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Stability of hybrid differential equations in the sense of Hyers–Ulam using Gronwall lemma

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Abstract. In this paper, for the first time, we use Gronwall's lemma to show the Hyers–Ulam and Hyers–Ulam–Rassias stability of hybrid differential equations, offering a fresh and rigorous framework for studying these complex structures. Two examples are provided to illustrate the correctness of our method, and graphical illustrations are used to graphically represent the results.

1. Introduction

The stability of solutions is a key concept in the qualitative theory of differential and integral equations. It is generally known that this idea has been used in literature to mean different things, one of which being the Hyers–Ulam stability. In Ulam's open problem, this idea initially surfaced in relation to homomorphisms (see the book [32]), and answered by Hyers in [12]. The initial definition of stability (Hyers–Ulam–Rassias stability, generalized Hyers–Ulam–Rassias stability, semi–Hyers–Ulam–Rassias stability, and Mittag–Leffler–Hyers–Ulam stability) was then generalized by numerous authors, opening up a vast field of research and yielding results for various classes of functional equations (see, for example, [10, 11, 16, 17, 31] and the references therein). The topic is presented in a systematic manner in [8, 15].

It was Obloza [22], and Alsina and Ger's [3] articles that initiated the investigation of Hyers–Ulam stability for ordinary differential equations. In addition, numerous intriguing outcomes were discovered for systems of equations or linear differential equations in publications such as [9, 13, 20, 23, 24, 29] or for certain particular differential equation classes in [1, 2]. We include [14, 18, 19, 21, 25] in our list of articles on partial differential equations. The book [30] offers a broad overview of the subject.

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In this paper, we study the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability, for the following hybrid differential equation,

$$\varpi'(\xi) = \vartheta(\xi, \varpi(\xi)) + \varrho(\xi, \varpi(\xi))\varphi(\xi), \quad \xi \in I : [a, b], \quad b > 0 \tag{1}$$

$$\varpi(\xi) = \psi(\xi), \quad \xi \in [a - h, a], \tag{2}$$

where $\vartheta \in C([a-h,b] \times \mathbb{R}, \mathbb{R}), \varrho \in C([a-h,b] \times \mathbb{R}, \mathbb{R}), \varphi \in C([a-h,b], \mathbb{R})$, M is the positive constant, $\max_{\xi \in [a-h,b]} = |\varphi(\xi)|$, $\psi \in C([a-h,a], \mathbb{R})$ is the initial function.

Hybrid differential equations, which simulate complex systems with switching dynamics, non–smooth behavior, and memory effects (see, e.g., earlier studies [4–7, 27, 28]), have extensive applications in many different fields. Hybrid differential equations improve control techniques and provide strong system stabilization in control systems. Switching device circuits, such as power electronics, are described in electrical engineering. Hybrid differential equations are also used to model mechanical systems, including impact dynamics and robotic systems. In addition, population dynamics, disease modeling, and gene regulatory networks are all described by hybrid differential equations in biology. In order to describe switching dynamics and non-smooth behavior, hybrid differential equations are frequently used in economic systems, neural networks, finance, medical imaging, and materials research. In these varied domains, hybrid differential equations facilitate creative solutions, better performance, and better decision-making by precisely modeling these intricate systems.

Here is a quick summary of this paper's originality and contributions: According to the relevant sources, a large number of books and papers have been published about the Hyers–Ulam and Hyers–Ulam–Rassias stability of ordinary differential equations. As far as we are aware, no work has been done on the Hyers–Ulam and Hyers–Ulam–Rassias stability of hybrid differential equation (1) in the literature. This study is the first contribution to the stability of the hybrid differential equations concerning Hyers–Ulam and Hyers–Ulam–Rassias. The preceding results for ordinary differential equations are refined and expanded upon in this work.

