Filomat 39:4 (2025), 1419–1426
https://doi.org/10.2298/FIL2504419R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on the approximation of conjugate functions in the H_p^ω space

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Abstract. In this paper we use the trigonometric mean $R_1(\beta)$, for $1 < \beta < 2$, to estimate a sharper degree of approximation of conjugate functions in H_p^ω–space. This paper also generalizes some results of Nigam [12] and [16]. In addition, a particular result is derived from our results as corollary.

1. Introduction

Das et al. [4] developed a new space H_p^ω and generalized the result of Leindler [9]. Leindler [10] extended the result "degree of approximation of the Fourier series in H_p^ω space". Again Das et al. [5] estimated the rate of convergence by using delayed arithmetic mean. Further, Nigam and Hadish [13] estimated the degree of convergence in H_p^ω space using TC^1 method and replacing Fourier series by the conjugate Fourier series. Dutta and Das [1] estimated the degree of convergence by the Zeweier-Euler product summability method. There are several generalizations of the above estimations in the H_p^ω space, see, for example, Mursaleen [11, 14, 15], Krasniqi [7, 8]. Recently, Nigam and Rani [16] estimated the convergence rate of conjugate Fourier series in H_p^ω space by using Matrix-Hausdorff product mean. We note that approximation using the trigonometric mean $R_1(\beta)$, for $1 < \beta < 2$, of Fourier series of functions belonging to the H_p^ω space have not been studied so far, which motivated us to study the problem further.

2. Definition and Notation

Let f be a periodic function with period 2π such that $f \in L_p[0, 2\pi]$, $p \geq 1$. Let the Fourier series associated with f at any point x be defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)), \forall\, k \geq 1,$$

where a_k, b_k are Fourier coefficients.
Then its conjugate series is given by

$$\sum_{k=1}^{\infty} (a_k \cos(kx) - b_k \sin(kx)).$$

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 42A10.
Keywords. Degree of approximation, Trigonometric mean, Conjugate Fourier series, Generalized Hölder space.
Received: 18 July 2024; Accepted: 11 November 2024
Communicated by Miodrag Spalević
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Let n-th partial sum of the above conjugate series be denoted by

$$\tilde{s}_n(x) = \sum_{k=1}^{n} (a_k \cos(kx) - b_k \sin(kx)).$$

It is known that [17]

$$\tilde{s}_n(x) = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \tilde{D}_n(t) dt,$$

where

$$\psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\text{and } \tilde{D}_n(t) = \sum_{\nu=1}^{n} \sin \nu t = \frac{\cos \frac{1}{2}t - \cos(n+\frac{1}{2})t}{2 \sin \frac{1}{2}t}.$$

Conjugate of a function f is denoted as $\tilde{f}$ and defined as follows:

$$\tilde{f} = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi_x(t) \cot(t/2) dt.$$

Note that every conjugate Fourier series is not a Fourier series. For example, consider the Fourier series $\sum_{q=2}^{\infty} \frac{\cos qx}{\ln q}$. Then the corresponding conjugate Fourier series can be deduced as $\sum_{q=2}^{\infty} \frac{\sin qx}{\ln q}$, which is not a Fourier series[[17], p.186].

Hence, conjugate Fourier series needs to be studied separately.

Let $\sum a_n$ be a given infinite series and let (s_n) be the sequence of partial sums of the given series. Then $R_1(\beta)$-mean, for $1 < \beta < 2$ which was introduced by Das, Nath and Ray [3] is as follows:

$$\begin{aligned} B(x) &= \sum_{\nu=1}^{\infty} b_\nu(x) s_\nu \\ &= \frac{(\beta-1)x^{2-\beta}}{\sin \pi \frac{(\beta-1)}{2}} \sum_{\nu=1}^{\infty} A_\nu^{1-\beta} \left(\frac{\sin \nu x}{\nu x} \right) s_\nu, \ \nu = 1, 2, 3, \ldots, \end{aligned}$$

where $\sum_{n=0}^{\infty} A_n^{\beta} x^n = \frac{1}{(1-x)^{1+\beta}}$.

Note that the method of summation is Fourier effective and regular [3].

Das et al. [4] have introduced the space

$$H_p^{\omega} := \{f \in L_p[0, 2\pi], p \geq 1 : A(f, \omega) < \infty\},$$

where ω is the integral modulus of continuity and

$$A(f, \omega) := \sup_{t \neq 0} \frac{\|f(.+t) - f(.)\|_p}{\omega(|t|)}.$$

The H_p^{ω} space is equiped with the norm $\|f\|_p^{\omega} := \|f\|_p + A(f, \omega)$ and with this norm the space H_p^{ω} is a Banach space [4].

