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Proximity structures via the class of bipolar fuzzy soft sets

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Abstract. In this paper, we first define the concept of a proximity with the help of bipolar fuzzy soft sets and establish some of its properties. Then, we demonstrate the process of generating a bipolar fuzzy soft topology with the aid of a bipolar fuzzy soft proximity (for short BFS-proximity). Moreover, we give a new definition of bipolar fuzzy soft neighborhood based on BFS-proximity, enabling an alternative framework for analyzing the notion of BFS-proximity. Next, by using a family of BFS-proximities, we present the initial BFS-proximity structure. Finally, taking into account the proximity structure in the classical set theory, we derive a BFS-proximity structure and study on its related results.

1. Introduction

Due to a number of uncertainties, classical approaches are insufficient for addressing complex issues in the domains of engineering, economics, and the environment. Many hypotheses have been adopted to deal with these uncertainties. The two most well-known theories are fuzzy set theory, which was first presented in 1965 by Zadeh [35], and rough set theory, which was first presented in 1982 by Pawlak [26]. Both theories are useful for handling uncertainties. Nevertheless, as noted by Molodtsov [23], they have their own difficulties and inadequacies because the parameterization tool is not sufficient. In order to deal with ambiguities and imprecisions in parametric ways, Molodtsov [23] developed a novel concept called the soft set. Then, this idea has been applied by several researchers as a potent tool for defining uncertainty. For instance, Maji et al. [22] building upon Molodtsov's [23] foundational work, significantly enriched the realm of soft set theory with their insightful contributions. Aktaş and Çağman [2] defined the soft group and compared soft sets to fuzzy sets and rough sets. Ali et al. [5] developed new algebraic operations on soft sets and investigated their properties. Also, Shabir and Naz [31] studied soft topological spaces. Moreover, Al-Shami [6] proposed a new idea to investigate a novel class of soft sets based on the generalizations of open subsets in the parametric topological spaces. Alcantud [3] introduced the formal model consisting of convex soft geometries and studied how he can associate a convex geometry with each convex soft geometry, and conversely. On the other hand, Maji et al. [21] introduced a more general concept, which is a combination of fuzzy set and soft set; the fuzzy soft set. Then, Kharal and Ahmad [18] studied the notion of a mapping on fuzzy soft classes. Also, Demir et al. [11] investigated convergence of fuzzy soft filters in

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a fuzzy soft topological space. Recently, many papers concerning soft set theory and fuzzy soft set theory have been published [4, 7, 15, 34].

Fuzzy sets cannot indicate the degree of satisfaction with a counter-property while they can reflect uncertainty in membership degree designation. To overcome this issue, Lee [19] proposed the idea of a bipolar valued fuzzy set, where positivity and negativity coexist within a membership degree range of [-1, 1]. In a bipolar valued fuzzy set, the membership value 0 of an element shows that the element is irrelevant to the corresponding property, the membership degree (0, 1] of an element means that the element somewhat satisfies the property, and the membership degree [-1, 0) of an element shows that the element somewhat satisfies the implicit counter-property. Then, Abdullah et al. [1] and Naz and Shabir [25] independently introduced the concept of a bipolar fuzzy soft set (BFS-set), merging the properties of both bipolar fuzzy sets and soft sets. Riaz and Tehrim [28] indicated the concept of mappings between BFS-sets and applied this concept to the problem of medical diagnosis. Also, they [29] initiated bipolar fuzzy soft topology (BFS-topology). Afterwards, Mahmood et al. [20] defined the concept of a bipolar complex fuzzy soft set and offered a decision-making algorithm to show the effectiveness and usefulness of the concept. Sarwar et al. [30] applied the technique of BFS-sets to hypergraphs. In the recent years, this concept has attracted a lot of interest because of its theoretical outcomes and real-world applicability [13, 33, 36].

Efremovic [14] introduced proximity structure in 1951. This structure provide a clear and conceptual solution to many topological difficulties such as compactification and extension problems as well as axiomatizations of geometric notions. Then, Naimpally and Warrack [24] laid the foundation for the study of proximity spaces, which led to many developments for subsequent studies. For instance, with the help of fuzzy sets, Katsaras [17] established an approach about proximity structures. Also, Artico and Moresco [8] introduced the different notion of a fuzzy proximity while Ramadan et al. [27] contributed to the field by presenting fuzzifying proximity structures. On the other hand, numerous researchers have explored extensions of proximity structures to both soft sets and fuzzy soft sets. Kandil et al. [16] defined soft proximity spaces and investigated some of their properties. Singh and Singh [32] gave the concepts of soft proximity base and subbase. Moreover, Çetkin et al. [9] introduced soft fuzzy proximity spaces based on the axioms proposed by Katsaras [17]. Later, Demir and Özbakır [10] studied fuzzy soft proximity spaces and demonstrated how a fuzzy soft topology is derived from a fuzzy soft proximity.