We first introduce two fundamental inequalities that are necessary for both the Hyers–Ulam and the Hyers–Ulam–Rassias stability of this problem.

let $\epsilon > 0$ and $\phi(\xi) = C([a-h, b], \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, be a positive non-decreasing continuous function. Consider the estimates

$$|\wp'(\xi) - \vartheta(\xi, \wp(\xi)) - \varrho(\xi, \wp(\xi))\varphi(\xi)| \leq \epsilon\phi(\xi), \quad \xi \in I \tag{3}$$

and

$$|\wp'(\xi) - \vartheta(\xi, \wp(\xi)) - \varrho(\xi, \wp(\xi))\varphi(\xi)| \leq \epsilon, \quad \xi \in I. \tag{4}$$

The structure of this paper is as follows: In Section 2, we give some fundamental definitions, some remarks, and Gronwall's lemma that are crucial to demonstrating our key findings. The Hyers–Ulam and Hyers–Ulam–Rassias stability of the hybrid differential equation (1) is covered in Section 3. We give two examples in Section 4 to demonstrate our findings, and we conclude in Section 5.

2. Preliminaries

In this section, we introduce some definitions, remarks, and the well–known Gronwall lemma that are used in this paper.

Definition 2.1. *Hybrid differential equation* (1) *is Hyers–Ulam–Rassias stable with respect to the positive, non-decreasing continuous function* $\phi(\xi) : [a-h, b] \to \mathbb{R}_+$, *if there exists a positive constant* c_ϕ *such that, for all* $\epsilon > 0$ *and every solution* $\wp \in C^1([a-h, b], \mathbb{R}$ *of* (3), *there exist a solution* $\varpi \in C^1([a-h, b], \mathbb{R})$ *of hybrid differential equation* (1) *with* $|\wp(\xi) - \varpi(\xi)| \leq c_\phi \epsilon \phi(\xi)$ *for* $\xi \in [a-h, b]$.

Definition 2.2. *Hybrid differential equation* (1) *is said to be Hyers–Ulam stable if there exists a positive constant c such that for all $\epsilon > 0$ and every solution $\wp \in C^1([a-h,b],\mathbb{R})$, of* (4) *there exists a solution $\varpi \in C^1([a-h,b],\mathbb{R})$ of hybrid differential equation* (1) *with $|\wp(\xi) - \varpi(\xi)| \leq c\epsilon$ for $\xi \in [a-h,b]$.*

Remark 2.3. *A function $\wp \in C^1(I,\mathbb{R})$ is a solution to* (3) *if and only if there exists a function $\Omega \in C(I,\mathbb{R})$ (which depends on $\wp$) such that*

- $|\Omega(\xi)| \leq \epsilon\phi(\xi)$ *for all* $\xi \in I$;
- $\wp'(\xi) = \vartheta(\xi,\wp(\xi)) + \varrho(\xi,\wp(\xi))\varphi(\xi) + \Omega(\xi)$ *for all* $\xi \in I$.

A $\wp \in C^1(I,\mathbb{R})$ is a solution to (4) *if and only if there exists a function $\nu \in C(I,\mathbb{R})$ (which depends on $\wp$) such that*

- $|\nu(\xi)| \leq \epsilon$ *for all* $\xi \in I$;
- $\wp'(\xi) = \vartheta(\xi,\wp(\xi)) + \varrho(\xi,\wp(\xi))\varphi(\xi) + \nu(\xi)$ *for all* $\xi \in I$.

Remark 2.4. *If $\wp \in C^1(I,\mathbb{R})$ is a solution to* (3)*, then it is a solution to the following integral inequality:*

$$\Big|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\Big| \leq \int_a^\xi \phi(\tau)d\tau, \quad \xi \in I.$$

If $\wp$ is a solution to (4)*, then it is a solution to the following integral inequality:*

$$\Big|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\Big| \leq (\xi - a)\epsilon, \quad \xi \in I.$$

Regardless, one may have the above definitions and remarks for the case $I = [a,\infty)$ then the interval $[a-h,b]$ should be changed by $[a-h,\infty)$. Gronwall's lemma, which is widely known, is provided here. It is crucial for the stability proofs of Hyers–Ulam and Hyers–Ulam–Rassias; for more information, see, for example, Rus [26].

Lemma 2.5. *Let $\hbar, \Im \in C([a,b],\mathbb{R}_+)$. Suppose that $\hbar$ is increasing. If $\varpi \in C([a,b],\mathbb{R}_+)$ is a solution to the inequality*

$$\varpi(\xi) \leq \hbar(\xi) + \int_a^b \Im(\tau)\varpi(\tau)d\tau, \quad \xi \in [a,b],$$

then

$$\varpi(\xi) \leq \hbar(\xi) exp\Big(\int_a^b \Im(\tau)d\tau\Big), \quad \xi \in [a,b].$$

3. Main results

In this section, we are going to prove the Hyers–Ulam–Rassias and the Hyers–Ulam stability of the hybrid differential equation (1).