It should be noted that if $\omega(t)/v(t)$ is non-decreasing, then $\|f\|_p^{v} \leq \max\left\{1, \frac{\omega(2\pi)}{v(2\pi)}\right\} \|f\|_p^{\omega}$. Thus

$$H_p^{\omega} \subseteq H_p^{v} \subseteq L_p, p \geq 1.$$

Let $\omega(t)$ and $v(t)$ be the integral moduli of continuity, then the following lemma holds[4]

Lemma 2.1. *Let* $0 \leq \delta \leq 1$ *and* $\frac{\omega(t)}{\nu(t)}$ *be non decreasing. Then*

$$n^{\delta} \int_0^{\pi/n} \frac{\omega(t)}{\nu(t)} t^{\delta-1} dt \leq C \int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)} dt,$$

where C is a constant.

By using the notation $K = \frac{2}{\pi}\left(\frac{\beta-1}{\sin \pi \frac{(\beta-1)}{2}}\right)$, we define the followings:

$$\tilde{K}_n(t) = \frac{K}{\pi n^{2-\beta}} \sum_{\nu=1}^{n} A_{\nu}^{1-\beta} \left(\frac{\sin(\nu/n)}{(\nu/n)}\right) sin\left(\nu + \frac{1}{2}\right) t.$$

$$\tilde{G}_n(t) = \frac{K}{n^{2-\beta}} \sum_{\nu=1}^{n} A_{\nu}^{1-\beta} \left(\frac{\sin(\nu/n)}{(\nu/n)}\right) sin\left(\nu + \frac{1}{2}\right) t.$$

$$\tilde{H}_n(t) = \frac{K}{n^{2-\beta}} \sum_{\nu=n+1}^{n} A_{\nu}^{1-\beta} \left(\frac{\sin(\nu/n)}{(\nu/n)}\right) sin\left(\nu + \frac{1}{2}\right) t.$$

3. Known Results

Dealing with the degree of approximation by matrix-Hausdorff product mean of conjuate Fourier series in $H_r^{(\eta)}$ space, Nigam and Rani [16] proved the following theorem.

Theorem 3.1. *If* $\tilde{f} \in H_p^{\omega}$, *for* $p \geq 1$ *and* ω *and* ν *be integral moduli of continuity, such that* $\frac{\omega(s)}{\nu(s)}$ *is non decreasing; then the degree of approximation of* $\tilde{f}$ *by matrix-Haausdorff product mean of its conjugate Fourier series is given by*

$$\|\tilde{\omega}_n - \tilde{f}\|_r^{\nu} = O\left(\frac{\log(n+1)+1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(s)}{t^2 \nu(s)} ds\right).$$

Subsequently J. Kim [6] replace the matrix mean by even-type delayed arithmetic mean and the following result is obtained:

Theorem 3.2. *Let* ω *and* ν *be integral moduli of continuity such that* $\frac{\omega}{\nu}$ *is nondecreasing. If* $f \in H_p^{\omega}$, $p \geq 1$ *then for* $k = 2c$, *where* $c \in \mathbb{N}$,

$$\|W_n(x;f) - f\|_p^{\nu} = O\left(\frac{1}{kn}\right) + O\left(\frac{k}{n^2}\right) \int_{\pi/n}^{\pi} \frac{\omega(t)}{\nu(t)} \frac{1}{t^3} dt.$$

4. Main Results

In this paper we have estimated the rate of convergence of conjugate Fourier series of functions belonging to the generalized Hölder metric space

$$H_p^{\omega} = \{f \in L_p(0, 2\pi), p \geq 1 : A(f, \omega) < \infty\},$$

with the norm $\|f\|_p^{(\omega)} := \|f\|_p + A(f, \omega)$, by using a trigonometric mean $R_1(\beta)$, for $1 < \beta < 2$.
First we prove the following lemmas which are used in proving our theorem.

Lemma 4.1. $\tilde{K}_n(t) = O(tn^{1-\beta}) + O(t^{\delta} n^{\delta-\beta})$, $0 < \delta < 1$.

