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Categorical and convergent properties of convex spaces

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Abstract. A convex structure (dually, a concave structure) and a topological structure have many common characters. This paper aims to apply the topological methods to the theory of convex structures. From a categorical aspect, this paper first deals with the extensionality and productivity of quotient maps in the category of convex spaces. It is shown that the category of convex spaces is not extensional, but productive for finite quotient maps. Then the paper introduced the convergence approach via co-Scott closed sets on powerset and proposed the concept of (preconcave, concave) convergence structures in concave spaces. It is proved that the category of concave convergence spaces is isomorphic to that of concave spaces and the latter can be embedded in the category of convergence spaces as a full and reflective subcategory. Finally, it is shown that the category of convergence spaces is extensional and productive for finite quotient maps.

1. Introduction

A convex structure (also called an algebraic closure system) via abstracting three basic properties of convex sets is an important mathematical structure. Explicitly, a convex structure on a set X is a subset C of the powerset of X satisfying: $\emptyset, X \in C$; C is closed for any intersections; C is closed for any directed unions. As a topology-like structure, convex structures are closely related to many other mathematical structures [22]. Adopting the lattice-valued approach in topological structures, convex structures are also studied in a lattice-valued viewpoint, which leads to several types of lattice-valued convex structures [11, 17, 19, 20]. To date, lattice-valued convex structures have been extensively studied in a topological approach, such as closure operators [14, 18, 31], interval operators [12, 13, 21, 23], categorical relationship [9, 24] and so on. This demonstrates the feasibility of applying the studying methods in the theory of topological structures to that of convex structures.

From a categorical aspect, extensionality and productivity of quotient maps are important categorical properties of topological categories [15]. But the category of topological spaces satisfies neither the extensionality nor the productivity of quotient maps. This motivates us to consider if the category of convex spaces satisfies these two kinds of categorical properties. Besides, convergence structures via filters [4, 16], or lattice-valued convergence structures via lattice-valued filters [2, 5, 10, 25, 27–30] serve as an important

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tool of characterizing topological structures and possess better categorical properties than topological structures. This motivates us to introduce the concept of convergence structures in the framework of convex spaces and study its relationship with convex structures as well as its categorical properties.

The aim of this paper is to apply the topological methods to the theory of convex structures. Concretely, we will discuss the extensionality and productivity of quotient maps in the category of convex spaces from a categorical aspect. Then we will propose convergence structures via filter analogues in a concave space and study its categorical relationship with concave spaces as well as its extensionality and productivity of quotient maps in a categorical sense.

The content is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we study the extensionality and productivity of quotient maps in the category of convex spaces. In Section 4, we focus on co-Scott closed sets on powerset. In Section 5, we propose the concept of convergence structures via co-Scott closed sets and establish its categorical relationship with concave spaces. In Section 6, we explore the extensionality and productivity of quotient maps in the category of convergence spaces.

2. Preliminaries

Throughout this paper, let X be a nonempty set and $\mathcal{P}(X)$ be the powerset of X. We say that $\{A_j\}_{j\in J}$ is a directed (co-directed) subset of $\mathcal{P}(X)$, in symbols $\{A_j\}_{j\in J} \subseteq^{dir} \mathcal{P}(X)$ ($\{A_j\}_{j\in J} \subseteq^{cdir} \mathcal{P}(X)$), if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_1}, A_{j_2} \subseteq A_{j_3}$ ($A_{j_3} \subseteq A_{j_1}, A_{j_2}$). Let $f : X \longrightarrow Y$ be a map. Define $f^{\rightarrow} : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ by $f^{\rightarrow}(A) = \{f(x) \mid x \in A\}$ for each $A \in \mathcal{P}(X)$ and $f^{\leftarrow} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ by $f^{\leftarrow}(B) = \{x \mid f(x) \in B\}$ for each $B \in \mathcal{P}(Y)$.

In [22], Van De Vel introduced the concept of convex spaces.

Definition 2.1. ([22]) A subset $\mathcal{C}^X \subseteq \mathcal{P}(X)$ is called a convex structure on X if it satisfies

(CE1) $\emptyset, X \in \mathcal{C}^X$;

(CE2) $\forall \{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}^X$, $\bigcap_{\lambda\in\Lambda} A_\lambda \in \mathcal{C}^X$;

(CE3) $\forall \{A_j\}_{j\in J} \subseteq^{dir} \mathcal{C}^X$, $\bigcup_{j\in J} A_j \in \mathcal{C}^X$.

For a convex structure $\mathcal{C}^X$ on X, the pair $(X, \mathcal{C}^X)$ is called a convex space.

A map $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{D}^Y)$ between two convex spaces is called convexity-preserving if $f^{\leftarrow}(D) \in \mathcal{C}^X$ for each $D \in \mathcal{D}^Y$.

It is easy to check that convex spaces and their convexity-preserving maps form a category, denoted by **Convex**.

Definition 2.2. ([22]) Let $(X, \mathcal{C}^X)$ be a convex space and $\mathbb{B} \subseteq \mathcal{C}^X$. If $\mathbb{B}$ satisfies

$$\forall\, C \in \mathcal{C},\ \exists\, \mathbb{B}_C \subseteq^{dir} \mathbb{B},\ s.t.\ C = \bigcup \mathbb{B}_C,$$

then $\mathbb{B}$ is called a base of $(X, \mathcal{C}^X)$.

Definition 2.3. ([22]) Let $(X, \mathcal{C}^X)$ be a convex space and $\mathbb{A} \subseteq \mathcal{C}^X$. If

$$\mathbb{B}_{\mathbb{A}} = \left\{ \bigcap_{i\in I} A_i \mid \{A_i \mid i \in I\} \subseteq \mathbb{A},\ I \neq \emptyset \right\}$$

is a base of $(X, \mathcal{C}^X)$, then $\mathbb{A}$ is called a subbase of $(X, \mathcal{C}^X)$.

Definition 2.4. ([1]) A concrete category $\mathbb{C}$ is called a topological category over **Set** with respect to the usual forgetful functor from $\mathbb{C}$ to **Set** if it satisfies the following conditions:

(TC1) Existence of initial structures: For any set X, any class Λ, any family $\{(X_\lambda, \xi_\lambda)\}_{\lambda\in\Lambda}$ of $\mathbb{C}$-object and any family $\{f_\lambda : X \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ of maps, there exists a unique $\mathbb{C}$-structure ξ on X which is initial with respect to the source $\{f_\lambda : X \longrightarrow (X_\lambda, \xi_\lambda)\}_{\lambda\in\Lambda}$, this means that for a $\mathbb{C}$-object (Y, η), a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a $\mathbb{C}$-morphism if and only if for all $\lambda \in \Lambda$, $f_\lambda \circ g : (Y, \eta) \longrightarrow (X_\lambda, \xi_\lambda)$ is a $\mathbb{C}$-morphism.

(TC2) Fibre-smallness: For any set X, the $\mathbb{C}$-fibre of X, i.e., the class of all $\mathbb{C}$-structures on X is a set.

Proposition 2.5. *The category* **Convex** *is topological over* **Set**.

Proof. We only note that for a set X, the initial structure $\mathcal{C}^X$ on X with respect to a class $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$ of convex spaces and a family $\{f_\lambda : X \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ of maps, is generated by the subbase

$$\Big\{\bigcup_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(A_\lambda) \mid \forall\, \lambda \in \Lambda,\ A_\lambda \in \mathcal{C}^{X_\lambda}\Big\}.$$

□

Since **Convex** is topological over **Set**, there are the product spaces and the subspaces of convex spaces in **Convex**. Next, we recall the concepts of product spaces and subspaces of convex spaces.

Definition 2.6. ([22]) Let $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$ be a family of convex spaces, $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ be a family of projection maps. The convex structure $\prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}$ on $\prod_{\lambda\in\Lambda} X_\lambda$ generated by the subbase $\bigcup_{\lambda\in\Lambda} p_\lambda^{\leftarrow}(\mathcal{C}^{X_\lambda})$, is called the product structure, the pair $(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda})$ is called the product space of $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$.

Proposition 2.7. ([22]) *Suppose that Λ is a finite index set. Let $\{(X_\lambda, \mathcal{C}^{X_\lambda}) \mid \lambda \in \Lambda\}$ be a family of convex spaces. Then*

$$\prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda} = \Big\{\prod_{\lambda\in\Lambda} C_\lambda \mid \forall\, \lambda \in \Lambda,\ C_\lambda \in \mathcal{C}^{X_\lambda}\Big\}.$$

Definition 2.8. ([22]) Let $(X, \mathcal{C}^X)$ be a convex space and $Y \subseteq X$. Define $\mathcal{C}^X|_Y \subseteq \mathcal{P}(X)$ by

$$\mathcal{C}^X|_Y = \{A \cap Y \mid A \in \mathcal{C}^X\}.$$

Then $(Y, \mathcal{C}^X|_Y)$ is a convex space, which is called a subspace of $(X, \mathcal{C}^X)$.

By Proposition 2.5, final structures also exist in **Convex**. Let X be a nonempty set, $\{(X_\lambda, \mathcal{C}^{X^\lambda})\}_{\lambda\in\Lambda}$ be a family of convex spaces and $\{f_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ be a family of maps. Then $\mathcal{C}^X \subseteq \mathcal{P}(X)$ defined by

$$B \in \mathcal{C}^X \Longleftrightarrow \forall\lambda \in \Lambda, f_\lambda^{\leftarrow}(B) \in \mathcal{C}^{X^\lambda},$$

is the final structure with respect to the sink $\{f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow X\}_{\lambda\in\Lambda}$. In particular, a quotient space of a convex space can be defined.

Definition 2.9. ([22]) Let $(X, \mathcal{C}^X)$ be a convex space and $f : X \longrightarrow Y$ be a surjective map. Define $\mathcal{C}^Y \subseteq \mathcal{P}(Y)$ by

$$B \in \mathcal{C}^Y \Longleftrightarrow f^{\leftarrow}(B) \in \mathcal{C}^X.$$

Then $(Y, \mathcal{C}^Y)$ is called a quotient space of $(X, \mathcal{C}^X)$ and f is called a quotient map.

