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Cesàro-recurrent operators

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Abstract. In this paper we introduce and study the notion of Cesàro-recurrent operators. We give some necessary and sufficient conditions for an operator T acting on a Banach space X to be Cesàro-recurrent.

1. Introduction

We start by recalling some notions of linear dynamics. Let X be a topological vector space. Taking $B(X)$ the set of linear bounded operators acting on X, an operator $T \in B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the sequence

$$\mathcal{O}(x,T) := \{T^n x : \ n \in \mathbb{N}\}$$

is dense in X; the vector x is then called hypercyclic for T. We will denote by $HC(T)$ the set of hypercyclic vectors for the operator T. Examples of hypercyclic operators includes the translation operator $Tf(z) = f(z+\alpha)$ acting on the space of entire functions $H(\mathbb{C})$, the differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, some unilateral and bilateral weighted shifts, and composition operators on certain function spaces (see the survey article [13]). If we have a sequence of bounded operators $(T_n)_{n\in\mathcal{I}}$ that act on X, then this sequence if hypercyclic if there is $x \in X$ such that the set

$$\mathcal{O}(x,(T_n)_{n\in\mathcal{I}}) := \{T_n x : \ n \in \mathcal{I}\}$$

is dense in X. In case we can find a vector $x \in X$ and a sequence $(\lambda_n)_{n\in\mathcal{I}} \subset \mathbb{C}$ such that

$$\mathbb{C}.\mathcal{O}(x,(T_n)_{n\in\mathcal{I}}) := \{\lambda_n T_n x : \ n \in \mathcal{I}\}$$

is a dense subset of X, then the sequence $(T_n)_{n\in\mathcal{I}}$ is said to be supercyclic. An operator $T \in B(X)$ is said to be supercyclic, if the sequence of its iterates $(T^n)_{n\in\mathbb{N}}$ is supercyclic, that is, there is $x \in X$ such that

$$\mathbb{C}.\mathcal{O}(x,T) := \{\lambda T^n x : \ n \in \mathbb{N}, \ \lambda \in \mathbb{C}\}$$

is dense in X; such a vector x is said to be supercyclic and the set of supercyclic vectors for T is denoted $SC(T)$. For more recent results on hypercyclicity and supercyclicity we refer to the books [7], [14]. If the

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main space is Banach the hypercyclicity of $T \in B(X)$ is equivalent to its transitivity, that is, for every two open non-empty subsets $U, V \subset X$, there is $n \in \mathbb{N}$ such that

$$T^n U \cap V \neq \emptyset.$$

Similarity, the operator T is supercyclic if and only if there are a scalar $\lambda \in \mathbb{C}$ and an integer n such that

$$\lambda T^n U \cap V \neq \emptyset.$$

Another notions that will be discussed here is that of recurrence and its near sibling super-recurrence. A vector $x \in X$ is said to be recurrent for T if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$T^{n_k} x \to x \quad \text{as} \quad k \to \infty.$$

and it is called super-recurrent if there is a pair of sequences $(\lambda_k, n_k)_{k\in\mathbb{N}} \subset \mathbb{C} \times \mathbb{N}$ such that

$$\lambda_k T^{n_k} x \to x \quad \text{as} \quad k \to \infty.$$

In context of linear operators, the operator T is recurrent (super-recurrent) is it has a dense set of recurrent vectors (of super-recurrent vectors) for T. An operator T is called recurrent (super-recurrent) if for each nonempty and open set U of X there is an integer n (a scalar λ and an integer n) such that

$$T^n U \cap U \neq \emptyset, \quad (\lambda T^n U \cap U \neq \emptyset).$$

Motivated by some questions related to ergodic theory (see [12], [22], [26], [21] for instance), Leon-Saavedra [23] introduced the notion of Cesàro hypercyclicity. Following his terminology, we denote by $M_n(T)$ the arithmetic means of an operator $T \in B(X)$, that is,

