after calculating the hyperbolic curvature of C_h and the Euclidean curvature of P. We point out that certain calculations require software, and we make use of Wolfram Alpha.

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1. The differential geometry of Euclidean ovals

A brief overview of the differential geometry of ovals is given in the first part. The Euclidean vector space $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ is hence the initial setting with the canonical inner product defined as follows:

$$\langle u, v\rangle = x^1y^1 + x^2y^2, \quad u = (x^1, x^2) \in \mathbb{R}^2, \quad v = (y^1, y^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{1.1}$$

Fix an open interval $I \subseteq \mathbb{R}$ and consider a regular parametrized curve $C \subset \mathbb{E}^2$ given by the equation:

$$C : r(t) = (x(t), y(t)), \quad r \in C^{\infty}, \quad \|r'(t)\| > 0, \quad t \in I. \tag{1.2}$$

Remember that C will be referred to as *oval* if it is closed, simple, and strictly convex. A smooth *support function*

$$p : I = [0, L > 0] \to \mathbb{R}$$

provides it and has the following properties:

$$p(0) = p(L), \quad p(t) + p''(t) > 0, \quad t \in I \tag{1.3}$$

through the relations:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := R(t) \cdot \begin{pmatrix} p(t) \\ p'(t) \end{pmatrix}, \quad R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1, \quad \|r(t)\|^2 = (p(t))^2 + (p'(t))^2. \tag{1.4}$$

We remark that the function

$$t \to \|r(t)\|$$

is exactly the first Legendre transformation of the convex function p. Let $\mathcal{F}(C) = \{T, N\}$ be the Frenet frame of C and

$$k : I = [0, L] \to \mathbb{R}^*_+ = (0, +\infty)$$

be its curvature function. Then, it is well known that these main functions are given by:

$$p(t) := -\langle r(t), N(t)\rangle \geq -\|r(t)\|, \quad k(t) := \frac{1}{p(t) + p''(t)} = \frac{1}{\|r'(t)\|} > 0, \tag{1.5}$$

since:

$$T(t) = (-\sin t, \cos t) = ie^{it}, \quad N(t) = iT(t) = -e^{it} = (-\cos t, -\sin t) \tag{1.6}$$

which means that the Frenet frame is universal for the set of ovals defined on the same interval I.

There are two famous integral relations in the geometry of ovals:
i) the Cauchy formula, [1, p. 233]:

$$L = \int_0^{2\pi} p(t)dt \tag{1.7}$$

ii) the Blaschke formula for the area $\mathcal{A}(C)$ enclosed by C, [1, p. 234]:

$$\mathcal{A}(C) = \frac{1}{2}\int_0^{2\pi} [(p(t))^2 - (p'(t))^2]dt \leq \frac{1}{2}\int_0^{2\pi} \|r(t)\|^2 dt, \quad 4\pi\mathcal{A}(C) \leq L^2 \tag{1.8}$$

with equality in the isoperimetric inequality (1.8) provided by the circle.

Remarks 1.1 i) The decomposition of the position vector field r in the Frenet basis is:

$$r(t) = p'(t)T(t) - p(t)N(t). \tag{1.9}$$

A plane curve satisfying [4]

$$k(t) = \frac{1}{\|r'(t)\|}, \forall t$$

is called *flat-flow curve*. Therefore, any oval is a curve of this type, which accounts for the equality with 2π of its overall curvature.
ii) A notion of *oval* in the Minkowski plane is defined in [6, p. 116] using the contact of the given curve with lines.
iii) An important tool in one-dimensional dynamics is the Fermi-Walker derivative. Let $\mathfrak{X}(C)$ be the set of vector fields along the curve C. Then the Fermi-Walker derivative is the map ([4, p. 420])

$$\nabla^{FW} : \mathfrak{X}(C) \to \mathfrak{X}(C)$$

given by

$$\nabla^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N\rangle T - \langle X, T\rangle N]. \tag{1.10}$$

The Frenet frame is conserved by Fermi-Walker:

$$\nabla^{FW}(T) = \nabla^{FW}(N) = 0.$$

For our oval C we derive:

$$\nabla^{FW}(r)(t) = r'(t) - \|r'(t)\|k(t)[p(t)T(t) + p'(t)N(t)] = p''(t)T(t) - p'(t)N(t). \tag{1.11}$$