Concavity is dual to convexity. In a natural way, the concept of concave spaces can be defined as follows.

Definition 2.10. A subset $\mathcal{C}^X \subseteq \mathcal{P}(X)$ is called a concave structure on X if it satisfies

(CA1) $\emptyset, X \in \mathcal{C}^X$;

(CA2) $\forall\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}^X$, $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathcal{C}^X$;

(CA3) $\forall \{A_j\}_{j\in J} \subseteq^{cdir} \mathcal{C}^X$, $\bigcap_{j\in J} A_j \in \mathcal{C}^X$.

For a concave structure $\mathcal{C}^X$ on X, the pair $(X, \mathcal{C}^X)$ is called a concave space.

A map $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{D}^Y)$ between two concave spaces is called concavity-preserving if $f^{\leftarrow}(D) \in \mathcal{C}^X$ for each $D \in \mathcal{D}^Y$.

It is easy to check that concave spaces and their concavity-preserving maps form a category, denoted by **Concave**.

In a convex space, there is the concept of the closure of a subset A of X. Dually, the concept of the interior of a subset A of X can be defined in a concave space.

Definition 2.11. Let $(X, \mathcal{C}^X)$ be a concave space. Define $\text{int}(A) \in \mathcal{P}(X)$ by

$$\text{int}(A) = \bigcup_{B\in\mathcal{C}^X,\ B\subseteq A} B$$

for each $A \in \mathcal{P}(X)$. Then $\text{int}(A)$ is called the interior of A.

Convex and **Concave** are isomorphic in a categorical sense, so in the following we will not distinguish them when it comes to categorical properties.

3. Categorical properties of convex spaces

In this section, we will discuss the categorical properties of **Convex**, including extensionality and productivity of quotients maps. We first recall the concept of partial morphisms in a topological category.

In a topological category $\mathbb{C}$, a partial morphism from X to Y is a $\mathbb{C}$-morphism $f : Z \longrightarrow Y$ whose domain is a subobject of X.

Definition 3.1. ([15]) A topological category $\mathbb{C}$ is called extensional if every $\mathbb{C}$-object X has a one-point extension $\overline{X}$, in the sense that every $\mathbb{C}$-object X can be embedded via the addition of a single point ∞ into a $\mathbb{C}$-object $\overline{X}$ such that for every partial morphism $f : Z \longrightarrow X$ from Y to X , the map $\overline{f} : Y \longrightarrow \overline{X}$ defined by

$$\overline{f}(x) = \begin{cases} f(x), & if\ x \in Z, \\ \infty, & if\ x \notin Z \end{cases}$$

is a $\mathbb{C}$-morphism.

It is well known that if a category is extensional, then quotient maps in this category are hereditary. Now, we will show quotient maps in **Convex** are not necessarily hereditary via the following example.

Example 3.2. Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $\mathcal{C}^X = \{\emptyset, \{a, c\}, \{b, d\}, X\}$ and $\mathcal{C}^Y = \{\emptyset, Y\}$. Then $(X, \mathcal{C}^X)$ and $(Y, \mathcal{C}^Y)$ are convex spaces. Define $f : X \longrightarrow Y$ by

$$f(x) = \begin{cases} a, & if\ x = a, \\ b, & if\ x = b, \\ c, & if\ x = c, d. \end{cases}$$

Then f is a surjective map and $D \in \mathcal{C}^Y$ if and only if $f^{\leftarrow}(D) \in \mathcal{C}^X$ for each $D \in \mathcal{P}(Y)$. So f is a quotient map.

Let $A = B = \{a, b\}$ and let $(A, \mathcal{C}^X|_A)$ and $(B, \mathcal{C}^Y|_B)$ be the subspaces of $(X, \mathcal{C}^X)$ and $(Y, \mathcal{C}^Y)$, respectively. Then $\mathcal{C}^X|_A = \{\emptyset, \{a\}, \{b\}, A\}$ and $\mathcal{C}^Y|_B = \{\emptyset, B\}$. The restriction of f on A, denoted by $f|_A : A \longrightarrow B$, is defined by

$$f|_A(x) = \begin{cases} a, & if\ x = a, \\ b, & if\ x = b. \end{cases}$$

Take any $\{a\} \in \mathcal{P}(B)$. Then it is easy to check that $f^{\leftarrow}(\{a\}) = \{a\} \in \mathcal{C}^X|_A$ and $\{a\} \notin \mathcal{C}^Y|_B$. This shows that $f|_A : (A, \mathcal{C}^X|_A) \longrightarrow (B, \mathcal{C}^Y|_B)$ is not a quotient map.

By Example 3.2, we can obtain the following proposition.

Proposition 3.3. *In* **Convex** *quotient maps are not hereditary.*

Since quotient maps in an extensional category must be hereditary, we have

Theorem 3.4. *The category* **Convex** *is not extensional.*

In the following, we will go on exploring the productivity of quotient maps in **Convex**. The following theorem illustrates that **Convex** is closed under the formation of finite products of quotient maps.

Theorem 3.5. *Suppose that Λ is a finite index set. Let $\{(X_\lambda, \mathcal{C}^{X_\lambda}) \mid \lambda \in \Lambda\}$ be a family of convex spaces. If $\{f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in* **Convex***, then the product map*

$$\prod_{\lambda\in\Lambda} f_\lambda : \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}\Big) \longrightarrow \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{Y_\lambda}\Big)$$

is a quotient map in **Convex***.*

Proof. Define

$$f := \prod_{\lambda\in\Lambda} f_\lambda,\ (X, \mathcal{C}^X) := \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}\Big),\ (Y, \mathcal{C}^Y) := \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{Y_\lambda}\Big).$$

Let

$$\begin{array}{ccc} (X, \mathcal{C}^X) & \xrightarrow{\quad f \quad} & (Y, \mathcal{C}^Y) \\ {\scriptstyle p_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle q_\lambda} \\ (X_\lambda, \mathcal{C}^{X_\lambda}) & \xrightarrow[\quad f_\lambda \quad]{} & (Y_\lambda, \mathcal{C}^{Y_\lambda}) \end{array}$$

be the product communication diagram with respect to sets. Since $\{f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in **Convex**, for each $B_\lambda \in \mathcal{P}(Y_\lambda)$, we have

$$B_\lambda \in \mathcal{C}^{Y_\lambda} \Longleftrightarrow f_\lambda^{\leftarrow}(B_\lambda) \in \mathcal{C}^{X_\lambda}.$$

Let $\mathcal{C}_*^Y$ be the quotient structure of $(X, \mathcal{C}^X)$ with respect to f. Then

$$\mathcal{C}_*^Y = \{B \in \mathcal{P}(Y) \mid f^{\leftarrow}(B) \in \mathcal{C}^X\}.$$

It suffices to verify that $\mathcal{C}^Y = \mathcal{C}_*^Y$.

On the one hand, take any $B \in \mathcal{P}(Y)$. Then

$$\begin{aligned} B \in \mathcal{C}^Y \quad &\Longleftrightarrow \quad \forall \lambda \in \Lambda, \exists\, B_\lambda \in \mathcal{C}^{Y_\lambda}, s.t.\ B = \prod_{\lambda\in\Lambda} B_\lambda \\ &\Longleftrightarrow \quad \forall \lambda \in \Lambda, \exists\, B_\lambda \in \mathcal{C}^{Y_\lambda}, s.t.\ f^{\leftarrow}(B) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(\prod_{\lambda\in\Lambda} B_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(B_\lambda). \end{aligned}$$

It follows that

$$f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(B_\lambda) \in \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda} = \mathcal{C}^X.$$

This shows that $\mathcal{C}^Y \subseteq \mathcal{C}^Y_*$.

On the other hand, take any $B \in \mathcal{P}(Y)$. Then

$$\begin{aligned} B \in \mathcal{C}^Y_* &\iff f^{\leftarrow}(B) \in \mathcal{C}^X \\ &\iff \forall \lambda \in \Lambda, \exists A_\lambda \in \mathcal{C}^{X_\lambda}, s.t.\ f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} A_\lambda \\ &\iff \forall \lambda \in \Lambda, \exists A_\lambda \in \mathcal{C}^{X_\lambda}, s.t.\ B = f^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} A_\lambda\Big) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} A_\lambda\Big) = \prod_{\lambda\in\Lambda}\Big(f_\lambda^{\rightarrow}(A_\lambda)\Big) \\ &\iff \forall \lambda \in \Lambda, \exists A_\lambda \in \mathcal{C}^{X_\lambda}, s.t.\ f^{\leftarrow}(B) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(\prod_{\lambda\in\Lambda} f_\lambda^{\rightarrow}(A_\lambda)\Big) = \prod_{\lambda\in\Lambda}\Big(f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda))\Big). \end{aligned}$$

This implies that

$$f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} A_\lambda = \prod_{\lambda\in\Lambda}\Big(f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda))\Big).$$

Then it follows that $f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda)) = A_\lambda \in \mathcal{C}^{X_\lambda}$ for each $\in \Lambda$. Since $f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})$ is a quotient map, we have $f_\lambda^{\rightarrow}(A_\lambda) \in \mathcal{C}^{Y_\lambda}$. This implies that $B = \prod_{\lambda\in\Lambda}\Big(f_\lambda^{\rightarrow}(A_\lambda)\Big) \in \mathcal{C}^Y$. By the arbitrariness of B, we have $\mathcal{C}^Y_* \subseteq \mathcal{C}^Y$. □

Extensionality is an important categorical property. Regretly, **Convex** is not extensional. This motivates us to find an extensional structure that is closely related to convex or concave structures. Inspired by filter-based convergence structures in topological spaces, we will consider convergence structures in convex spaces or concave spaces. To this end, we need to determine the filter analogues as the tools to define a convergence structure in a convex or concave space, which is exactly the co-Scott closed sets in the following section.

4. Co-Scott closed sets on $\mathcal{P}(X)$

In this section, we will focus on co-Scott closed sets on $\mathcal{P}(X)$.