3.1. Hyers–Ulam–Rassias stability of hybrid differential equation

In this subsection, we will prove the Hyers–Ulam–Rassias stability of the hybrid differential equation (1).

Theorem 3.1. *Suppose that*
(a) $\vartheta \in C([a-h,b]\times\mathbb{R},\mathbb{R})$, $\varrho \in ([a-h,b]\times\mathbb{R},\mathbb{R})$, $\varphi \in ([a-h,b],\mathbb{R})$, $\nu(\xi) \le \xi$, $\nu > 0$;
(b) *There exist positive constants* L_ϑ, L_ϱ, M, *and* $\varpi_1, \varpi_2 \in \mathbb{R}$, *such that for* $\xi \in I$,

$$M = \max_{\xi\in[a-h,b]} |\varphi(\xi)|,$$

$$|\vartheta(\xi,\varpi_1) - \vartheta(\xi,\varpi_2)| \le L_\vartheta|\varpi_1 - \varpi_2|,$$

$$|\varrho(\xi,\varpi_1)\varphi(\xi) - \varrho(\xi,\varpi_2)\varphi(\xi)| \le ML_\varrho|\varpi_1 - \varpi_2|;$$

(c) *There exist a positive, non-decreasing, continuous function* $\phi : [a-h,b] \to \mathbb{R}_+$ *and a positive constant* $\aleph$ *such that*

$$\int_0^\xi \epsilon\phi(\tau)d\tau \le \aleph\epsilon\phi(\xi), \xi \in I.$$

Then, problem (1)–(2) *has a unique solution in* $C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ *and Equation* (1) *is Hyers–Ulam–Rassias stable with respect to the function* ϕ.
Proof. *Let* $\wp \in C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$ *be a solution to* (3). *Equation* (1) *has a unique solution in* $C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$. *We denote by* $\varpi \in C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$ *the unique solution to the Cauchy problem*

$$\varpi^{'}(\xi) = \vartheta(\xi,\varpi(\xi)) + \varrho(\xi,\varpi(\xi))\varphi(\xi), \quad \xi \in I,$$

$$\varpi(a) = \wp(\xi), \quad \xi \in [a-h,a].$$

So,

$$\varpi(\xi) = \begin{cases} \wp(\xi), & \xi \in [a-h,a], \\ \wp(a) + \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau + \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau, & \xi \in I. \end{cases}$$

Remark 2.4 gives

$$\left|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right| \le \int_a^\xi \varepsilon\phi(\tau)d\tau \le \aleph\varepsilon\phi(\xi), \quad \xi \in I.$$

From the above relations, for $\xi \in [a-h,a]$ *we have* $\wp(\xi) - \varpi(\xi) = 0$ *and for* $\xi \in [a,b]$, *we obtain*