Proof.

$$\begin{aligned}\tilde{K}_n(t) &= \frac{2K}{n^{2-\beta}}\sum_{\nu=1}^{\infty} A_\nu^{1-\beta}\frac{\sin(\nu/n)}{\nu/n}\sin(\nu+\frac{1}{2})t\sin(\frac{\nu t}{2})\\ &= \frac{2K}{n^{2-\beta}}\left(\sum_{\nu=1}^{n}+\sum_{\nu=n+1}^{\infty}\right)A_\nu^{1-\beta}\frac{\sin(\nu/n)}{\nu/n}\sin\left(\nu+\frac{1}{2}\right)t\sin(\frac{\nu t}{2})\\ &= I_1+I_2. \end{aligned}\tag{1}$$

Now,

$$\begin{aligned}I_1 &= \frac{2K}{n^{2-\beta}}\sum_{\nu=1}^{n} A_\nu^{1-\beta}\frac{\sin(\nu/n)}{\nu/n}\sin\left(\nu+\frac{1}{2}\right)t\sin(\frac{\nu t}{2})\\ &\leq \frac{2K}{n^{2-\beta}}\frac{\sin(1/n)}{1/n}\sum_{\nu=1}^{n}|A_\nu^{1-\beta}\sin\left(\nu+\frac{1}{2}\right)t\sin(\frac{\nu t}{2})|\\ &= O(1)\sum_{\nu=1}^{n}A_\nu^{1-\beta}\nu t\\ &= O(tn^{1-\beta}). \end{aligned}\tag{2}$$

$$I_2 = \frac{2K}{n^{2-\beta}}\sum_{\nu=n+1}^{\infty} A_\nu^{1-\beta}\frac{\sin(\nu/n)}{\nu/n}\sin\left(\nu+\frac{1}{2}\right)t\sin(\frac{\nu t}{2}).$$

Using the fact that, for any $0<\delta\leq 1$

$$\left|\sin\left(\frac{\nu t}{2}\right)\right| = \left|\sin\left(\frac{\nu t}{2}\right)\right|^{\delta}\left|\sin\left(\frac{\nu t}{2}\right)\right|^{1-\delta} = O(\nu^\delta t^\delta).$$

Now, we obtain

$$\begin{aligned}I_2 &= O(1)\sum_{\nu=n+1}^{\infty}\nu^{1-\beta}\nu^{-1}\nu^{\delta}t^{\delta}\\ &= O(1)\sum_{\nu=n+1}^{\infty}\nu^{-\beta+\delta}t^{\delta}\\ &= O(t^\delta n^{\delta-\beta}). \end{aligned}\tag{3}$$

Using 2 and 3 for $\tilde{K}_n(t)$ in 1, we get

$$\tilde{K}_n(t) = O(tn^{1-\beta}) + O(t^\delta n^{\delta-\beta}).$$

□

Lemma 4.2. $\tilde{H}_n(t) = O(n^{-\beta+1}(2-3tn)^{-1})$.

Proof.

$$\begin{aligned}\tilde{H}_n(t) &= \sum_{\nu=1}^{\infty}\frac{A_\nu^{1-\beta}}{\nu}\sin(\nu/n)\cos\left(\nu+\frac{1}{2}\right)t\\ &= \frac{1}{2}\sum_{\nu=1}^{\infty}\frac{A_\nu^{1-\beta}}{\nu}2\sin(\nu/n)\cos\left(\nu+\frac{1}{2}\right)t \end{aligned}$$

$$= \frac{1}{2}\sum_{\nu=1}^{\infty}\frac{A_\nu^{1-\beta}}{\nu}\sin\left(\frac{\nu}{n}+\left(\nu+\frac{1}{2}\right)t\right)+\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{A_\nu^{1-\beta}}{\nu}\sin\left(\frac{\nu}{n}-\left(\nu+\frac{1}{2}\right)t\right)$$
$$= O\left(n^{-\beta}\frac{1}{\left(\frac{1}{n}+\frac{3t}{2}\right)}\right)+O\left(n^{-\beta}\frac{1}{\left(\frac{1}{n}-\frac{3t}{2}\right)}\right)$$
$$= O(n^{-\beta})\left(\frac{2n}{2+3tn}+\frac{2n}{2-3tn}\right)$$
$$= O(n^{-\beta+1}(2-3tn)^{-1}).$$

□

Lemma 4.3. $\tilde{G}_n(t) = O(\frac{1}{n^{2-\beta}})$.