Definition 4.1. A subset $\mathbb{F} \subseteq \mathcal{P}(X)$ is called co-Scott closed on $\mathcal{P}(X)$ if it satisfies

(CSC1) $A \in \mathbb{F}$ and $A \subseteq B$ imply $B \in \mathbb{F}$;

(CSC2) $\forall \{A_j\}_{j\in J} \subseteq^{cdir} \mathbb{F}$, $\bigcap_{j\in J} A_j \in \mathbb{F}$.

The set of all co-Scott closed sets on $\mathcal{P}(X)$ is denoted by $\mathcal{C}_S(X)$ and for a co-Scott closed set $\mathbb{F}$ on $\mathcal{P}(X)$, the pair $(X, \mathbb{F})$ is called a co-Scott closed set space. An order on $\mathcal{C}_S(X)$ can be defined by $\mathbb{F} \leq \mathbb{G}$ if and only if $\mathbb{F} \subseteq \mathbb{G}$, then $(\mathcal{C}_S(X), \leq)$ is a poset. The infimum of a family of co-Scott closed sets $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ is given by $\bigwedge_{\lambda\in\Lambda}\mathbb{F}_\lambda = \bigcap_{\lambda\in\Lambda}\mathbb{F}_\lambda$. Since $\mathcal{P}(X)$ is the maximal element in $\mathcal{C}_S(X)$, it follows that $\mathcal{C}_S(X)$ is a complete lattice.

Remark 4.2. Considering a directed complete poset as the background, Scott closed sets are extensively discussed [6]. It is well known that $\mathcal{P}(X)$ is a directed complete poset. With the inclusion order between subsets, a co-Scott closed subset in Definition 4.1 is exactly the duality of a Scott closed subset in $\mathcal{P}(X)$.

Example 4.3. Let X be a nonempty set. Then

(1) $\mathbb{F} = \{X\}$ is a co-Scott closed set on $\mathcal{P}(X)$;

(2) If $\emptyset \neq A \subseteq X$, then $\mathbb{F} = \{F \subseteq X \mid A \subseteq F\}$ is a co-Scott closed set on $\mathcal{P}(X)$, which denoted by (A). In particular, if A is a singleton set, i.e., $A = \{x\}$, then we will use $\dot{x}$ to denote $(\{x\})$.

Proposition 4.4. *Let $\mathbb{F}_{\lambda_1}, \mathbb{F}_{\lambda_2}$ be two co-Scott closed sets on $\mathcal{P}(X)$. Then $\mathbb{F}_{\lambda_1} \cup \mathbb{F}_{\lambda_2}$ is a co-Scott closed set in $\mathcal{C}_S(X)$.*

Proof. It suffices to verify that $\mathbb{F}_{\lambda_1} \cup \mathbb{F}_{\lambda_2}$ satisfies (CSC1) and (CSC2). (CSC1) is straightforward.

For (CSC2), take each $\{A_j\}_{j\in J} \subseteq^{cdir} \mathbb{F}_{\lambda_1} \cup \mathbb{F}_{\lambda_2}$. Let $\mathcal{A}_{J_1} = \{A_k \in \{A_j\}_{j\in J} \mid A_k \in \mathbb{F}_{\lambda_1}\}$ and $\mathcal{A}_{J_2} = \{A_l \in \{A_j\}_{j\in J} \mid A_l \in \mathbb{F}_{\lambda_2}\}$. Then $\{A_j\}_{j\in J} = \mathcal{A}_{J_1} \cup \mathcal{A}_{J_2}$. If $\mathcal{A}_{J_1} \nsubseteq \mathcal{A}_{J_2}$ and $\mathcal{A}_{J_2} \nsubseteq \mathcal{A}_{J_1}$. Then there exist $A_k \in \mathcal{A}_{J_1} \subseteq \mathbb{F}_{\lambda_1}$ such that $A_k \notin \mathcal{A}_{J_2}$, and $A_l \in \mathcal{A}_{J_2} \subseteq \mathbb{F}_{\lambda_2}$ such that $A_l \notin \mathcal{A}_{J_1}$. By the co-directedness of $\{A_j\}_{j\in J}$, there exists $A_j \in \{A_j\}_{j\in J}$ such that $A_j \subseteq A_k$ and $A_j \subseteq A_l$. If $A_j \in \mathcal{A}_{J_1}$, then $A_l \in \mathbb{F}_{\lambda_1}$. This implies $A_l \in \mathcal{A}_{J_1}$, which is a contradiction. Similarly, if $A_j \in \mathcal{A}_{J_2}$, then $A_k \in \mathbb{F}_{\lambda_2}$. This implies $A_k \in \mathcal{A}_{J_2}$, which is a contradiction. So we have $\mathcal{A}_{J_1} \subseteq \mathcal{A}_{J_2}$ or $\mathcal{A}_{J_2} \subseteq \mathcal{A}_{J_1}$. Then $\{A_j\}_{j\in J} = \mathcal{A}_{J_2} \subseteq^{cdir} \mathbb{F}_{\lambda_2}$ or $\{A_j\}_{j\in J} = \mathcal{A}_{J_1} \subseteq^{cdir} \mathbb{F}_{\lambda_1}$. This implies that $\bigcap_{j\in J} A_j \in \mathbb{F}_{\lambda_2}$ or $\bigcap_{j\in J} A_j \in \mathbb{F}_{\lambda_1}$. □

For a family of co-Scott closed sets $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{P}(X)$, $\bigcup_{\lambda\in\Lambda} \mathbb{F}_\lambda$ in not the supremum of $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ in $C_S(X)$. The supremum of a family of co-Scott closed sets $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ is given by

$$\bigvee_{\lambda\in\Lambda} \mathbb{F}_\lambda = \bigcap \left\{ \mathbb{F} \in C_S(X) \mid \bigcup_{\lambda\in\Lambda} \mathbb{F}_\lambda \subseteq \mathbb{F} \right\}.$$

Proposition 4.5. *Let $f : X \longrightarrow Y$ be a map and $\mathbb{F} \in C_S(X)$. Define $f^{\Rightarrow}(\mathbb{F}) = \{G \mid \exists\, F \in \mathbb{F},\ s.t.\ f^{\rightarrow}(F) \subseteq G\}$. Then $f^{\Rightarrow}(\mathbb{F}) \in C_S(Y)$.*

Proof. It suffices to verify that $f^{\Rightarrow}(\mathbb{F})$ satisfies (CSC1) and (CSC2). (CSC1) is straightforward.

For (CSC2), let $\{A_j\}_{j\in J} \subseteq^{cdir} f^{\Rightarrow}(\mathbb{F})$. Then for each $j \in J$, there exists $F_j \in \mathbb{F}$ such that $f^{\rightarrow}(F_j) \subseteq A_j$, or equivalently, for each $j \in J$, there exists $F_j \in \mathbb{F}$ such that $F_j \subseteq f^{\leftarrow}(A_j)$. Since $\{A_j\}_{j\in J} \subseteq^{cdir} f^{\Rightarrow}(\mathbb{F})$, we have $\{f^{\leftarrow}(A_j)\}_{j\in J} \subseteq^{cdir} \mathbb{F}$. Then it follows that $f^{\leftarrow}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} f^{\leftarrow}(A_j) \in \mathbb{F}$. By $f^{\rightarrow}(f^{\leftarrow}(\bigcap_{j\in J} A_j)) \subseteq \bigcap_{j\in J} A_j$, we have $\bigcap_{j\in J} A_j \in f^{\Rightarrow}(\mathbb{F})$. □

By Proposition 4.5, we know $B \in f^{\Rightarrow}(\mathbb{F})$ if and only if $f^{\leftarrow}(B) \in \mathbb{F}$. The co-Scott closed set $f^{\Rightarrow}(\mathbb{F})$ is called the image of $\mathbb{F}$ under f.

Proposition 4.6. *Let $f : X \longrightarrow Y$ be a map and $\mathbb{G} \in C_S(Y)$. Define $f^{\Leftarrow}(\mathbb{G}) = \{F \mid \exists\, G \in \mathbb{G},\ s.t.\ f^{\leftarrow}(G) \subseteq F\}$. Then $f^{\Leftarrow}(\mathbb{G}) \in C_S(X)$.*

Proof. It suffices to verify that $f^{\Leftarrow}(\mathbb{G})$ satisfies (CSC1) and (CSC2). (CSC1) is straightforward.

For (CSC2), let $\{A_j\}_{j\in J} \subseteq^{cdir} f^{\Leftarrow}(\mathbb{G})$. Then for each $j \in J$, there exists $G_j \in \mathbb{G}$ such that $f^{\leftarrow}(G_j) \subseteq A_j$. Let $B_j = \bigcup\{G_j \mid f^{\leftarrow}(G_j) \subseteq A_j\}$. Then $\{B_j\}_{j\in J} \subseteq^{cdir} \mathbb{G}$. This implies that $\bigcap_{j\in J} B_j \in \mathbb{G}$. Since $f^{\leftarrow}(B_j) = \bigcup\{f^{\leftarrow}(G_j) \mid f^{\leftarrow}(G_j) \subseteq A_j\} \subseteq A_j$, we have $f^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{\leftarrow}(B_j) \subseteq \bigcap_{j\in J} A_j$. This shows that $\bigcap_{j\in J} A_j \in f^{\Leftarrow}(\mathbb{G})$. □

The co-Scott closed set $f^{\Leftarrow}(\mathbb{G})$ is called the inverse image of $\mathbb{G}$ under f.

Proposition 4.7. *Let $f : X \longrightarrow Y$ be a map, $\mathbb{F} \in C_S(X), \mathbb{G} \in C_S(Y)$. Then*

(1) $f^{\Leftarrow}(f^{\Rightarrow}(\mathbb{F})) \subseteq \mathbb{F}$. *If f is injective, then* $f^{\Leftarrow}(f^{\Rightarrow}(\mathbb{F})) = \mathbb{F}$;

(2) $\mathbb{G} \subseteq f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$. *If f is surjective, then* $\mathbb{G} = f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$.