$$M_n(T) := \frac{I + T + \ldots + T^{n-1}}{n}, \; n = 1, 2, \ldots.$$

The operator T will be called Cesàro hypercyclic if there is a vector x whose Cesàro orbit under T, $\{M_n(T)x : \; n = 1, 2, \ldots\}$, is dense in X. Recent studies on Cesàro hypercyclic operators and Cesàro orbits can be found in [10], [23], [24]. Let us also mention that Leon-Saavedra [23] showed that hypercyclicity does not imply Cesàro hypercyclicity or vice versa. In [23] it is shown that an operator T is Cesàro hypercyclic if and only if there exists a vector y such that $\{n^{-1}T^n y : \; n = 1, 2, \ldots\}$ is dense. We call the vector y Cesàro hypercyclic with respect to the sequence $\{n^{-1}T^n\}$.

Recently, many authors gave an important contribution in the field of linear dynamics see as example [1–6, 8, 9, 17–20, 28–31].

In this paper we introduce and study the Cesàro recurrence of an operator acting on a Banach space X.

In Section 2, we introduce the concept of Cesàro recurrent operators. We give some examples and we prove some proprieties of this class of operators. In particular, we prove that the Cesàro recurrence is stable under some operators transformation and we give some sufficient and necessary condition to ensure the Cesàro recurrence. A condition on the point spectrum of the Banach adjoint operator of a Cesàro recurrent operator is also provided.

2. Cesàro recurrence of operators

Definition 2.1. *Let T be a bounded linear operator on a complex Banach space X. A vector $x \in X$ is said to be Cesàro-recurrent vector for T if there exists a strictly increasing sequence (n_k) of natural numbers such that $M_{n_k}(T)x$ converges to x as k goes towards infinity. The set of Cesàro-recurrent vectors for T will be noted $CMRec(T)$. The operator T is said to be Cesàro-recurrent if the set $CMRec(T)$ is dense in X.*

Definition 2.2. *Let T be a bounded linear operator on a complex Banach space B. T is topologically Cesàro-recurrent if for every non-empty open subset $U \subset B$ there is $n \in \mathbb{N}$ such that*

$$M_n(T)U \cap U \neq \emptyset.$$

First of all, we deal with the initial matter in linear dynamics, which is the case of finite-dimensional spaces. Recall that hypercyclicity cannot exist in this type of space [32], and supercyclicity exists only in spaces of dimension one [15]. However, the recurrence case is quite different. In fact, there is a large class of operators in finite-dimensional spaces which are recurrent [11]. In the following result, we prove that unlike in the case of recurrence, there are no Cesàro-recurrent operators on nonzero finite-dimensional spaces.

Proposition 2.3. *There are no Cesàro-recurrent operators on nonzero finite-dimensional spaces.*

Proof. Assume that $A \in \mathcal{M}_n(\mathbb{C})$ is Cesàro-recurrent, where $\mathcal{M}_n(\mathbb{C})$ is the set of all square complex matrices. By Jordan decomposition theorem, the matrix A is similar to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a Cesàro-recurrent for A. Then for some increasing sequence of natural numbers (n_k),

$$\frac{1 + \lambda_i + \cdots + \lambda_i^{n_k - 1}}{n_k} x_i \longrightarrow x_i, \quad \text{for all } 1 \leq i \leq n.$$

When $|\lambda| > 1$, $\frac{1 + \lambda_i + \cdots + \lambda_i^{n_k - 1}}{n_k}$ diverges to infinity. On the other hand, when $|\lambda| \leq 1$, $\frac{1 + \lambda_i + \cdots + \lambda_i^{n_k - 1}}{n_k}$ stays within the closed unit disk for any positive integer n. This clearly leads to a contradiction. Therefore, A cannot have any Cesàro-recurrent vector. Hence, A is not a Cesàro-recurrent matrix. □

Given the proposition stated earlier and the results concerning recurrence in finite-dimensional spaces, one can deduce the existence of a recurrent operator that is not Cesàro-recurrent. This highlights the distinction between recurrence and Cesàro-recurrence, demonstrating that not all recurrent operators possess the Cesàro-recurrence property.