Proof.

$$G_n(t) = 1-\frac{K}{n^{2-\beta}}\sum_{\nu=1}^{n}A_\nu^{1-\beta}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)$$
$$\leq \frac{1}{n^{2-\beta}}\left(\frac{\sin(1/n)}{(1/n)}\right).$$

Which implies

$$\tilde{G}_n(t) = O\left(\frac{1}{n^{2-\beta}}\right).$$

□

Theorem 4.4. *If $\tilde{f} \in H_p^\omega$, for $p \geq 1$ and $\beta \in (1, 2)$; then the degree of approximation of $\tilde{f}$ by $R_1(\beta)$ trigonometric mean of its conjugate Fourier series is given by*

$$\|\tilde{\omega}_n(\tilde{f}) - \tilde{f}(\pi/n)\|_p^\nu = \|\tilde{l}_n(x)\|_p^\nu$$
$$= O(1)\max\left\{\frac{1}{n^{2-\beta}}, \frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2\nu(t)}dt\right\},$$

where $\omega(t)$ and $\nu(t)$ are integral moduli of continuity such that $\frac{\omega(t)}{\nu(t)}$ is nondecreasing.

Proof. We know [17]

$$\tilde{s}_\nu(\tilde{f}; x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi_x(t)\frac{\cos(\nu+1/2)t}{\sin(t/2)}dt.$$

By using $R_1(\beta), (1 < \beta < 2)$ mean on $\tilde{s}_n(x)$, we have

$$\tilde{\omega}_n(\tilde{f}) = \frac{K}{n^{2-\beta}}\sum_{\nu=1}^{\infty}A_\nu^{1-\beta}\frac{\sin(\nu/n)}{(\nu/n)}\int_0^\pi \psi_x(t)\frac{\cos(t/2)-\cos\left(\nu+\frac{1}{2}\right)t}{2\sin(t/2)}dt$$
$$= \frac{K}{n^{2-\beta}}\int_0^\pi \frac{\psi_x(t)}{2\sin(t/2)}\sum_{\nu=1}^{\infty}A_\nu^{1-\beta}\frac{\sin(\nu/n)}{(\nu/n)}\left(\cos(t/2)-\cos\left(\nu+\frac{1}{2}\right)t\right)dt$$
$$= \int_0^\pi \frac{\psi_x(t)}{2\sin(t/2)}\tilde{K}_n(t)dt.$$

Now,

$$\begin{aligned}\tilde{l}_n(x) &= \tilde{\omega}_n(\tilde{f}) - \tilde{f}(x,\pi/n)\\ &= \frac{2}{\pi}\int_0^{\pi/n}\frac{\psi_x(t)}{2\sin(t/2)}\tilde{K}_n(t)dt - \frac{2}{\pi}\int_{\pi/n}^{\pi}\psi_x(t)\cot(t/2)dt\\ &= \frac{2}{\pi}\left\{\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right\}\frac{\psi_x(t)}{2\sin(t/2)}\tilde{K}_n(t)dt - \frac{2}{\pi}\int_{\pi/n}^{\pi}\psi_x(t)\cot(t/2)dt\\ &= \frac{2}{\pi}\int_0^{\pi/n}\frac{\psi_x(t)}{2\sin(t/2)}\tilde{K}_n(t)dt - \frac{2Kn^{\beta-1}}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_x(t)}{2\sin(t/2)}\tilde{H}_n(t)dt - \frac{2}{\pi}\int_{\pi/n}^{\pi}\psi_x(t)\cot(t/2)\tilde{G}_n(t)dt.\end{aligned}$$

Then,

$$\begin{aligned}\|\tilde{l}_n(x+y)-\tilde{l}_n(x)\|_p &\le \frac{2}{\pi}\int_0^{\pi/n}\|\psi_{x+y}(t)-\psi_x(t)\|_p\left|\frac{\tilde{K}_n(t)}{2\sin(t/2)}\right|dt\\ &\quad - \frac{2Kn^{\beta-1}}{\pi}\int_{\pi/n}^{\pi}\|\psi_{x+y}(t)-\psi_x(t)\|_p\left|\frac{\tilde{H}_n(t)}{2\sin(t/2)}\right|dt\\ &\qquad - \frac{2}{\pi}\int_{\pi/n}^{\pi}\|\psi_{x+y}(t)-\psi_x(t)\|_p\,|\tilde{G}_n(t)|\cos(t/2)dt\\ &= I_1+I_2+I_3 \end{aligned}\tag{4}$$

Using Lemma 4.1, we get

$$\begin{aligned}I_1 &= \frac{2}{\pi}\int_0^{\pi/n}\|\psi_{x+y}(t)-\psi_x(t)\|_p\left|\frac{\tilde{K}_n(t)}{2\sin(t/2)}\right|dt\\ &= O(1)\nu(y)\left[\int_0^{\pi/n}\frac{\omega(t)}{\nu(t)}n^{1-\beta}ds + n^{\delta-\beta}\int_0^{\pi/n}\frac{\omega(t)}{\nu(t)}t^{\delta-1}dt\right].\end{aligned}\tag{5}$$