Proof. (1) Take any $F \in \mathcal{P}(X)$. Then

$$\begin{aligned} F \in f^{\Leftarrow}(f^{\Rightarrow}(\mathbb{F})) &\iff \exists\, G \in f^{\Rightarrow}(\mathbb{F})\ s.t.\ f^{\leftarrow}(G) \subseteq F \\ &\iff f^{\leftarrow}(G) \in \mathbb{F}, f^{\leftarrow}(G) \subseteq F \\ &\Longrightarrow F \in \mathbb{F}. \end{aligned}$$

This shows that $f^{\Leftarrow}(f^{\Rightarrow}(\mathbb{F})) \subseteq \mathbb{F}$. If f is injective, then $F = f^{\leftarrow}(f^{\rightarrow}(F)) \in \mathbb{F}$. This implies that $\mathbb{F} \subseteq f^{\Leftarrow}(f^{\Rightarrow}(\mathbb{F}))$.

(2) Take any $G \in \mathcal{P}(X)$. Then

$$\begin{aligned} G \in \mathbb{G} &\iff \exists\, G_1 \in \mathbb{G},\ s.t.\ G_1 \subseteq G \\ &\Longrightarrow \exists\, G_1 \in \mathbb{G},\ s.t.\ f^{\leftarrow}(G_1) \subseteq f^{\leftarrow}(G) \\ &\iff f^{\Leftarrow}(G) \in f^{\Leftarrow}(\mathbb{G}) \\ &\iff G \in f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G})). \end{aligned}$$

This shows that $\mathbb{G} \subseteq f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$. If f is surjective, then $G_1 = f^{\rightarrow}(f^{\leftarrow}(G_1)) \in \mathbb{G}$. This implies that $f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G})) \subseteq \mathbb{G}$. □

Remark 4.8. By Proposition 4.7, we know $(f^{\Leftarrow}, f^{\Rightarrow}) : \mathcal{C}_S(Y) \longrightarrow \mathcal{C}_S(X)$ is a Galois correspondence between $\mathcal{C}_S(Y)$ and $\mathcal{C}_S(X)$. Moreover, $f^{\Leftarrow}$ is the left adjoint and $f^{\Rightarrow}$ is the right adjoint.

Definition 4.9. A map $f : (X, \mathbb{F}) \longrightarrow (Y, \mathbb{G})$ between co-Scott closed set spaces is called continuous if $f^{\Leftarrow}(\mathbb{G}) \subseteq \mathbb{F}$.

It is easy to check that co-Scott closed set spaces and their continuous maps form a category, denoted by **CSCS**.

For $\mathbb{F} \in \mathcal{C}_S(X)$ and $\mathbb{G} \in \mathcal{C}_S(Y)$, by Propositions 4.4 and 4.6, we can obtain a co-Scott closed set $\mathbb{F} \times \mathbb{G}$ on $\mathcal{P}(X \times Y)$ in the following way:

$$\mathbb{F} \times \mathbb{G} = p_X^{\Leftarrow}(\mathbb{F}) \cup p_Y^{\Leftarrow}(\mathbb{G}),$$

where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are the projection maps.

Definition 4.10. For $\mathbb{F} \in \mathcal{C}_S(X)$ and $\mathbb{G} \in \mathcal{C}_S(Y)$, $\mathbb{F} \times \mathbb{G}$ is called the product of $\mathbb{F}$ and $\mathbb{G}$.

Definition 4.11. For two co-Scott closed sets $\mathbb{F}$ and $\mathbb{G}$ on $\mathcal{P}(X)$, $(X, \mathbb{G})$ is called coarser than $(X, \mathbb{F})$ if $id_X : (X, \mathbb{F}) \longrightarrow (X, \mathbb{G})$ is continuous.

It is easy to verify that $(X \times Y, \mathbb{F} \times \mathbb{G})$ is the coarsest co-Scott closed set space on $\mathcal{P}(X \times Y)$ such that $p_X : (X \times Y, \mathbb{F} \times \mathbb{G}) \longrightarrow (X, \mathbb{F})$ and $p_Y : (X \times Y, \mathbb{F} \times \mathbb{G}) \longrightarrow (Y, \mathbb{G})$ are continuous. The next proposition shows that $(X \times Y, \mathbb{F} \times \mathbb{G})$ is exactly the product object in the category **CSCS**.

Proposition 4.12. *Let $(X, \mathbb{F})$, $(Y, \mathbb{G})$ be two co-Scott closed set spaces. Then the pair $(X \times Y, \mathbb{F} \times \mathbb{G})$ is the product object of $(X, \mathbb{F})$ and $(Y, \mathbb{G})$ in* **CSCS**.

Proof. It suffices to verify that for each co-Scott closed set space $(Z, \mathbb{H})$ and two continuous maps $f : (Z, \mathbb{H}) \longrightarrow (X, \mathbb{F})$ and $g : (Z, \mathbb{H}) \longrightarrow (Y, \mathbb{G})$, there exists a unique continuous map $h : (Z, \mathbb{H}) \longrightarrow (X \times Y, \mathbb{F} \times \mathbb{G})$ such that $p_X \circ h = f$ and $p_Y \circ h = g$. Let $h = f \times g$, where $(f \times g)(z) = (f(z), g(z))$ for each $z \in Z$. By Definition 4.9, we need to show $h^{\Leftarrow}(\mathbb{F} \times \mathbb{G}) \subseteq \mathbb{H}$.

Since $f^{\Leftarrow}(\mathbb{F}) \subseteq \mathbb{H}$ and $g^{\Leftarrow}(\mathbb{G}) \subseteq \mathbb{H}$, we have

$$\begin{aligned} h^{\Leftarrow}(\mathbb{F} \times \mathbb{G}) &= h^{\Leftarrow}(p_X^{\Leftarrow}(\mathbb{F}) \cup p_Y^{\Leftarrow}(\mathbb{G})) \\ &= h^{\Leftarrow}(p_X^{\Leftarrow}(\mathbb{F})) \cup h^{\Leftarrow}(p_Y^{\Leftarrow}(\mathbb{G})) \quad \text{(by Remark 4.8)} \\ &= (p_X \circ h)^{\Leftarrow}(\mathbb{F}) \cup (p_Y \circ h)^{\Leftarrow}(\mathbb{G}) \\ &= f^{\Leftarrow}(\mathbb{F}) \cup g^{\Leftarrow}(\mathbb{G}) \\ &\subseteq \mathbb{H}. \end{aligned}$$

This shows that $h^{\Leftarrow}(\mathbb{F} \times \mathbb{G}) \subseteq \mathbb{H}$. This completes the proof. □

Adopting Definition 4.10, the product of arbitrary finite co-Scott closed sets can be defined.

Definition 4.13. Suppose that Λ is a finite index set. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu \in \Lambda} X_\mu \longrightarrow X_\lambda$ be the projection maps, $\mathbb{F}_\lambda \in \mathcal{C}_S(X_\lambda)$ $(\lambda \in \Lambda)$. Then $\prod_{\lambda \in \Lambda} \mathbb{F}_\lambda = \bigcup_{\lambda \in \Lambda} p_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)$ is a co-Scott closed set on $\mathcal{P}(\prod_{\lambda \in \Lambda} X_\lambda)$, which is called the product of $\{\mathbb{F}_\lambda\}_{\lambda \in \Lambda}$.

Proposition 4.14. *Suppose that Λ is a finite index set. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu \in \Lambda} X_\mu \longrightarrow X_\lambda$ be the projection maps, $\mathbb{F}_\lambda \in \mathcal{C}_S(X_\lambda)$ $(\lambda \in \Lambda)$ and $\mathbb{F} \in \mathcal{C}_S(\prod_{\lambda \in \Lambda} X_\lambda)$. Then the following statements hold:*

(1) $\prod_{\lambda \in \Lambda} p_\lambda^{\Rightarrow}(\mathbb{F}) \subseteq \mathbb{F}$;

(2) $\mathbb{F}_\mu \subseteq p_\mu^{\Rightarrow}(\prod_{\lambda \in \Lambda} \mathbb{F}_\lambda)$;

(3) $p_\mu^{\Rightarrow}(\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathbb{F})) = p_\mu^{\Rightarrow}(\mathbb{F})$.

Proof. (1) Take any $A \in \mathcal{P}(\prod_{\lambda\in\Lambda} X_\lambda)$. Then

$$\begin{aligned} A \in \prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathbb{F}) &\iff A \in \bigcup_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(p_\lambda^{\Rightarrow}(\mathbb{F})) \\ &\iff \exists\, \lambda_0 \in \Lambda,\ s.t.\ A \in p_{\lambda_0}^{\Leftarrow}(p_{\lambda_0}^{\Rightarrow}(\mathbb{F})) = \mathbb{F} \quad \text{(by Proposition 4.7)} \\ &\Longrightarrow A \in \mathbb{F}. \end{aligned}$$

(2) Take any $A \in \mathcal{P}(X_\mu)$. Then

$$\begin{aligned} A \in \mathbb{F}_\mu &\Longrightarrow p_\mu^{\leftarrow}(A) \in p_\mu^{\Leftarrow}(\mathbb{F}_\mu) \\ &\Longrightarrow p_\mu^{\leftarrow}(A) \in \bigcup_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathbb{F}_\lambda) \\ &\iff A \in p_\mu^{\Rightarrow}\Big(\bigcup_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)\Big) \\ &\iff A \in p_\mu^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} \mathbb{F}_\lambda\Big). \end{aligned}$$

(3) It follows immediately from (1) and (2). □

5. Convergence spaces and their relationship with concave spaces

In this section, we will use co-Scott closed sets to define convergence structures and study their relationship with concave structures.

5.1. Convergence spaces

Definition 5.1. A binary relation $\lim^X \subseteq \mathcal{C}_S(X) \times X$ is called a convergence structure on X if it satisfies

(CS1) $(\dot{x}, x) \in \lim^X$;

(CS2) If $(\mathbb{F}, x) \in \lim^X$ and $\mathbb{F} \subseteq \mathbb{G}$, then $(\mathbb{G}, x) \in \lim^X$.

For a convergence structure $\lim^X$ on X, the pair $(X, \lim^X)$ is called a convergence space.