Example 2.4. *Let A be a square diagonal matrix over $\mathbb{C}$, i.e., $A = diag(\lambda_1, \dots, \lambda_n)$. By [11], A is recurrent if and only if all λ_i are in the unit cirecle. On the other hand, A cannot be Cesàro-recurrent by Proposition 2.3*

Next, we aim to demonstrate the stability of Cesàro recurrence under certain straightforward operator transformations. To commence, we present the following result.

Proposition 2.5. *Let T and S be two operators such that $TS = ST$. If $x \in CMRec(T)$, then $Sx \in CMRec(T)$.*

Proof. Let $x \in CMRec(T)$. Then there exists a sequence (n_k) of positive integers such that

$$\frac{x + Tx + \cdots + T^{n_k - 1}x}{n_k} \xrightarrow{k \to \infty} x.$$

Since S is continuous, it follows that

$$\frac{Sx + STx + \cdots + ST^{n_k - 1}x}{n_k} \xrightarrow{k \to \infty} Sx.$$

Utilizing the fact that $TS = ST$ and a simple induction, one can deduce that $T^nS = ST^n$ for all n. Hence,

$$\frac{Sx + TSx + \cdots + T^{n_k-1}Sx}{n_k} \xrightarrow{k\to\infty} Sx.$$

This implies that Sx is a Cesàro-recurrent vector for T. □

Corollary 2.6. *If x is a Cesàro-recurrent vector for an operator T and λ is a nonzero complex scalar, then λx is a Cesàro-recurrent vector for T.*

Proof. Assume that x is a Cesàro-recurrent vector for T and let λ be a nonzero complex scalar. Take $S = \lambda I$, where I represents the identity operator on X. It is clear that S is an operator such that $TS = ST$. By using Proposition 2.5, one can easily deduce the desired result. □

In fact, by utilizing Proposition 2.5, we can even provide a stronger result than Corollary 2.6. This result is the analogue of the Herrero-Bourdon theorem in the case of hypercyclicity; see, for example, [16].

Theorem 2.7. *If x is a Cesàro-recurrent vector for T, then*

$$\{p(T)x \mid p \text{ is a polynomial}\} \subset CMRec(T).$$

In particular, if T has a nonzero Cesàro-recurrent vector, then it admits an invariant subspace consisting of Cesàro-recurrent vectors.

Proof. Pick any polynomial p and let $S = p(T)$. Then it is easy to see that S commutes with T. Then, we use Proposition 2.5 to obtain the desired result. □

The second transformation, which will prove that Cesàro recurrence is stable under it, is the quasi-similarity.

Recall that if X and Y are two Banach spaces, and T and S are operators acting on X and Y respectively, then T is called quasi-conjugate or quasi-similar to S if there exists some continuous map $\phi : X \longrightarrow Y$ with a dense range such that $S \circ \phi = \phi \circ T$. If ϕ can be chosen to be a homeomorphism, then T and S are called conjugate or similar, see [14, Definition 1.5].

Proposition 2.8. *Assume that T is quasi-similar to S, if x is a Cesàro-recurrent vector for T, then $\phi(x)$ is a Cesàro-recurrent vector for S. This implies that*

$$\phi(CMRec(T)) \subset CMRec(S).$$

Proof. Let x be a Cesàro-recurrent vector for T. Then, there exists a sequence (n_k) of positive integers such that

$$M_{n_k}(T)x \longrightarrow x.$$

Since ϕ is continuous, it follows that

$$\phi(M_{n_k}(T)x) \longrightarrow \phi(x).$$

Utilizing that $S \circ \phi = \phi \circ T$ and a simple induction, we deduce that

$$M_{n_k}(S)\phi(x) \longrightarrow \phi(x).$$

Thus, $\phi(x)$ is a Cesàro-recurrent vector for S. □

Remark 2.9. *The result in Proposition 2.8 remains valid even if we assume that the map ϕ is just continuous without the condition of dense range.*

Corollary 2.10. *Assuming that T and S are similar. Then x is a Cesàro-recurrent vector for T if and only if $\phi(x)$ is a Cesàro-recurrent vector for S. This means that*

$$\phi(CMRec(T)) = CMRec(S).$$

The final transformation we will consider is the restriction. In this context, we will demonstrate that Cesàro-recurrence is preserved under restriction.