$$
\begin{aligned}
|\wp(\xi)-\varpi(\xi)| &= \left|\wp(\xi)-\wp(a)-\int_a^{\xi}\vartheta(\tau,\varpi(\tau))d\tau-\int_a^{\xi}\varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\leq \left|\wp(\xi)-\wp(a)-\int_a^{\xi}\vartheta(\tau,\wp(\tau))d\tau-\int_a^{\xi}\varrho(\tau,\wp(\tau))\varphi(\tau)d\tau+\int_a^{\xi}\vartheta(\tau,\wp(\tau))d\tau\right. \\
&\quad \left.+\int_a^{\xi}\varrho(\tau,\wp(\tau))\varphi(\tau)d\tau-\int_a^{\xi}\vartheta(\tau,\varpi(\tau))d\tau-\int_a^{\xi}\varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\leq \left|\wp(\xi)-\wp(a)-\int_a^{\xi}\vartheta(\tau,\wp(\tau))d\tau-\int_a^{\xi}\varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right|+\left|\int_a^{\xi}\vartheta(\tau,\wp(\tau))d\tau\right. \\
&\quad \left.-\int_a^{\xi}\vartheta(\tau,\varpi(\tau))d\tau\right|+\left|\int_a^{\xi}\varrho(\tau,\wp(\tau))\varphi(\tau)d\tau-\int_a^{\xi}\varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\leq \aleph\varepsilon\phi(\xi)+\left|\int_a^{\xi}\vartheta(\tau,\wp(\tau))d\tau-\int_a^{\xi}\vartheta(\tau,\varpi(\tau))d\tau\right|+\left|\int_a^{\xi}\varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right. \\
&\quad \left.-\int_a^{\xi}\varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\leq \aleph\varepsilon\phi(\xi)+\int_a^{\xi}\left|\vartheta(\tau,\wp(\tau))-\vartheta(\tau,\varpi(\tau))\right|d\tau+\int_a^{\xi}\left|\varrho(\tau,\wp(\tau))\varphi(\tau)\right. \\
&\quad \left.-\varrho(\tau,\varpi(\tau))\varphi(\tau)\right|d\tau \\
&\leq \aleph\varepsilon\phi(\xi)+L_{\vartheta}\int_a^{\xi}|\wp(\tau)-\varpi(\tau)|d\tau+L_{\varrho}\int_a^{\xi}|\wp(\tau)-\varpi(\tau)|\left|\varphi(\tau)\right|d\tau \\
&\leq \aleph\varepsilon\phi(\xi)+L_{\vartheta}\int_a^{\xi}|\wp(\tau)-\varpi(\tau)|d\tau+ML_{\varrho}\int_a^{\xi}\left|\wp(\tau)-\varpi(\tau)\right|d\tau \\
&= \aleph\varepsilon\phi(\xi)+(L_{\vartheta}+ML_{\varrho})\int_a^{\xi}|\wp(\tau)-\varpi(\tau)|d\tau.
\end{aligned}
$$

Using the Gronwall lemma [26], we obtain

$$|\wp(\xi)-\varpi(\xi)| \leq \aleph\varepsilon\phi(\xi)e^{(L_{\vartheta}+ML_{\varrho})(b-a)} = c_{\phi}\varepsilon\phi(\xi), \quad \xi\in I,$$

where $c_{\phi}=\aleph e^{(L_{\vartheta}+ML_{\varrho})(b-a)}$, *i.e., Equation* (1) *is Hyers–Ulam–Rassias stable.*

3.2. *Hyers–Ulam stability of hybrid differential equation*

In this subsection, we will prove the Hyers–Ulam stability of the hybrid differential equation (1).

Theorem 3.2. *Suppose that*
(*a*) $\vartheta\in C([a-h,b]\times\mathbb{R},\mathbb{R})$, $\varrho\in([a-h,b]\times\mathbb{R},\mathbb{R})$, $\varphi\in([a-h,b],\mathbb{R})$, $\nu(\xi)\leq\xi$, $\nu>0$;
(*b*) *There exist positive constants* L_{ϑ}, L_{ϱ}, M, *and* $\varpi_1,\varpi_2\in\mathbb{R}$, *such that for* $\xi\in I$,

$$M=\max_{\xi\in[a-h,b]}|\varphi(\xi)|,$$

$$|\vartheta(\xi,\varpi_1)-\vartheta(\xi,\varpi_2)|\leq L_{\vartheta}|\varpi_1-\varpi_2|,$$

$$|\varrho(\xi,\varpi_1)\varphi(\xi)-\varrho(\xi,\varpi_2)\varphi(\xi)|\leq ML_{\varrho}|\varpi_1-\varpi_2|;$$

(*c*) $(b-a)(L_{\vartheta}+ML_{\varrho})<1$.
Then, problem (1)–(2) *has a unique solution in* $C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ *and Equation* (1) *is Hyers–Ulam stable.*

Proof. *Let $\wp \in C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$ be a solution to* (4). *Equation* (1) *has a unique solution in $C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$. We denote by $\varpi \in C([a-h,b],\mathbb{R}) \cap C^1([a,b],\mathbb{R})$ the unique solution to the Cauchy problem*

$$\varpi'(\xi) = \vartheta(\xi,\varpi(\xi)) + \varrho(\xi,\varpi(\xi))\varphi(\xi), \quad \xi \in I,$$
$$\varpi(a) = \wp(\xi), \quad \xi \in [a-h,a].$$

So,

$$\varpi(\xi) = \begin{cases} \wp(\xi), & \xi \in [a-h,a], \\ \wp(a) + \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau + \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau, & \xi \in I. \end{cases}$$

Remarks 2.4 gives

$$\left|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right| \le (\xi-a)\varepsilon, \quad \xi \in I.$$