A map $f : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ between two convergence spaces is called continuous if $(f^{\Rightarrow}(\mathbb{F}), f(x)) \in \lim^Y$ for each $(\mathbb{F}, x) \in \lim^X$.

It is easy to check that convergence spaces and their continuous maps form a category, denoted by **CS**.

Proposition 5.2. *The category* **CS** *is topological over* **Set**.

Proof. Firstly, we prove the existence of initial structures. Let $\{(X_\lambda, \lim_\lambda)\}_{\lambda\in\Lambda}$ be a family of convergence spaces and X be a nonempty set. Let further $\{f_\lambda : X \longrightarrow (X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$ be a source. Define $\lim^X \subseteq \mathcal{C}_S(X) \times X$ by

$$(\mathbb{F}, x) \in \lim^X \iff \forall \lambda \in \Lambda, (f_\lambda^{\Rightarrow}(\mathbb{F}), f_\lambda(x)) \in \lim^{X_\lambda}$$

for each $\mathbb{F} \in \mathcal{C}_S(X)$ and $x \in X$. Then $(X, \lim^X)$ is a convergence space. Let $(Y, \lim^Y)$ be a convergence space and $g : Y \longrightarrow X$. If $f_\lambda \circ g$ is continuous for each $\lambda \in \Lambda$, then we have $(f_\lambda^{\Rightarrow} \circ g^{\Rightarrow}(\mathbb{G}), f_\lambda \circ g(y)) \in \lim^{X_\lambda}$ for each $(\mathbb{G}, y) \in \lim^Y$, or equivalently, $(g^{\Rightarrow}(\mathbb{G}), g(y)) \in \lim^X$. So g is continuous.

Secondly, we prove the fibre-smallness. The class of all convergence structures on a fixed set X is a subset of $\mathcal{PP}(X) \times X$, which means that the **CS**-fibre of X is a set. □

Since **CS** is a topological category over **Set**, there are the product spaces and the subspaces of convergence spaces in **CS**. Next, we introduce the concepts of product spaces and subspaces of convergence spaces.

Definition 5.3. Let $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$ be a family of convergence spaces and $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ be the family of the projection maps $\{p_\lambda\}_{\lambda\in\Lambda}$. The initial structure with respect to the source $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow (X_\lambda, \lim^\lambda)\}_{\lambda\in\Lambda}$ is called the product of $\{\lim^{X_\lambda}\}_{\lambda\in\Lambda}$, denoted by $\prod_{\lambda\in\Lambda}\lim^{X_\lambda}$. The pair $(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda}\lim^{X_\lambda})$ is called the product space of $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$. Concretely, for each $\mathbb{F} \in \mathcal{C}_S(\prod_{\lambda\in\Lambda} X_\lambda)$ and $x \in \prod_{\lambda\in\Lambda} X_\lambda$,

$$(\mathbb{F}, x) \in \prod_{\lambda\in\Lambda} \lim^{X_\lambda} \iff \forall \lambda \in \Lambda,\ (p_\lambda^{\Rightarrow}(\mathbb{F}), p_\lambda(x)) \in \lim^{X_\lambda}.$$

Definition 5.4. Let $(X, \lim^X)$ be a convergence space, $Y \subseteq X$ and $i_Y : Y \longrightarrow X$ be the inclusion map. The initial structure with respect to the source $\{i_Y : Y \longrightarrow (X, \lim^X)\}$ is called the sub-convergence structure, denoted by $\lim^X|_Y$. The pair $(Y, \lim^X|_Y)$ is called the subspace of $(X, \lim^X)$. Concretely, for each $\mathbb{G} \in \mathcal{C}_S(Y)$ and $y \in Y$,

$$(\mathbb{G}, y) \in \lim^X|_Y \iff (i_Y^{\Rightarrow}(\mathbb{G}), y) \in \lim^X.$$

By Proposition 5.2, final structures also exist in **CS**. Let X be a nonempty set, $\{(X_\lambda, \lim_\lambda)\}_{\lambda\in\Lambda}$ be a family of convergence spaces and $\{f_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ be a family of maps. Then the binary relation $\lim^X \subseteq \mathcal{C}_S(X)\times X$ defined by

$$(\mathbb{F}, x) \in \lim^X \iff [x] \subseteq \mathbb{F} \textit{ or } \exists\, \lambda \in \Lambda \textit{ and } \mathbb{F}_\lambda \in \mathcal{C}_S(X_\lambda) \textit{ s.t. } f_\lambda(x_\lambda) = x, f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda) \subseteq \mathbb{F} \textit{ and } (\mathbb{F}_\lambda, x_\lambda) \in \lim^{X_\lambda},$$

is the final structure with respect to the sink $\{f_\lambda : (X_\lambda, \lim^{X_\lambda}) \longrightarrow X\}_{\lambda\in\Lambda}$. In particular, a quotient space of a convergence space can be defined.

Definition 5.5. Let $(X, \lim^X)$ be a convergence space and $f : X \longrightarrow Y$ be a surjective map. Define $\lim^Y \subseteq \mathcal{C}_S(Y) \times Y$ by

$$(\mathbb{G}, y) \in \lim^Y \iff \exists\, x \in X \textit{ and } \mathbb{F} \in \mathcal{C}_S(X) \textit{ s.t. } f(x) = y, f^{\Rightarrow}(\mathbb{F}) \subseteq \mathbb{G} \textit{ and } (\mathbb{F}, x) \in \lim^X.$$

Then $(Y, \lim^Y)$ is called a quotient space of $(X, \lim^X)$ and f is called a quotient map.

5.2. Concave convergence spaces

In this subsection, we will propose the concept of concave convergence spaces and establish its relationship with concave spaces.

In a convergence space $(X, \lim^X)$, a special co-Scott closed set that is a counterpart of neighborhood filter in a topological space can be defined in the following way.

Proposition 5.6. *Let $(X, \lim^X)$ be a convergence space and $x \in X$. Define $\mathcal{N}^x_{\lim^X} \subseteq \mathcal{P}(X)$ by*

$$\mathcal{N}^x_{\lim^X} = \bigcap \Big\{\mathbb{F} \in \mathcal{C}_S(X) \mid (\mathbb{F}, x) \in \lim^X\Big\}.$$

Then $\mathcal{N}^x_{\lim^X} \in \mathcal{C}_S(X)$ and $\mathcal{N}^x_{\lim^X} \subseteq \dot{x}$.

Proof. It follows immediately from (CS1) in Definition 5.1. □

Definition 5.7. A convergence space $(X, \lim^X)$ is called preconcave if it satisfies

(P) $(\mathcal{N}^x_{\lim^X}, x) \in \lim^X$ for each $x \in X$.

Definition 5.8. A preconcave convergence space $(X, \lim^X)$ is called concave if it satisfies

(T) For each $U \in \mathcal{N}^x_{\lim^X}$, there exists $V \in \mathcal{N}^x_{\lim^X}$ such that $U \in \mathcal{N}^y_{\lim^X}$ for each $y \in V$.

(**T**) has an equivalent form which can be stated as follows:

(T') For each $U \in \mathcal{N}^x_{\lim^X}$, there exists $V \in \mathcal{P}(X)$ such that $x \in V \subseteq U$ and $V \in \mathcal{N}^y_{\lim^X}$ for each $y \in V$.

The full subcategory of **CS** consisting of concave convergence spaces is denoted by **CaCS**.

Lemma 5.9. *Let $(X, \mathcal{C})$ be a concave space. Define $\mathbb{U}_{\mathcal{C}}(x) \subseteq \mathcal{P}(X)$ by*

$$\mathbb{U}_{\mathcal{C}}(x) = \{A \in \mathcal{P}(X) \mid \exists\, B \in \mathcal{C},\ s.t.\ x \in B \subseteq A\}.$$

Then the following statements hold:

(1) $\mathbb{U}_{\mathcal{C}}(x) \in \mathcal{C}_S(X)$;

(2) $A \in \mathcal{C}$ *if and only if* $A \in \mathbb{U}_{\mathcal{C}}(x)$ *for each* $x \in A$.

Proof. (1) (CSC1) is straightforward. For (CSC2), let $\{A_j\}_{j\in J} \subseteq^{cdir} \mathbb{U}_{\mathcal{C}}(x)$. Then for each $j \in J$, there exists $B_j \in \mathcal{C}$ such that $x \in B_j \subseteq A_j$. Let $B_j = \mathrm{int}(A_j)$. Then $\{B_j\}_{j\in J} \subseteq^{cdir} \mathcal{C}$. So we have $\bigcap_{j\in J} B_j \in \mathcal{C}$ and $x \in \bigcap_{j\in J} B_j \subseteq \bigcap_{j\in J} A_j$. This shows that $\bigcap_{j\in J} A_j \in \mathbb{U}_{\mathcal{C}}(x)$.

(2) The necessity is obvious. It remains to verify the sufficiency. For each $x \in A$, there exists $B_x \in \mathcal{C}$ such that $x \in B_x \subseteq A$. Then it follows that

$$A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x\in A} B_x \subseteq A.$$

This shows that $A = \bigcup_{x\in A} B_x \in \mathcal{C}$. □

Next, we establish the relationship between convergence structures and concave structures.

Proposition 5.10. *Let $(X, \mathcal{C})$ be a concave space. Define $\lim^{\mathcal{C}} \subseteq \mathcal{C}_S(X) \times X$ by*

$$\lim{}^{\mathcal{C}} = \{(\mathbb{F}, x) \mid \mathbb{U}_{\mathcal{C}}(x) \subseteq \mathbb{F}\}.$$

Then $(X, \lim^{\mathcal{C}})$ is a concave convergence space.

Proof. It suffices to verify that $\lim^{\mathcal{C}}$ satisfies (CS1), (CS2), (P) and (T). (CS1), (CS2) and (P) are straightforward.