Using Lemma 4.2, we get

$$\begin{aligned}I_2 &= \frac{2Kn^{\beta-1}}{\pi}\int_{\pi/n}^{\pi}\|\psi_{x+y}(t)-\psi_x(t)\|_p\left|\frac{\tilde{H}_n(t)}{2\sin(t/2)}\right|dt\\ &= O(\nu(y))\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}(2-3tn)^{-1}dt.\end{aligned}\tag{6}$$

By Lemma 4.3,

$$\begin{aligned}I_3 &= \frac{2}{\pi}\int_{\pi/n}^{\pi}\|\psi_{x+y}(t)-\psi_x(t)\|_p\,|\tilde{G}_n(t)|\cos(t/2)dt.\\ &= O(1)\frac{\nu(y)}{n^{2-\beta}}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}dt.\end{aligned}\tag{7}$$

Collecting the estimates for I_1, I_2, I_3 and using (4), we obtain

$$\|\tilde{l}_n(x+y)-\tilde{l}_n(x)\|_p = O(1)\nu(y)\frac{1}{n^{2-\beta}}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}dt.\tag{8}$$

Thus,

$$\|\tilde{l}_n(x)\|_p^{\nu} = \|\tilde{l}_n(x)\|_p + \sup_{y\neq 0}\frac{\|\tilde{l}_n(x+y)-\tilde{l}_n(x)\|_p}{\nu(y)}$$

$$= O(1)\frac{1}{n^{1-\beta}}\int_0^{\pi/n}\frac{\omega(t)}{t\nu(t)}(2-3tn)^{-1}dt + O(1)\frac{1}{n^{2-\beta}}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}dt$$

$$= O(1)\max\left\{\frac{1}{n^{2-\beta}}, \frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2\nu(t)}dt\right\}. \tag{9}$$

□

Corollary 4.5. *If we take $\omega(s) = \frac{s^\alpha}{\left(\log\frac{1}{s}\right)^a}$, $\nu(s) = \frac{s^\eta}{\left(\log\frac{1}{s}\right)^b}$, in theorem 4.4, where $a, b \in \mathbb{R}$ and $0 \le \eta \le \alpha \le 1$. Then*

$$\|\tilde{\omega}_n(f) - \tilde{f}(x)\|_p^\nu = \begin{cases} \frac{O(1)}{n} & \text{if } \alpha > \eta, \text{ any } a, b \in \mathbb{R}, \\ O(1)\frac{1}{n(\log n)^{a-b-1}} & \text{if } \alpha = \eta, \text{ any } a-b < 1, \\ O(1)\frac{\log\log n}{n} & \text{if } \alpha = \eta, \text{ any } a-b = 1, \\ \frac{O(1)}{n} & \text{if } \alpha = \eta, \text{ any } a-b > 1. \end{cases}$$

Proof. We have

$$\|\tilde{\omega}_n(\tilde{f}) - \tilde{f}\|_p^\nu = O(1)\max\left\{\frac{1}{n}\int_{\pi/n}^{\pi}\frac{s^{\alpha-\eta-2}}{\left(\log\frac{1}{s}\right)^{a-b}}ds, \frac{1}{n^{2-\beta}}\right\}.$$

Since

$$\int_{\pi/n}^{\pi} s^{\alpha-\eta-2}\left(\log\frac{1}{s}\right)^{b-a} ds = \begin{cases} O(1) & \text{if } \alpha > \eta, \text{ any } a, b \in \mathbb{R}, \\ O(1)\frac{1}{(\log n)^{a-b-1}} & \text{if } \alpha = \eta, \text{ any } a-b < 1, \\ O(1)\log\log n & \text{if } \alpha = \eta, \text{ any } a-b = 1, \\ O(1) & \text{if } \alpha = \eta, \text{ any } a-b > 1. \end{cases}$$

The result follows. □

Remark 4.6. *If $\omega(\delta) = \delta^\eta, 0 < \eta \le 1$, then H_p^ω space becomes $H(\eta, p)$ space, which was introduced by Das et al. [2].*

Remark 4.7. *If $\omega(\delta)/\delta$ tends to 0 as $\delta \to 0^+$, then $f'(x) = 0$, $\forall x$, and $f(x)$ is a constant function.*

Acknowledgements

The first author is grateful to CSIR(India) for providing Senior Research Fellowship (SRF) (Ref. No: 18112/2016 (ii) EU-V) to carry out the current research work and also thanks to Utkal university for providing the research facilities.

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