Definition 2.11. *Let T be an operator on X. Then a subset $Y \subset X$ is called T-invariant or invariant under T if $T(Y) \subset Y$. In this sense, if $Y \subset X$ is T-invariant, then $T_{|Y} : Y \to Y$ is also an operator.*

Proposition 2.12. *Let M_1 and M_2 be two closed T-invariant subspaces of X such that $X = M_1 \oplus M_2$. If $x \oplus y \in M_1 \oplus M_2$ is Cesàro-recurrent for T, then x and y are Cesàro-recurrent vectors for $T_1 = T_{|M_1}$ and $T_2 = T_{|M_2}$, respectively. This means that*

$$CMRec(T) \subset CMRec(T_1) \oplus CMRec(T_2).$$

Proof. Let $x \oplus y \in M_1 \oplus M_2$ be Cesàro-recurrent for T. Then there exists a sequence of positive integers (n_k) such that

$$M_{n_k}(T)(x \oplus y) \longrightarrow (x \oplus y).$$

On the other hand, we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} M_n(T_1 \oplus T_2)(x \oplus y) &= \frac{(x \oplus y) + (T_1 \oplus T_2)(x \oplus y) + \cdots + (T_1^{n-1} \oplus T_2^{n-1})(x \oplus y)}{n} \\ &= \frac{x + T_1 x + \cdots + T_1^{n-1} x}{n} \oplus \frac{y + T_2 y + \cdots + T_2^{n-1} y}{n} \\ &= M_n(T_1)x \oplus M_n(T_2)y. \end{aligned}$$

Hence, $M_{n_k}(T_1)x \longrightarrow x$ and $M_{n_k}(T_2)y \longrightarrow y$, which implies the desired result. □

Proposition 2.13. *Let $T \in B(X)$ be a bounded linear operator acting on a Banach space X. The following are equivalent:*

1. *for every non-empty open subset U of X there exists a strictly increasing sequence (n_k) of natural numbers such that*
$$M_{n_k}(T)(U) \cap U \neq \emptyset$$
for each k.
2. *T is Cesàro-recurrent.*

Proof. First, assume that the T has a dense set of Cesàro mean recurrent vectors and let U be an open set in X. Then there is a Cesàro recurrent vector y in U, let $\varepsilon > 0$ such that $B := B(y, \varepsilon) \subset U$. Since y is a Cesàro-recurrent vector for T, there exists a strictly increasing sequence (n_k) of natural numbers such that $M_{n_k}(T)y$ converges to y as k goes towards $infinity$.

Hence, we may without loss of generality assume that $M_{n_k}(T)y \in B \subset U$ for all k. Thus,

$$M_{n_k}(T)(U) \cap U \neq \emptyset \text{ for each } k.$$

For the converse, suppose that (1) is satisfied and fix an open ball $B := B(x, \varepsilon)$ for some $x \in X$ and $0 < \varepsilon < 1$. Our purpose is to show that there is a Cesàro recurrent vector in B. By hypothesis there exists a positive integer k_1 such that

$$M_{k_1}(T)^{-1}B \cap B \neq \emptyset.$$

Let $x_1 \in M_{k_1}(T)^{-1}B \cap B \neq \emptyset$. Since T is continuous, so is $M_{k_1}(T)$, and there exists $\varepsilon_1 < \frac{1}{2}$ such that

$$B_2 := B(x_1, \varepsilon_1) \subset B \cap M_{k_1}(T)^{-1}B.$$

Then, there is $k_2 > k_1$ such that $x_2 \in M_{k_2}(T)^{-1}B_2 \cap B_2$ for some $x_2 \in X$. By continuity again there exists $\varepsilon_2 < \frac{1}{2^2}$ such that

$$B_3 := B(x_2, \varepsilon_2) \subset B_2 \cap M_{k_2}(T)^{-1}B_2.$$

Continuing inductively we construct a sequence of positive real numbers $\varepsilon_n < \frac{1}{2^n}$, such that

$$B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}), \text{ and } M_{k_{n-1}}(T)B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}). \tag{1}$$