For (T), since $(X, \mathcal{C})$ is a concave space, we have $\mathbb{U}_{\mathcal{C}}(x) \in \mathcal{C}_S(X)$. Furthermore, we have

$$\mathcal{N}^x_{\lim^{\mathcal{C}}} = \bigcap_{(\mathbb{F},x)\in\lim^{\mathcal{C}}} \mathbb{F} = \bigcap_{\mathbb{U}_{\mathcal{C}}(x)\subseteq\mathbb{F}} \mathbb{F} = \mathbb{U}_{\mathcal{C}}(x).$$

Then for each $A \in \mathcal{N}^x_{\lim^{\mathcal{C}}} = \mathbb{U}_{\mathcal{C}}(x)$, there exists $B \in \mathcal{C}$ such that $x \in B \subseteq A$. By Lemma 5.9, we have $A \in \mathbb{U}_{\mathcal{C}}(y) = \mathcal{N}^y_{\lim^{\mathcal{C}}}$ for each $y \in B$. This shows that for each $A \in \mathcal{N}^x_{\lim^{\mathcal{C}}}$, there exists $B \in \mathcal{P}(X)$ such that $x \in B \subseteq A$ and $A \in \mathcal{N}^y_{\lim^{\mathcal{C}}}$ for each $y \in B$. □

Proposition 5.11. *Let $(X, \lim^X)$ be a concave convergence space. Define $\mathcal{C}^{\lim^X} \subseteq \mathcal{P}(X)$ by*

$$\mathcal{C}^{\lim^X} = \{A \in \mathcal{P}(X) \mid \forall x \in A, A \in \mathcal{N}^x_{\lim^X}\}.$$

Then $(X, \mathcal{C}^{\lim^X})$ is a concave space.

Proof. It is straightforward and is omitted. □

Proposition 5.12.

(1) If $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{C}^Y)$ is concavity-preserving, then $f : (X, \lim^{\mathcal{C}^X}) \longrightarrow (Y, \lim^{\mathcal{C}^Y})$ is continuous.

(2) If $f : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ is continuous, then $f : (X, \mathcal{C}^{\lim^X}) \longrightarrow (Y, \mathcal{C}^{\lim^Y})$ is concavity-preserving.

Proof. (1) Take any $(\mathbb{F}, x) \in \lim^{\mathcal{C}^X}$, i.e., $\mathbb{U}_{\mathcal{C}^X}(x) \subseteq \mathbb{F}$. In order to prove $(f^{\Rightarrow}(\mathbb{F}), f(x)) \in \lim^{\mathcal{C}^Y}$, i.e., $\mathbb{U}_{\mathcal{C}^Y}(f(x)) \subseteq f^{\Rightarrow}(\mathbb{F})$, take any $M \in \mathbb{U}_{\mathcal{C}^Y}(f(x))$. Then there exists $D \in \mathcal{C}^Y$ such that $f(x) \in D \subseteq M$. This implies that $x \in f^{\leftarrow}(D) \subseteq f^{\leftarrow}(M)$. Since $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{C}^Y)$ is concavity-preserving, it follows that $f^{\leftarrow}(D) \in \mathcal{C}^X$. By definition of $\mathbb{U}_{\mathcal{C}^X}(x)$, we obtain $f^{\leftarrow}(M) \in \mathbb{U}_{\mathcal{C}^X}(x) \subseteq \mathbb{F}$. Then it follows that $M \in f^{\Rightarrow}(\mathbb{F})$. By the arbitrariness of M, we obtain $\mathbb{U}_{\mathcal{C}^Y}(f(x)) \subseteq f^{\Rightarrow}(\mathbb{F})$. That is, $(f^{\Rightarrow}(\mathbb{F}), f(x)) \in \lim^{\mathcal{C}^Y}$. This shows that $f : (X, \lim^{\mathcal{C}^X}) \longrightarrow (Y, \lim^{\mathcal{C}^Y})$ is continuous.

(2) Take any $D \in \mathcal{C}^{\lim^Y}$, i.e., $\forall y \in D, D \in \mathcal{N}^y_{\lim^Y} = \bigcap_{(\mathbb{G},y)\in\lim^Y} \mathbb{G}$. In order to prove $f^{\leftarrow}(D) \in \mathcal{C}^{\lim^X}$, take any $x \in f^{\leftarrow}(D)$, i.e., $f(x) \in D$. Then

$$D \in \mathcal{N}^{f(x)}_{\lim^Y} = \bigcap_{(\mathbb{G},f(x))\in\lim^Y} \mathbb{G} \subseteq \bigcap_{(f^{\Rightarrow}(\mathbb{F}),f(x))\in\lim^Y} f^{\Rightarrow}(\mathbb{F}) \subseteq \bigcap_{(\mathbb{F},x)\in\lim^X} f^{\Rightarrow}(\mathbb{F}) = f^{\Rightarrow}\Big(\bigcap_{(\mathbb{F},x)\in\lim^X} \mathbb{F}\Big) = f^{\Rightarrow}(\mathcal{N}^x_{\lim^X}).$$

This implies that $f^{\leftarrow}(D) \in \mathcal{N}^x_{\lim^X}$ for each $x \in f^{\leftarrow}(D)$. That is, $f^{\leftarrow}(D) \in \mathcal{C}^{\lim^X}$. By the arbitrariness of D, we obtain $f : (X, \mathcal{C}^{\lim^X}) \longrightarrow (Y, \mathcal{D}^{\lim^Y})$ is concavity-preserving. □

Proposition 5.13. *Suppose that $(X, \mathcal{C})$ is a concave space and $(X, \lim)$ is a concave convergence space, then $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$ and $\lim^{\mathcal{C}^{\lim}} = \lim$.*

Proof. For $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$, take any $A \in \mathcal{P}(X)$. Then

$$\begin{aligned} A \in \mathcal{C}^{\lim^{\mathcal{C}}} &\iff \forall x \in A,\ A \in \mathcal{N}^x_{\lim^{\mathcal{C}}} \\ &\iff \forall x \in A,\ A \in \bigcap_{(\mathbb{F},x)\in\lim^{\mathcal{C}}} \mathbb{F} = \bigcap_{\mathbb{U}_{\mathcal{C}}(x)\subseteq\mathbb{F}} \mathbb{F} = \mathbb{U}_{\mathcal{C}}(x) \\ &\iff \forall x \in A,\ A \in \mathbb{U}_{\mathcal{C}}(x) \\ &\iff A \in \mathcal{C}.\quad \text{(by Proposition 5.9)} \end{aligned}$$

For $\lim^{\mathcal{C}^{\lim}} = \lim$, take any $A \in \mathcal{P}(X)$. On the one hand,

$$\begin{aligned} A \in \mathbb{U}_{\mathcal{C}^{\lim}}(x) &\iff \exists\, B \in \mathcal{C}^{\lim},\ x \in B \subseteq A \\ &\iff \forall y \in B, B \in \mathcal{N}^y_{\lim}, x \in B \subseteq A \\ &\implies A \in \mathcal{N}^x_{\lim}. \end{aligned}$$

It follows that $\mathbb{U}_{\mathcal{C}^{\lim}}(x) \subseteq \mathcal{N}^x_{\lim}$. On the other hand, since $(X, \lim)$ is concave, for each $U \in \mathcal{N}^x_{\lim}$, there exists $V \in \mathcal{P}(X)$ such that $x \in V \subseteq U$ and $V \in \mathcal{N}^y_{\lim}$ for each $y \in V$. This implies $V \in \mathcal{C}^{\lim}$ and $x \in V \subseteq U$. Then it follows that $U \in \mathbb{U}_{\mathcal{C}^{\lim}}(x)$. This shows that $\mathcal{N}^x_{\lim} \subseteq \mathbb{U}_{\mathcal{C}^{\lim}}(x)$. So we obtain $\mathcal{N}^x_{\lim} = \mathbb{U}_{\mathcal{C}^{\lim}}(x)$. This implies that

$$(\mathbb{F}, x) \in \lim \iff \mathcal{N}^x_{\lim} \subseteq \mathbb{F} \iff \mathbb{U}_{\mathcal{C}^{\lim}}(x) \subseteq \mathbb{F} \iff (\mathbb{F}, x) \in \lim^{\mathcal{C}^{\lim}}.$$

Hence, we obtain $\lim^{\mathcal{C}^{\lim}} = \lim$. □

Now we obtain the main result in this subsection.

Theorem 5.14. *The categories* **Concave** *and* **CaCS** *are isomorphic.*

Proof. It follows from Propositions 5.10–5.13. □

Note that using the inducing methods between concave spaces $(X, \mathcal{C})$ and concave convergence spaces in Propositions 5.10 and 5.11, concave spaces and convergence spaces can also be induced by the other. Also, it is easily observed in Proposition 5.13 that $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$ and $\lim^{\mathcal{C}^{\lim}} \subseteq \lim$ for a concave space $(X, \mathcal{C})$ and a convergence space $(X, \lim)$. Then combining the compatibility with respect to morphisms in Proposition 5.12, we can obtain the following result.

Theorem 5.15. *The category* **Concave** *can be embedded in the category* **CS** *as a full and reflective subcategory.*

6. Categorical properties of convergence spaces

In this section, we will discuss the categorical properties of **CS**, including extensionality and productivity of quotients maps.

Firstly, let us explore the extensionality of the category of convergence spaces.

For convenience, let $(X, \lim^X)$ be a convergence space, $\overline{X} = X \cup \{\infty\}$ with $\infty \notin X$ and $i_X : X \longrightarrow \overline{X}$ denote the inclusion map.

Proposition 6.1. *Let* $(X, \lim^X)$ *be a convergence space. Define* $\lim^{\overline{X}} \subseteq C_S(\overline{X}) \times \overline{X}$ *by*

$$(\mathbb{F}, x) \in \lim^{\overline{X}} \Longleftrightarrow x = \infty \text{ or } (i_X^{\Leftarrow}(\mathbb{F}), x) \in \lim^X.$$

Then $(\overline{X}, \lim^{\overline{X}})$ *is a convergence space.*

Proof. It suffices to verify that $\lim^{\overline{X}}$ satisfies (CS1) and (CS2).

For (CS1), if $x = \infty$, then $(\dot{\infty}, \infty) \in \lim^{\overline{X}}$. If $x \in X$, then $i_X^{\Leftarrow}(\dot{x}) = \dot{x}$ and $(\dot{x}, x) \in \lim^X$. So $(\dot{x}, x) \in \lim^{\overline{X}}$.