Hence if we denote by $r = Rotation(p)$, then the curve $t \to \nabla^{FW}(r)(t)$ is exactly the curve $Rotation(p')$. □

2. The spacelike mate of an oval

The second setting of this paper is the Lorentz plane $\mathbb{R}^2_1 := (\mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$:

$$\langle u, v\rangle_L = -u^1v^1 + u^2v^2, \quad u = (u^1, u^2) \in \mathbb{R}^2, \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \le \|u\|_L^2 = |\langle u, u\rangle_L|. \tag{2.1}$$

The infinitesimal generator of the Lorentz rotations in $\mathbb{R}^2_1$ is the linear vector field:

$$\xi_L(u) := u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi_L(u) = j \cdot u = j \cdot (u^1 + iu^2) \tag{2.2}$$

where $(\mathbb{R}^2, j)$, $j^2 = 1$, is the two-dimensional paracomplex algebra [3]. The first integrals of ξ_L are the functions: $f_\alpha(x, y) = \alpha(x^2 - y^2)$, $\alpha \in \mathbb{R}$.

Fix again an open interval $I \subseteq \mathbb{R}$ and consider a spacelike parametrized curve $C_h \subset \mathbb{R}^2$ given by:

$$C_h : r_h(t) = (x_h(t), y_h(t)) = x_h(t)\bar{i} + y_h(t)\bar{j}, \quad \bar{i} = (1, 0), \bar{j} = (0, 1), \langle r'_h(t), r'_h(t)\rangle_L > 0, t \in I. \tag{2.3}$$

The Frenet apparatus of the curve C_h is provided by:

$$\begin{cases} T_h(t) = \frac{r'_h(t)}{\|r'_h(t)\|_L}, \quad N_h(t) = j \cdot T_h(t) = \frac{1}{\|r'_h(t)\|_L}(y'_h(t), x'_h(t)), \\ \langle T_h(t), T_h(t)\rangle_L = 1 = -\langle N_h(t), N_h(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'_h(t)\|_L}\langle T'_h(t), N_h(t)\rangle_L = \frac{1}{\|r'_h(t)\|_L^3}\langle r''_h(t), jr'_h(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'_h(t)\|_L^3}[x'_h(t)y''_h(t) - y'_h(t)x''_h(t)]. \end{cases} \tag{2.4}$$

Thus, along C_h, T_h is a unit spacelike vector field and along C_h, N_h is a unit timelike vector field. We can use a 2×2 determinant to express the curvature function:

$$k_L(t) = \frac{1}{\|r'_h(t)\|_L^3} \det\begin{pmatrix} x'_h(t) & y'_h(t) \\ x''_h(t) & y''_h(t) \end{pmatrix} \tag{2.5}$$

and the difference to the Euclidean curvature consists in the ratio in front of this determinant; in the Euclidean case is the Euclidean norm $\|r_h'(t)\|^{-3}$. The Frenet equations can be unified by means of the column matrix $\mathcal{F}_h(t) = \begin{pmatrix} T_h \\ N_h \end{pmatrix}(t)$ as:

$$\frac{d}{dt}\mathcal{F}_h(t) = -\|r_h'(t)\|_L k_L(t) R_L'(0)\,\mathcal{F}_h(t), \quad R_L'(0) \in so(1,1) \tag{2.6}$$

with the Lorentz rotation $R_L(t) \in SO(1,1)$ given by the symmetric matrices:

$$R_L(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}, \quad R_L^{-1}(t) := \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}. \tag{2.7}$$

The Lorentz rotated curve

$$jC : r_j(t) := j \cdot r_h(t) = (y_h(t), x_h(t))$$

is a timelike curve since

$$< r_j'(t), r_j'(t) >_L = - < r_h'(t), r_h'(t) >_L .$$

The hyperbolic Fermi-Walker derivative for our spacelike curve is:

$$\nabla_h^{FW}(X) := \frac{d}{dt}X - \|r'(\cdot)\|_L k_L [\langle X, N_h\rangle_L T_h - \langle X, T_h\rangle_L N_h] \tag{2.8}$$

such that

$$\nabla_h^{FW}(T_h) = \nabla_h^{FW}(N_h) = 0.$$

Owing to the growing curiosity about the geometry of curves in the Lorentz plane (see, for example, [6] and [7]), this brief study establishes the hyperbolic mate for the given oval C:

Definition 2.1 The curve C_h is the *h-mate* of C if its parametrization satisfies:

$$r_h(t) = \begin{pmatrix} x_h \\ y_h \end{pmatrix}(t) := R_L(t)\begin{pmatrix} p \\ p' \end{pmatrix}(t) = \begin{pmatrix} p(t)\cosh t + p'(t)\sinh t \\ p'(t)\cosh t + p(t)\sinh t \end{pmatrix}, \quad t \in I = [0, L]. \tag{2.9}$$

Since the derivative of r_h is:

$$r_h'(t) = \left(2p'(t)\cosh t + \frac{\sinh t}{k(t)}, 2p'(t)\sinh t + \frac{\cosh t}{k(t)}\right) \tag{2.10}$$

it results:

$$\|r_h'(t)\|_L^2 = \frac{1}{k^2(t)} - 4(p'(t))^2. \tag{2.11}$$

We will restrict our study to a spacelike C_h provided:

$$(p + p'')^2 > (2p')^2. \tag{2.12}$$

It results immediately:

$$\langle r_h(t), T_h(t)\rangle_L = \frac{p'(t)(p''(t) - p(t))}{\sqrt{(p(t) + p''(t))^2 - (2p'(t))^2}}, \quad \langle r_h(t), N_h(t)\rangle_L = \frac{2(p'(t))^2 - p(t)(p(t) + p''(t))}{\sqrt{(p(t) + p''(t))^2 - (2p'(t))^2}}. \tag{2.13}$$

Remark 2.2 Using the methodology of [5], we note that C and C_h can be considered as the Euclidean and hyperbolic *deformations*, respectively, of the curve

$$t \to P(t) = (p(t), p'(t)).$$

The geometric meaning of the condition (2.12) is that the vector

$$P(t) + j \cdot P'(t) = (p(t) + p''(t), 2p'(t))$$

is timelike.

It is important to note that the timelike vectors $N_h(t)$ and $P(t) + j \cdot P'(t)$ have a strictly positive first component since the criterion (2.12) implies that they are both *positive*. There is a unique non-negative number θ for two positive timelike vectors $\overrightarrow{x}, \overrightarrow{y}$ such that:

$$< \overrightarrow{x}, \overrightarrow{y} >_L = \|j\overrightarrow{x}\|_L \|j\overrightarrow{y}\|_L \cosh\theta,$$

θ is known as *the timelike angle* between them. The timelike angle for the pair $(N_h(t), P(t) + j \cdot P'(t))$ is precisely $\theta(t) = t$. $\square$

Our main theoretical finding is a lengthy but simple computation of the curvature of the spacelike C_h:

Theorem 2.3 i) *If p is not a constant then the curve P is a regular one having the Euclidean curvature*:

$$k_P(t) = \frac{p'(t)p'''(t) - (p''(t))^2}{[(p'(t))^2 + (p''(t))^2]^{\frac{3}{2}}}. \tag{2.14}$$

ii) *Suppose that the h-mate C_h of the oval C is spacelike. Then its curvature is:*

$$k_L(t) = \frac{2p'(t)(3p'(t) + p'''(t)) - (p(t) + p''(t))(p(t) + 3p''(t))}{[(p(t) + p''(t))^2 - (2p'(t))^2]^{\frac{3}{2}}}. \tag{2.15}$$

The inequality (2.12) *yields a upper bound:*

$$k_L(t) < \frac{2[(p'(t))^2 + p'(t)p'''(t) - p''(t)(p(t) + p''(t))]}{[(p(t) + p''(t))^2 - (2p'(t))^2]^{\frac{3}{2}}}. \tag{2.16}$$

The decomposition of the vector field r_h in the hyperbolic Frenet frame is:

$$r_h(t) = \frac{p'(t)(p''(t) - p(t))}{\|r_h'(t)\|_L} T_h(t) + \frac{p(t)(p''(t) + p(t)) - 2(p'(t))^2}{\|r_h'(t)\|_L} N_h(t). \tag{2.17}$$

We focus now on some concrete examples.