By completeness of X we conclude by Cantor's theorem that

$$\bigcap_n B(x_n, \varepsilon_n) = \{y\},$$

where $y \in X$. Then to this end, we have $M_{k_n}(T)y \to y$, so y is a Cesàro recurrent vector in B. □

As a consequence of Proposition 2.8 and Proposition 2.13, we can easily deduce that Cesàro recurrence is stable under quasi-similarity and restriction.

Proposition 2.14. *Assume that $T \in \mathcal{B}(X)$ is quasi-similar to $S \in \mathcal{B}(Y)$. If T is Cesàro-recurrent on X, then S is Cesàro-recurrent on Y.*

Proof. On one hand, T is assumed to be Cesàro-recurrent on X, so by Proposition 2.13, $\overline{CMRec(T)} = X$.

On the other hand, T is quasi-similar to S, so there is a continuous map $\phi : X \to Y$ of dense range such that $\phi \circ T = S \circ \phi$.

By Proposition 2.8, we have $\phi(CMRec(T)) \subset CMRec(S)$.

We then have

$$\phi(X) = \phi\left(\overline{CMRec(T)}\right) \subset \overline{\phi\left(CMRec(T)\right)} \subset \overline{CMRec(S)}.$$

Finally, the desired result follows from the fact that the map ϕ has a dense range. □

Corollary 2.15. *Assume that $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$ are similar. Then, T is Cesàro-recurrent on X if and only if S is Cesàro-recurrent on Y.*

Proposition 2.16. *Let M_1 and M_2 be closed T-invariant subspaces of X such that $X = M_1 \oplus M_2$. If T is Cesàro-recurrent on X, then $T_1 = T_{|M_1}$ and $T_2 = T_{|M_2}$ are Cesàro-recurrent on M_1 and M_2, respectively.*

Proof. Assume that the operator T is Cesàro-recurrent on X. Firstly, by Proposition 2.13, the set $CMRec(T)$ is dense in the entire space X. Secondly, by Proposition 2.12, we have that $CMRec(T) \subset CMRec(T_1) \oplus CMRec(T_2)$. Combining these two facts with the fact that $X = M_1 \oplus M_2$, one can deduce that $CMRec(T_1)$ and $CMRec(T_2)$ are dense in M_1 and M_2, respectively. This means that T_1 and T_2 are Cesàro-recurrent on M_1 and M_2, respectively. □

Let us now give some equivalent conditions for Cesàro-recurrence.

Theorem 2.17. *Let T be an operator acting on a Banach space X. The following conditions are equivalent:*

1. *T is Cesàro-recurrent;*
2. *For all $x \in X$, there exist sequences $(x_k) \subset X$ and $(n_k) \subset \mathbb{N}$ such that both (x_k) and $M_{n_k}(T)x_k$ converge to x;*
3. *For all $x \in X$ and for every neighborhood W of 0, there exist $z \in X$ and $n \in \mathbb{N}$ such that both $x - z$ and $M_n(T)z - x$ are in W.*