From the above relations, for $\xi \in [a-h,a]$ we have $\wp(\xi) - \varpi(\xi) = 0$ and for $\xi \in [a,b]$, we obtain

$$\begin{aligned}
|\wp(\xi) - \varpi(\xi)| &= \left|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau - \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\le \left|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau + \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau\right. \\
&\quad \left. + \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau - \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau - \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\le \left|\wp(\xi) - \wp(a) - \int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right| + \left|\int_a^\xi \vartheta(\tau,\wp(\tau))d\tau\right. \\
&\quad \left. - \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau\right| + \left|\int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau - \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\le (\xi-a)\epsilon + \left|\int_a^\xi \vartheta(\tau,\wp(\tau))d\tau - \int_a^\xi \vartheta(\tau,\varpi(\tau))d\tau\right| + \left|\int_a^\xi \varrho(\tau,\wp(\tau))\varphi(\tau)d\tau\right. \\
&\quad \left. - \int_a^\xi \varrho(\tau,\varpi(\tau))\varphi(\tau)d\tau\right| \\
&\le (\xi-a)\epsilon + \int_a^\xi \left|\vartheta(\tau,\wp(\tau)) - \vartheta(\tau,\varpi(\tau))\right|d\tau + \int_a^\xi \left|\varrho(\tau,\wp(\tau))\varphi(\tau)\right. \\
&\quad \left. - \varrho(\tau,\varpi(\tau))\varphi(\tau)\right|d\tau \\
&\le (\xi-a)\epsilon + L_\vartheta \int_a^\xi |\wp(\tau) - \varpi(\tau)|d\tau + L_\varrho \int_a^\xi |\wp(\tau) - \varpi(\tau)|\left|\varphi(\tau)\right|d\tau \\
&\le (\xi-a)\epsilon + L'_\vartheta \int_a^\xi |\wp(\tau) - \varpi(\tau)|d\tau + ML_\varrho \int_a^\xi \left|\wp(\tau) - \varpi(\tau)\right|d\tau \\
&= (\xi-a)\epsilon + (L_\vartheta + ML_\varrho) \int_a^\xi |\wp(\tau) - \varpi(\tau)|d\tau.
\end{aligned}$$

Using the Gronwall lemma [26], we obtain that

$$|\wp(\xi) - \varpi(\xi)| \le (b-a)\varepsilon e^{(L_\vartheta + ML_\varrho)(b-a)} = c\epsilon, \quad \xi \in I,$$

where $c = (b-a)e^{(L_\vartheta + ML_\varrho)(b-a)}$, *i.e., Equation* (1) *is Hyers–Ulam stable.*

4. Applications

To illustrate that the conditions of the above results are possible to attain, we will present some examples.

Example 4.1. *Consider the Hybrid differential equation of the form:*

$$\begin{aligned} \varpi'(\xi) &= \frac{2\arcsin\varpi(\xi)}{375} - \frac{1}{2} - \Big(\frac{\sin(\xi)\varpi(\xi) + \cos^2(\xi)}{30} + \frac{1}{10}\Big)\frac{\xi}{25}, \quad \xi \in [0,2], && (5)\\ \varpi(0) &= \frac{1}{2}, \quad \xi \in [-1,0]. && (6)\end{aligned}$$

We know that the exact solution of that equation is $\varpi(\xi) = \sin(\xi)$. *Using MATLAB, the exact solution* $\varpi(\xi)$ *of Equation* (5) *is calculated, and Figure 1 shows the result.*

We realize that all the conditions of Theorem 3.1 are here satisfied. Comparing with problem (1), *we have*

$$\vartheta(\xi,\varpi(\xi)) = \frac{2\arcsin\varpi(\xi)}{375} - \frac{1}{2}, \quad \varrho(\xi,\varpi(\xi)) = -\Big(\frac{\sin(\xi)\varpi(\xi)+\cos^2(\xi)}{30} + \frac{1}{10}\Big), \quad \varphi(\xi) = \frac{\xi}{25}.$$