For (CS2), let $(\mathbb{F}, x) \in \lim^{\overline{X}}$ and $\mathbb{F} \subseteq \mathbb{G}$. If $x = \infty$, then $(\mathbb{G}, x) \in \lim^{\overline{X}}$. If $x \neq \infty$, then $(i_X^{\Leftarrow}(\mathbb{F}), x) \in \lim^X$. By $i_X^{\Leftarrow}(\mathbb{F}) \subseteq i_X^{\Leftarrow}(\mathbb{G})$, it follows that $(i_X^{\Leftarrow}(\mathbb{G}), x) \in \lim^X$. So $(\mathbb{G}, x) \in \lim^{\overline{X}}$. □

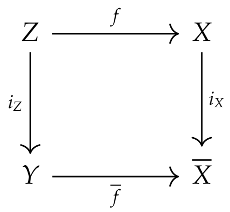
Theorem 6.2. *The category* **CS** *is extensional.*

Proof. Let $(X, \lim^X)$ be a convergence space. By Proposition 6.1, we obtain a convergence structure $\lim^{\overline{X}}$ on $\overline{X}$. It suffices to show that $(\overline{X}, \lim^{\overline{X}})$ is a one-point extension of $(X, \lim^X)$.

Firstly, we show that $(X, \lim^X)$ is a subspace of $(\overline{X}, \lim^{\overline{X}})$, that is, $\lim^X = \lim^{\overline{X}}|_X$. Take any $\mathbb{F} \in C_S(X)$ and $x \in X$. Since $i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F})) = \mathbb{F}$, we have

$$(\mathbb{F}, x) \in \lim^{\overline{X}}|_X \Longleftrightarrow (i_X^{\Rightarrow}(\mathbb{F}), x) \in \lim^{\overline{X}} \Longleftrightarrow (i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F})), x) \in \lim^X \Longleftrightarrow (\mathbb{F}, x) \in \lim^X.$$

Next, let $(Y, \lim^Y)$ be a convergence space, $(Z, \lim^Z)$ be a subspace of $(Y, \lim^Y)$ and $f : (Z, \lim^Z) \longrightarrow (X, \lim^X)$ be continuous. For the inclusion map $i_Z : Z \longrightarrow Y$ and the extensional map $\overline{f} : Y \longrightarrow \overline{X}$ of f defined by $\overline{f}(y) = f(y)$ for each $y \in Z$, and $\overline{f}(y) = \infty$ otherwise, there exists a commutative diagram in the category **Set** of sets as follows:



In order to prove $\overline{f} : (Y, \lim^Y) \longrightarrow (\overline{X}, \lim^{\overline{X}})$ is continuous, it suffices to verify that $(\overline{f}^{\Rightarrow}(\mathbb{G}), \overline{f}(y)) \in \lim^{\overline{X}}$ for each $(\mathbb{G}, y) \in \lim^Y$. Now we divide into two cases:

Case 1: $\overline{f}(y) = \infty$, i.e., $y \in Y/Z$;

Case 2: $\overline{f}(y) \neq \infty$, i.e., $y \in Z$.

For case 1, by the definition of $\lim^{\overline{X}}$, we have $(\overline{f}^{\Rightarrow}(\mathbb{G}), \overline{f}(y)) \in \lim^{\overline{X}}$.

For case 2, take any $(\mathbb{G}, y) \in \lim^Y$, it follows from $\mathbb{G} \subseteq i_Z^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G}))$ that $(i_Z^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G})), y) \in \lim^Y$. Since $(Z, \lim^Z)$ is a subspace of $(Y, \lim^Y)$, we have $(i_Z^{\Leftarrow}(\mathbb{G}), y) \in \lim^Z$. By the continuity of f, it follows that $(f^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G})), f(y)) \in \lim^X$. Since $i_X^{\leftarrow}(\overline{f}^{\rightarrow}(G)) \subseteq f^{\rightarrow}(i_Z^{\leftarrow}(G))$, we have $f^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G})) \subseteq i_X^{\Leftarrow}(\overline{f}^{\Rightarrow}(\mathbb{G}))$. This implies that $(i_X^{\Leftarrow}(\overline{f}^{\Rightarrow}(\mathbb{G})), f(y)) \in \lim^X$. By the definition of $\lim^{\overline{X}}$, we obtain $(\overline{f}^{\Rightarrow}(\mathbb{G}), \overline{f}(y)) \in \lim^{\overline{X}}$. Hence, we obtain that $\overline{f} : (Y, \lim^Y) \longrightarrow (\overline{X}, \lim^{\overline{X}})$ is continuous. □

Secondly, we will show that finite products of quotients maps are quotient maps in **CS**. At the beginning, we first give an important property of co-Scott closed sets.

Lemma 6.3. *Suppose that Λ is a finite index set. Let $\{f_\lambda : X_\lambda \longrightarrow Y_\lambda\}_{\lambda\in\Lambda}$ be a family of surjective maps and $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of co-Scott closed sets with $\mathbb{F}_\lambda \in \mathcal{C}_S(X_\lambda)$ for each $\lambda \in \Lambda$. Then*

$$\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda{}^{\Rightarrow}(\mathbb{F}_\lambda).$$

Proof. Let

$$\begin{array}{ccc} \prod\limits_{\lambda\in\Lambda} X_\lambda & \xrightarrow{\ \prod_{\lambda\in\Lambda} f_\lambda\ } & \prod\limits_{\lambda\in\Lambda} Y_\lambda \\ \Big\downarrow{\scriptstyle p_\lambda} & & \Big\downarrow{\scriptstyle q_\lambda} \\ X_\lambda & \xrightarrow[\ f_\lambda\ ]{} & Y_\lambda \end{array}$$

be the product commutation diagram with respect to sets. Take any $y \in \prod\limits_{\lambda\in\Lambda} Y_\lambda$, $\lambda_0 \in \Lambda$ and $F_{\lambda_0} \in \mathbb{F}_{\lambda_0}$. Then

$$\begin{aligned} y \in q_{\lambda_0}^{\leftarrow}(f_{\lambda_0}^{\rightarrow}(F_{\lambda_0})) &\iff \exists\, x_{\lambda_0} \in F_{\lambda_0},\ s.t.\ f_{\lambda_0}(x_{\lambda_0}) = q_{\lambda_0}(y) \\ &\iff \exists\, x \in p_{\lambda_0}^{\leftarrow}(F_{\lambda_0}),\ s.t.\ \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)(x) = y \\ &\qquad \text{(Since } \{f_\lambda : X_\lambda \longrightarrow Y_\lambda\}_{\lambda\in\Lambda} \textit{ are surjective maps)} \\ &\iff y \in \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(p_{\lambda_0}^{\leftarrow}(F_{\lambda_0})\Big). \end{aligned}$$

This implies that $q_{\lambda_0}^{\leftarrow}(f_{\lambda_0}^{\rightarrow}(F_{\lambda_0})) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(p_{\lambda_0}^{\leftarrow}(F_{\lambda_0})\Big)$. Take any $A \in \mathcal{P}(\prod_{\lambda\in\Lambda} X_\lambda)$. Then

$$\begin{aligned} A \in \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda\Big) &\iff \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(A\Big) \in \prod_{\lambda\in\Lambda}\mathbb{F}_\lambda \\ &\iff \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(A\Big) \in \bigcup_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathbb{F}_\lambda) \\ &\iff \exists\, \lambda_0 \in \Lambda,\ s.t.\ \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(A\Big) \in p_{\lambda_0}^{\Leftarrow}(\mathbb{F}_{\lambda_0}) \\ &\iff \exists\, \lambda_0 \in \Lambda, F_{\lambda_0} \in \mathbb{F}_{\lambda_0},\ s.t.\ p_{\lambda_0}^{\leftarrow}(F_{\lambda_0}) \subseteq \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(A\Big). \\ &\iff \exists\, \lambda_0 \in \Lambda, F_{\lambda_0} \in \mathbb{F}_{\lambda_0},\ s.t.\ \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(p_{\lambda_0}^{\leftarrow}(F_{\lambda_0})\Big) \subseteq A. \\ &\iff \exists\, \lambda_0 \in \Lambda, F_{\lambda_0} \in \mathbb{F}_{\lambda_0},\ s.t.\ q_{\lambda_0}^{\leftarrow}(f_{\lambda_0}^{\rightarrow}(F_{\lambda_0})) \subseteq A. \\ &\iff \exists\, \lambda_0 \in \Lambda,\ s.t.\ A \in q_{\lambda_0}^{\Leftarrow}(f_{\lambda_0}^{\Rightarrow}(\mathbb{F}_{\lambda_0})) \\ &\iff A \in \bigcup_{\lambda\in\Lambda} q_\lambda^{\Leftarrow}(f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda)) \\ &\iff A \in \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda). \end{aligned}$$

This implies that

$$\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda).$$

□

Theorem 6.4. *Suppose that Λ is a finite index set. If $\{f_\lambda : (X_\lambda, \lim^{X_\lambda}) \longrightarrow (Y_\lambda, \lim^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in* **CS**, *then the product map*

$$\prod_{\lambda\in\Lambda} f_\lambda : \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \lim^{X_\lambda}\Big) \longrightarrow \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \lim^{Y_\lambda}\Big)$$

is a quotient map in **CS**.