Proof. $(1) \Rightarrow (2)$: Assume that T is Cesàro-recurrent on X and let $x \in X$. Consider $B\left(x, \frac{1}{k}\right)$, the ball centered at x with radius $\frac{1}{k}$. Since T is supposed to be Cesàro-recurrent, it follows that there exists $n_k \in \mathbb{N}$ such that $M_{n_k}(T)B\left(x, \frac{1}{k}\right) \cap B\left(x, \frac{1}{k}\right) \neq \emptyset$. Let x_k be in $B\left(x, \frac{1}{k}\right)$ such that $M_{n_k}(T)x_k$ is also in $B\left(x, \frac{1}{k}\right)$. This leads us to deduce that

$$\|x_k - x\| < \frac{1}{k} \qquad \text{and} \qquad \|M_{n_k}(T)x_k - x\| < \frac{1}{k}.$$

This easily implies that

$$x_k \longrightarrow x \qquad \text{and} \qquad M_{n_k}(T)x_k \longrightarrow x.$$

Thus, the desired result is proven.
$(2) \Rightarrow (3)$: Let $x \in X$. Consider (x_k) in X and (n_k) in $\mathbb{N}$ such that

$$x_k \longrightarrow x \qquad \text{and} \qquad M_{n_k}(T)x_k \longrightarrow x.$$

Let W be a neighborhood of 0. Then there is some N in $\mathbb{N}$ such that both $x - x_N$ and $M_{n_N}x_N - x$ are in W, since both $(x_k - x)$ and $(M_{n_k}(T)x_k - x)$ converge to 0.
$(3) \Rightarrow (1)$: Let U be an arbitrary nonempty open subset of X and consider x any vector in U. Since $B\left(x, \frac{1}{k}\right)$, the ball centered at x with radius $\frac{1}{k}$, is a neighborhood of x for all k, it follows that there exist $z_k \in B\left(x, \frac{1}{k}\right)$ and $n_k \in \mathbb{N}$ such that $M_{n_k}(T)z_k \in B\left(x, \frac{1}{k}\right)$. Thus, both (z_k) and $M_{n_k}(T)z_k$ converge to x. Finally, by utilizing the fact that U is open and x is in U, we obtain the desired result. □

Proposition 2.18. *Let T be an operator acting on a Banach space X. If T is Cesàro-recurrent then $\sigma_p(T^*) \subset \mathbb{T}$.*

Proof. Suppose that there is $\lambda \in \mathbb{C}\backslash\mathbb{T}$ such that $\overline{(T-\lambda I)(X)} \neq X$, then $X\backslash\overline{(T-\lambda I)(X)}$ is non empty open subset of X. Since T is Cesàro-recurrent, there is a Cesàro-recurrent vector x in $X\backslash\overline{(T-\lambda I)(X)}$. By Hahn-Banach theorem there exists $x^* \in X^*$ such that $x^*(x) \neq 0$ and $x^*(\overline{(T-\lambda I)(X)}) = \{0\}$. Then for every $y \in X$ we have $x^*(Ty) = \lambda x^*(y)$ and thus $x^*(M_n(T)y) = M_n(\lambda)x^*(y)$ for every $n = 1, 2, ...$ Since x is Cesàro-recurrent there exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ tending to infinity such that $M_{k_n}(T)x \to x$. Hence $M_{k_n}(\lambda)x^*(x) = x^*(M_{k_n}(T)x) \to x^*(x)$. But $x^*(x) \neq 0$ thus $M_{k_n}(\lambda) \to 1$, which is a contradiction with the fact that $\lambda \notin \mathbb{T}$. This completes the proof. □

One might ask what occurs if we replace the operator T with a multiple of T by a scalar on the unit circle. The hypercyclicity case was studied by León-Saavedra and Müller in [25], while the recurrence case was investigated by Costakis, Manoussos, and Parissis in [11]. Unfortunately, this result cannot be extended to the Cesàro-recurrence case.

Example 2.19. *It is clear that the idientity operator I on any Banach space X is Cesàro-recurrent. However, however λI cannot be Cesàro-recurrent for any λ in the unit cicrle, this since*

$$\frac{1 + \lambda + \cdots + \lambda^{n-1}}{n}$$

is even goes to infinity if $|\lambda| > 1$ or stays within the closed unit disk for any positive integer n when $|\lambda| \leq 1$.

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