Clearly, we obtain

$$\max|\varphi(\xi)| = \max_{\xi\in[0,2]}\Big|\frac{\xi}{25}\Big| = \frac{2}{25}, \quad \xi \in [0,2],$$

$$\begin{aligned} |\vartheta(\xi,\varpi_1) - \vartheta(\xi,\varpi_2)| &= \Big|\frac{2\arcsin\varpi_1}{375} - \frac{2\arcsin\varpi_2}{375}\Big| \\ &= \Big|\frac{2}{375}(\arcsin\varpi_1 - \arcsin\varpi_2)\Big| \\ &= \frac{2}{375}|\arcsin\varpi_1 - \arcsin\varpi_2| \\ &\le \frac{2}{375}|\varpi_1 - \varpi_2|, \quad \varpi_1,\varpi_2 \in \mathbb{R}, \quad \xi \in [0,2], \end{aligned}$$

and

$$\begin{aligned} |\varrho(\xi,\varpi_1)\varphi(\xi) - \varrho(\xi,\varpi_2)\varphi(\xi)| &= \Big|\Big(\frac{\sin(\xi)\varpi_1 + \cos^2(\xi)}{30}\Big)\frac{\xi}{25} - \Big(\frac{\sin(\xi)\varpi_2 + \cos^2(\xi)}{30}\Big)\frac{\xi}{25}\Big| \\ &= \Big|\frac{\xi}{25}\Big(\frac{\sin(\xi)\varpi_1}{30} - \frac{\sin(\xi)\varpi_2}{30}\Big)\Big| \\ &\le \Big|\frac{2}{25}\Big(\frac{\varpi_1}{30} - \frac{\varpi_2}{30}\Big)\Big| \\ &= \frac{2}{25}\times\frac{1}{30}\Big|\varpi_1 - \varpi_2\Big|, \quad \varpi_1,\varpi_2 \in \mathbb{R}, \quad \xi \in [0,2], \end{aligned}$$

with

$$L_\vartheta = \frac{2}{375}, \quad M = \frac{2}{25}, \quad L_\varrho = \frac{1}{30}.$$

Letting $\phi(\xi) = e^{11\xi}$, *and* $\epsilon = \frac{1}{2}$, *we obtain*

$$\Big|\int_0^\xi \epsilon\phi(\tau)d\tau\Big| = \Big|\int_0^\xi \frac{1}{2}e^{11\tau}d\tau\Big| = \frac{1}{2}\times\frac{1}{11}(e^{11\xi} - 1) \le \frac{1}{2}\times\frac{1}{11}e^{11\xi} = \aleph\epsilon\phi(\xi).$$

If we choose $\wp(\xi) = \frac{\sin(\xi)}{0.2}$ *, it follows,*

$$\left|\wp'(\xi) - \frac{\wp(\xi)}{100} - \Big(\frac{1}{30}\wp(\xi) + \frac{1}{10}\Big)\frac{\xi}{25}\right| \leq \frac{1}{2} \times \frac{1}{11} e^{11\xi}, \quad \xi \in [0, 2].$$

This exhibits the Hyers–Ulam–Rassias stability of the hybrid differential Equation (5). *The Hyers–Ulam–Rassias stability of Equation* (5) *is independent of the initial value condition. In addition, having in mind the exact solution* $\varpi(\xi) = \sin(\xi)$ *and* $L_\vartheta + ML_\varrho = \frac{2}{375} + \frac{2}{25} \times \frac{1}{30} = 0.008 < 1$, *we have*

$$|\wp(\xi) - \varpi(\xi)| \leq \frac{1}{22} e^{\frac{2}{125}} e^{11\xi}, \quad \xi \in [0, 2];$$

see Figure 2.

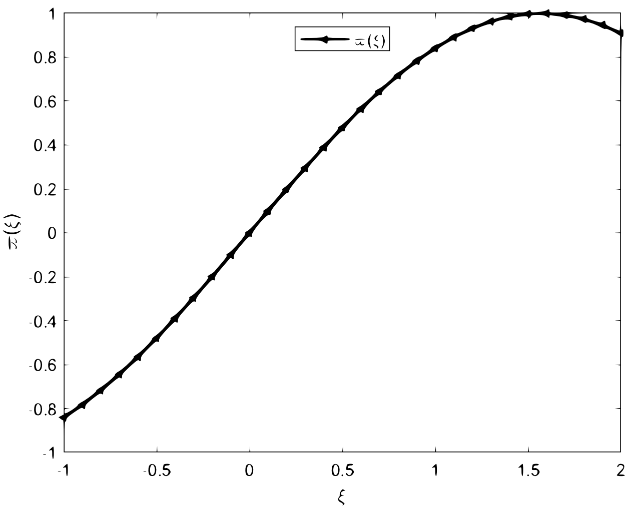

Figure 1: The solution $\varpi(\xi)$ of Equation (5).