Examples 2.4 The circle $C(O, R > 0)$ of Euclidean plane geometry is the oval provided by the constant support function $p \equiv R$. Its h-mate is the (part of) equilateral hyperbola $H_e(R)$ as integral curve of ξ_L:

$$\begin{cases} H_e(R) : x^2 - y^2 = R^2, \quad k_L \equiv -\frac{1}{R} = -\frac{1}{\|r_h'(t)\|_L} < 0, \\ T_h(t) = (\sinh t, \cosh t) = \textit{spacelike}, \quad N_h(t) = \frac{1}{R} r_h(t) = (\cosh t, \sinh t), \quad t \in [0, L(C) = 2\pi R] \\ (R_L(u) \circ r_h)(t) = r_h(t + u) = (R_L(t) \circ r_h)(u), \quad \nabla_h^{FW}(r_h)(t) = (R + 1)(\sinh t, \cosh t). \end{cases} \tag{2.18}$$

The Euclidean length of this arc of $H_e(R)$ is:

$$L(H_e(R)|_{[0, L(C)]}) = R \int_0^{2\pi R} \sqrt{\cosh(2t)} dt \tag{2.19}$$

and for $R = 1$ this value is approximately 378.051. The parametrization by arc-length of the complete $H_e(R)$ is:

$$r_e(s) = R\left(\cosh \frac{s}{R}, \sinh \frac{s}{R}\right), \quad s \in \mathbb{R}. \tag{2.20}$$

We note that:
i) the equilateral hyperbola $H_e(R)$ is called *pseudo-circle* [6, p. 110] and is denoted $H^1(-R)$,
ii) for the initial oval C of the first section we have the formula:

$$x(t)^2 - y(t)^2 = -\|r(t)\|_L^2 = [(p(t))^2 - (p'(t))^2]\cos 2t - 2p(t)p'(t)\sin 2t. \tag{2.21}$$

□

Example 2.5 Fix the smooth real function $p(t) := r - \cos 3t$; hence

$$p(t) = p(t + 2\pi).$$

In [2, p. 23], it is proved that if $r > 8$ then p is the support function of an oval C. With the derivatives:

$$p'(t) = 3\sin 3t, \quad p''(t) = 9\cos 3t, \quad p'''(t) = -27\sin 3t \tag{2.22}$$

it results in the following curvatures:

$$k_P(t) = \frac{-3}{[(\sin 3t)^2 + 9(\cos 3t)^2]^{\frac{3}{2}}} < 0, \quad k(t) = \frac{1}{r + 8\cos 3t}, \quad k_L(t) = -\frac{108 + r^2 + 34r\cos 3t + 100(\cos 3t)^2}{[(r + 8\cos 3t)^2 - 36(\sin 3t)^2]^{\frac{3}{2}}} \tag{2.23}$$

with k_L computed on the sub-interval of $[0, 2\pi]$ provided by the condition (2.12) which reads as:

$$(r + 8\cos 3t)^2 > (6\sin 3t)^2.$$

The Cauchy and the Blaschke formulae provide the following:

$$L(C) = 2\pi r, \quad \mathcal{A}(C) = \pi(r^2 - 4) > 60\pi. \tag{2.24}$$

For example, if $r = 9$ then:

$$k_L(t) = \frac{-2[95 + 153\cos 3t + 50(\cos 3t)^2]}{[(9 + 8\cos 3t)^2 - 36(\sin 3t)^2]^{\frac{3}{2}}}, \quad \mathcal{A}(C) = 77\pi. \tag{2.25}$$

The integral of the curvature of the oval $C(r = 9)$ is:

$$\int_0^{2\pi} \frac{dt}{9 + 8\cos 3t} = \frac{2\pi}{\sqrt{17}} \simeq 1.5239 < 2\pi. \tag{2.26}$$

□

Example 2.6 For $\alpha \in [1, +\infty)$, the 2π-periodic function $p_\alpha : [0, 2\pi] \to \mathbb{R}_+^*$ such as

$$p_\alpha(t) := \frac{1}{\alpha}\sqrt{\alpha^4\cos^2 t + \sin^2 t}$$

is the support function of an ellipse since:

$$p_\alpha(t) + p_\alpha''(t) = \frac{\alpha^3}{(\alpha^4\cos^2 t + \sin^2 t)^{\frac{3}{2}}} > 0. \tag{2.27}$$

The condition (2.12) restricts t to the values satisfying:

$$|\sin 2t(\alpha^4\cos^2 t + \sin^2 t)| < \frac{\alpha^4}{\alpha^4 - 1}. \tag{2.28}$$

□

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