Proof. Define

$$f := \prod_{\lambda\in\Lambda} f_\lambda,\ (X, \lim^X) := \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \lim^{X_\lambda}\Big),\ (Y, \lim^Y) := \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \lim^{Y_\lambda}\Big).$$

Let

$$\begin{array}{ccc} (X, \lim^X) & \xrightarrow{\quad f \quad} & (Y, \lim^Y) \\ \Big\downarrow{\scriptstyle p_\lambda} & & \Big\downarrow{\scriptstyle q_\lambda} \\ (X_\lambda, \lim^{X_\lambda}) & \xrightarrow[\quad f_\lambda \quad]{} & (Y_\lambda, \lim^{Y_\lambda}) \end{array}$$

be the product communication diagram with respect to sets. Since $\{f_\lambda : (X_\lambda, \lim^{X_\lambda}) \longrightarrow (Y_\lambda, \lim^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in **CS**, for each $\mathbb{H}_\lambda \in \mathcal{C}_S(Y_\lambda)$ and $y_\lambda \in Y_\lambda$, we have

$$(\mathbb{H}_\lambda, y_\lambda) \in \lim^{Y_\lambda} \Longleftrightarrow \exists\, x_\lambda \in X_\lambda, \mathbb{F}_\lambda \in \mathcal{C}_S(X_\lambda)\ s.t.\ f_\lambda(x_\lambda) = y_\lambda, f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda) \subseteq \mathbb{H}_\lambda \textit{ and } (\mathbb{F}_\lambda, x_\lambda) \in \lim^{X_\lambda}.$$

Suppose that $\lim^Y_*$ is the quotient structure with respect to f. Then

$$(\mathbb{H}, y) \in \lim^Y_* \Longleftrightarrow \exists\, x \in X, \mathbb{G} \in \mathcal{C}_S(X)\ s.t.\ f(x) = y, f^{\Rightarrow}(\mathbb{G}) \subseteq \mathbb{H} \textit{ and } (\mathbb{G}, x) \in \lim^X.$$

It suffices to verify that $\lim^Y_* = \lim^Y$.

On the one hand, if $(\mathbb{H}, y) \in \lim^Y_*$, then there exist $x \in X$ and $\mathbb{G} \in \mathcal{C}_S(X)$ such that $f(x) = y, f^{\Rightarrow}(\mathbb{G}) \subseteq \mathbb{H}$ and $(\mathbb{G}, x) \in \lim^X$. Since $f_\lambda \circ p_\lambda = q_\lambda \circ f$, we have

$$f_\lambda^{\Rightarrow} \circ p_\lambda^{\Rightarrow}(\mathbb{G}) = q_\lambda^{\Rightarrow} \circ f^{\Rightarrow}(\mathbb{G}) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H})$$

and

$$f_\lambda \circ p_\lambda(x) = q_\lambda \circ f(x) = q_\lambda(y)$$

for each $\lambda \in \Lambda$. It follows from the continuity of $f_\lambda \circ p_\lambda$ that $((f_\lambda \circ p_\lambda)^{\Rightarrow}(\mathbb{G}), f_\lambda \circ p_\lambda(x)) \in \lim^{Y_\lambda}$. This implies that $(q_\lambda^{\Rightarrow}(\mathbb{H}), q_\lambda(y)) \in \lim^{Y_\lambda}$. Thus $(\mathbb{H}, y) \in \lim^Y_*$ implies $(q_\lambda^{\Rightarrow}(\mathbb{H}), q_\lambda(y)) \in \lim^{Y_\lambda}$ for each $\lambda \in \Lambda$. That is, $(\mathbb{H}, y) \in \lim^Y_*$ implies $(\mathbb{H}, y) \in \lim^Y$. This shows that $\lim^Y_* \subseteq \lim^Y$.

On the other hand, let

$$\mathcal{G}_\lambda = \Big\{\mathbb{G}_\lambda \in \mathcal{C}_S(X_\lambda) \mid f_\lambda^{\Rightarrow}(\mathbb{G}_\lambda) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H})\Big\}$$

for each $\lambda \in \Lambda$ and let

$$\prod_{\lambda\in\Lambda} \mathcal{G}_\lambda = \Big\{g : \Lambda \longrightarrow \coprod \mathcal{G}_\lambda \mid \forall \lambda \in \Lambda, g(\lambda) \in \mathcal{G}_\lambda\Big\}$$

be the set of choice functions, that is,

$$\forall\, \lambda \in \Lambda,\ \exists\, \mathbb{G}_\lambda \in \mathcal{C}_S(X_\lambda),\ s.t.\ f_\lambda^{\Rightarrow}(\mathbb{G}_\lambda) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H}) \Longleftrightarrow \exists\, g \in \prod_{\lambda\in\Lambda} \mathcal{G}_\lambda,\ s.t.\ \forall\, \lambda \in \Lambda,\ f_\lambda^{\Rightarrow}(g(\lambda)) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H}).$$

Furthermore, we have

$$\prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(g(\lambda)) \subseteq \prod_{\lambda\in\Lambda} q_\lambda^{\Rightarrow}(\mathbb{H}) \subseteq \mathbb{H},$$

which implies

$$f^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} g(\lambda)\Big) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} g(\lambda)\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(g(\lambda)) \subseteq \mathbb{H}.$$

Let

$$H_\lambda = \{x_\lambda \in X_\lambda \mid f_\lambda(x_\lambda) = q_\lambda(y)\}$$

for each $\lambda \in \Lambda$ and let

$$\prod_{\lambda\in\Lambda} H_\lambda = \Big\{h : \Lambda \longrightarrow \coprod H_\lambda \mid \forall\, \lambda \in \Lambda,\ f_\lambda(h(\lambda)) = q_\lambda(y)\Big\}$$

be the set of choice functions, that is,

$$\forall\, \lambda \in \Lambda,\ \exists\, x_\lambda \in X_\lambda,\ s.t.\ f_\lambda(x_\lambda) = q_\lambda(y) \Longleftrightarrow \exists\, h \in \prod_{\lambda\in\Lambda} H_\lambda,\ s.t.\ \forall\, \lambda \in \Lambda,\ f_\lambda(h(\lambda)) = q_\lambda(y).$$

Furthermore, we have

$$f\Big((h(\lambda))_{\lambda\in\Lambda}\Big) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)\Big((h(\lambda))_{\lambda\in\Lambda}\Big) = \Big(f_\lambda(h(\lambda))\Big)_{\lambda\in\Lambda} = \Big(q_\lambda(y)\Big)_{\lambda\in\Lambda} = y.$$

Then for each $\mathbb{H} \in \mathcal{C}_S(Y)$ and $y \in Y$, we have

$$\begin{aligned}
& (\mathbb{H}, y) \in \lim{}^Y \\
\Longleftrightarrow\ & \forall\, \lambda \in \Lambda, \exists\, x_\lambda \in X_\lambda, \mathbb{G}_\lambda \in \mathcal{C}_S(X_\lambda)\ s.t.\ f_\lambda(x_\lambda) = q_\lambda(y), f_\lambda^{\Rightarrow}(\mathbb{G}_\lambda) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H})\ and\ (\mathbb{G}_\lambda, x_\lambda) \in \lim{}^{X_\lambda} \\
\Longleftrightarrow\ & \exists\, h \in \prod_{\lambda\in\Lambda} H_\lambda, \forall \lambda \in \Lambda, \exists\, \mathbb{G}_\lambda \in \mathcal{C}_S(X_\lambda),\ s.t.\ f_\lambda^{\Rightarrow}(\mathbb{G}_\lambda) \subseteq q_\lambda^{\Rightarrow}(\mathbb{H})\ and\ (\mathbb{G}_\lambda, h(\lambda)) \in \lim{}^{X_\lambda} \\
\Longleftrightarrow\ & \exists\, h \in \prod_{\lambda\in\Lambda} H_\lambda\ and\ g \in \prod_{\lambda\in\Lambda} \mathcal{G}_\lambda,\ s.t.\ \forall \lambda \in \Lambda, (g(\lambda), h(\lambda)) \in \lim{}^{X_\lambda} \\
\Longrightarrow\ & \exists\, h \in \prod_{\lambda\in\Lambda} H_\lambda\ and\ g \in \prod_{\lambda\in\Lambda} \mathcal{G}_\lambda,\ s.t.\ \forall \lambda \in \Lambda, \Big(p_\lambda^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} g(\lambda)\Big), p_\lambda\Big((h(\lambda))_{\lambda\in\Lambda}\Big)\Big) \in \lim{}^{X_\lambda} \\
\Longleftrightarrow\ & \exists\, h \in \prod_{\lambda\in\Lambda} H_\lambda\ and\ g \in \prod_{\lambda\in\Lambda} \mathcal{G}_\lambda,\ s.t.\ \Big(\prod_{\lambda\in\Lambda} g(\lambda), (h(\lambda))_{\lambda\in\Lambda}\Big) \in \lim{}^X \\
\Longrightarrow\ & \exists\, x \in X\ and\ \mathbb{G} \in \mathcal{C}_S(X),\ s.t.\ f(x) = y, f^{\Rightarrow}(\mathbb{G}) \subseteq \mathbb{H}\ and\ (\mathbb{G}, x) \in \lim{}^X \\
\Longleftrightarrow\ & (\mathbb{H}, y) \in \lim{}_*^Y.
\end{aligned}$$

This shows that $\lim^Y \subseteq \lim_*^Y$. As a consequence, we obtain $\lim^Y = \lim_*^Y$. □

7. Conclusions

In this paper, we first discussed the categorical properties of convex spaces, including extensionality and productivity of quotient maps. Then we introduced convergence structures in the framework of concave spaces and studied its categorical relationship with concave spaces as well as its categorical properties. Actually, we applied the method in topology to the theory of convex spaces (dually, concave spaces). Following this approach, we can further consider the following problems:

(1) Besides the extensionality and productivity of quotient maps, Cartesian-closedness is another important categorical property. Yao and Zhou [26] proved that the category of convex spaces is not Cartesian

closed. By the duality, the category of concave spaces is not Cartesian-closed. But we showed in Theorem 5.15 that the category of concave spaces can be embedded in the category of convergence spaces as a full and reflective subcategory. This relationship is similar to that between topological spaces and filter-based convergence spaces. It is well known that the category of filter-based convergence spaces is Cartesian closed. This motivates to consider if the category of convergence spaces is Cartesian closed.

(2) In Theorems 3.5 and 6.4, we only showed the productivity of finite quotient maps since we could only know the convex sets in the product space of a finite family of convex spaces and only define the finite product of co-Scott closed sets in the present stage. So we will go on considering the productivity of an arbitrary family of quotient maps in Theorems 3.5 and 6.4.

(3) In this paper, the categorical properties of convex spaces and its corresponding convergence spaces are discussed. As far as I know, convex structures have been generalized to the fuzzy case. So it is natural to consider the fuzzy counterparts of all the results of this paper.

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