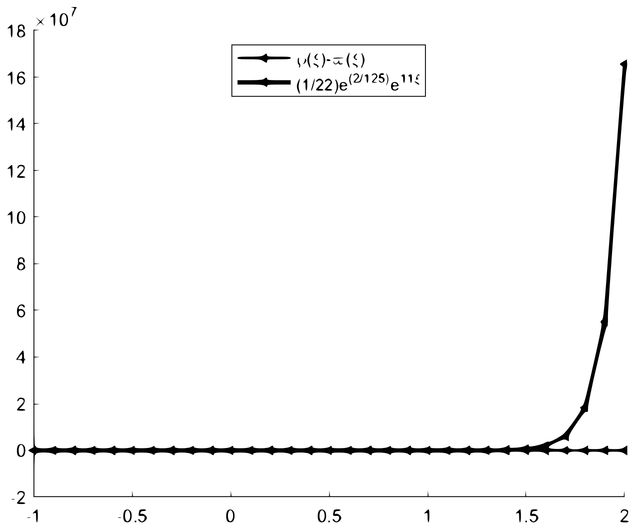

Figure 2: Functions $\wp(\xi) - \varpi(\xi)$, and $\frac{1}{22} e^{\frac{2}{125}} e^{11\xi}$.

Example 4.2. *Consider the hybrid differential equation of the form:*

$$\begin{aligned} \varpi'(\xi) &= \xi e^{\frac{\xi}{3}}\Big[\frac{1}{4}e^{\xi} - 3e^{\frac{2\xi}{3}} + \frac{1}{4}\Big] + \frac{\xi}{8}\varpi(\xi) + e^{\xi}(3\xi+1) \\ &\quad -\frac{\xi}{4}\Big[\frac{\varpi(\xi)}{2} + e^{\frac{\xi}{3}} + e^{\frac{4\xi}{3}}\Big], \quad \xi \in [0,2], \end{aligned} \tag{7}$$

$$\varpi(0) = 0, \quad \xi \in [-1,0]. \tag{8}$$

We know that the exact solution of that equation is $\varpi(\xi) = e^{\xi}$. Using MATLAB, the exact solution $\varpi(\xi)$ of Equation (7) *is calculated, and Figure 3 shows the result.*

We realize that all the conditions of Theorem 3.2 are here satisfied. Comparing with problem (1), *we have*

$$\vartheta(\xi,\varpi(\xi)) = \xi e^{\frac{\xi}{3}}\Big[\frac{1}{4}e^{\xi} - 3e^{\frac{2\xi}{3}} + \frac{1}{4}\Big] + \frac{\xi}{8}\varpi(\xi) + e^{\xi}(3\xi+1), \quad \varrho(\xi,\varpi(\xi)) = \frac{\varpi(\xi)}{2} + e^{\frac{\xi}{3}} + e^{\frac{4\xi}{3}}$$

$$\varphi(\xi) = -\frac{\xi}{4}.$$

Clearly, we obtain

$$\max|\varphi(\xi)| = \max_{\xi\in[0,2]}\Big|-\frac{\xi}{4}\Big| = \frac{1}{2}, \quad \xi \in [0,2],$$

$$\begin{aligned} |\vartheta(\xi,\varpi_1) - \vartheta(\xi,\varpi_2)| &= \Big|\frac{\xi}{8}\varpi_1 - \frac{\xi}{8}\varpi_2\Big| \\ &= \Big|\frac{\xi}{8}(\varpi_1 - \varpi_2)\Big| \\ &\le \frac{1}{4}|\varpi_1 - \varpi_2|, \quad \varpi_1,\varpi_2 \in \mathbb{R}, \quad \xi \in [0,2], \end{aligned}$$

and

$$\begin{aligned} |\varrho(\xi,\varpi_1)\varphi(\xi) - \varrho(\xi,\varpi_2)\varphi(\xi)| &= \Big|\frac{\xi}{4}\Big(\frac{\varpi_1}{2} - \frac{\varpi_2}{2}\Big)\Big| \\ &= \Big|\frac{\xi}{4} \times \frac{1}{2}(\varpi_1 - \varpi_2)\Big| \\ &\le \frac{1}{2} \times \frac{1}{2}|\varpi_1 - \varpi_2|, \quad \varpi_1,\varpi_2 \in \mathbb{R}, \quad \xi \in [0,2], \end{aligned}$$

with

$$L_{\vartheta} = \frac{1}{4}, \quad M = \frac{1}{2}, \quad L_{\varrho} = \frac{1}{2}.$$

If we choose $\wp(\xi) = \frac{e^{\xi}}{8}$, and $\epsilon = \frac{1}{5}$, it follows,

$$\Big|\varpi'(\xi) - \xi e^{\frac{\xi}{3}}\Big[\frac{1}{4}e^{\xi} - 3e^{\frac{2\xi}{3}} + \frac{1}{4}\Big] - \frac{\xi}{8}\varpi(\xi) - e^{\xi}(3\xi+1) + \frac{\xi}{4}\Big[\frac{\varpi(\xi)}{2} + e^{\frac{\xi}{3}} + e^{\frac{4\xi}{3}}\Big| \le \frac{1}{5}\xi, \quad \xi \in [0,2].$$

This exhibits the Hyers–Ulam stability of the hybrid differential Equation (7). *The Hyers–Ulam stability of Equation* (7) *is independent of the initial value condition. In addition, having in mind the exact solution $\varpi(\xi) = e^{\xi}$ and $L_{\vartheta} + ML_{\varrho} = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = 0.5 < 1$, we have*

$$|\wp(\xi) - \varpi(\xi)| \le \frac{2}{5}e, \quad \xi \in [0,2];$$

see Figure 4.

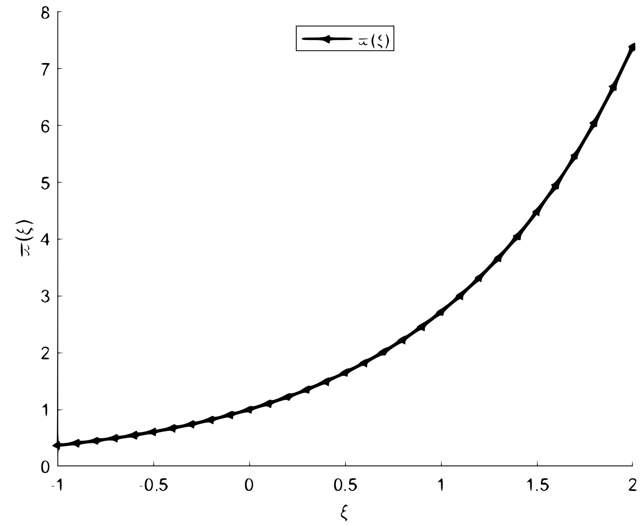

Figure 3: The solution $\varpi(\xi)$ of Equation (7).

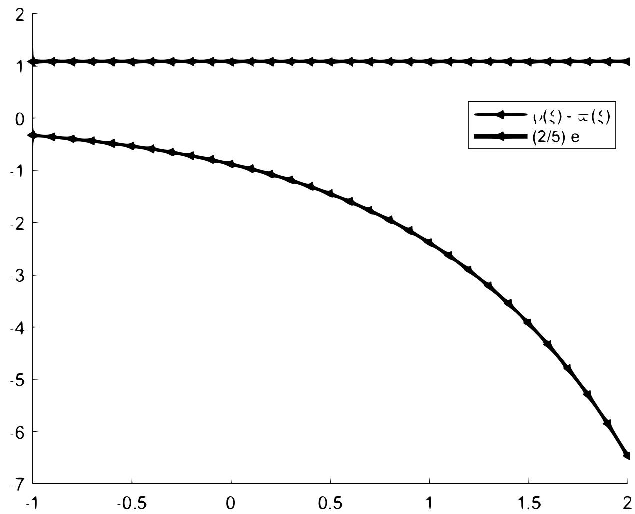

Figure 4: Function $\wp(\xi) - \varpi(\xi)$ and $\frac{2}{5}e$.

5. Conclusion

Using the Gronwall lemma, we demonstrated the Hyers–Ulam and the Hyers–Ulam–Rassias stability of the hybrid differential equation in this paper. We then illustrated our approach with a discussion of applications.

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