Inspired by these studies, we give the definition of a BFS-proximity space and establish some of its properties. Then, we demonstrate how a BFS-topology is derived from a BFS-proximity. Also, we introduce the notion of a bipolar fuzzy soft neighborhood in BFS-proximity space, which provides an alternative method for studying BFS-proximity spaces. Moreover, we present the notion of a BFS-proximity mapping and analyze its relationship with the BFS-continuous mappings. Next, we obtain the initial BFS-proximity structure. Finally, we investigate the connection between BFS-proximity structures and proximity structures.

2. Preliminaries

In this section, we review some basic notions of BFS-sets that we will use in the subsequent sections.

Throughout this paper, U be a universe of alternatives (objects) and E be a set of specified parameters (criteria or attributes) unless otherwise explicit.

Definition 2.1. ([19]) Consider a universal set U. A set having form

$$\eta = \{(u, \delta^+_\eta(u), \delta^-_\eta(u)) : u \in U\}$$

denotes a bipolar fuzzy set on U, where $\delta^+_\eta(u)$ denotes the positive memberships ranges over $[0, 1]$ and $\delta^-_\eta(u)$ denotes the negative memberships ranges over $[-1, 0]$.

Definition 2.2. ([19]) Let η_1 and η_2 be two bipolar fuzzy sets on U. Then, their intersection and union are defined as follows:

(*i*) $\eta_1 \wedge \eta_2 = \Big\{\Big(u, \min\{\delta^+_{\eta_1}(u), \delta^+_{\eta_2}(u)\}, \max\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}\Big) : u \in U\Big\}.$

(*ii*) $\eta_1 \vee \eta_2 = \Big\{\Big(u, \max\{\delta^+_{\eta_1}(u), \delta^+_{\eta_2}(u)\}, \min\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}\Big) : u \in U\Big\}$.

Definition 2.3. ([1, 25]) Consider a universal set U and a set of parameters E. Let $A \subseteq E$ and define a mapping $\Omega : E \to BF^U$, where BF^U represents the family of all bipolar fuzzy subsets of U. Then, Ω_A is called a BFS-set on U, where

$$\Omega_A = \{\langle e, \Omega(e)\rangle : e \in E\}$$

such that $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ for all $e \notin A$ and all $u \in U$.

Note that the set of all bipolar fuzzy soft sets on U with the attributes from E is denoted by $(BF^U)^E$.
Consider the following example to achieve a better understanding of the definition mentioned above.

Example 2.4. Suppose Mr. Kemal is thinking of buying a headset and let $E = \{e_1 = \textit{battery life},\ e_2 = \textit{latency},\ e_3 = \textit{microphone quality},\ e_4 = \textit{bass response}\}$ be the set of decision variables. Next, take into consideration the set of three model headset types $U = \{u_1, u_2, u_3\}$ while keeping in mind Mr. Kemal's needs. After research, we show that a website has assigned the numerical values for each decision variable to three model headsets, evaluating the positive and negative feedbacks based on customers. The tabular representation of these numerical values is as follows:

Table 1
Tabular reprentation of positive feedbacks

	e_1	e_2	e_3	e_4
u_1	0.6	0.2	0.58	0.7
u_2	0.4	0.53	0.74	0.35
u_3	0.43	0.36	0.35	0.55

Table 2
Tabular reprentation of negative feedbacks

	e_1	e_2	e_3	e_4
u_1	−0.35	−0.64	−0.32	−0.53
u_2	−0.34	−0.58	−0.72	−0.35
u_3	−0.25	−0.45	−0.62	−0.85

Therefore, the following bipolar fuzzy soft set on U with the set E of decision variables reporting the positive-negative informations is obtained:

$$\Omega_A = \left\{\begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.6, -0.35), (u_2, 0.4, -0.34), (u_3, 0.43, -0.25)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.2, -0.64), (u_2, 0.53, -0.58), (u_3, 0.36, -0.45)\}\rangle, \\ \langle e_3, \Omega(e_3) = \{(u_1, 0.58, -0.32), (u_2, 0.74, -0.72), (u_3, 0.35, -0.62)\}\rangle, \\ \langle e_4, \Omega(e_4) = \{(u_1, 0.7, -0.53), (u_2, 0.35, -0.35), (u_3, 0.55, -0.85)\}\rangle \end{array}\right\}.$$

Definition 2.5. ([36])

(*i*) A BFS-set $\Omega_E \in (BF^U)^E$ is called an absolute BFS-set, denoted by U_E, if $\delta^+_{\Omega(e)}(u) = 1$ and $\delta^-_{\Omega(e)}(u) = -1$ for all $u \in U$ and all $e \in E$.

(*ii*) *A* BFS-set $\Omega_A \in (BF^U)^E$ is called a null BFS-set, denoted by ϕ_A, if $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ for all $u \in U$ and all $e \in A$.

Definition 2.6. ([1, 25]) Let $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. Then,

(*i*) The union of $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ is a bipolar fuzzy soft set $\Omega^3_{A_3}$ over U such that for all $e \in E$, $\Omega^3(e) = \Omega^1(e) \vee \Omega^2(e)$ and denoted by $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cup} \Omega^2_{A_2}$.

(*ii*) The intersection of $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ is a bipolar fuzzy soft set $\Omega^3_{A_3}$ over U such that for all $e \in E$, $\Omega^3(e) = \Omega^1(e) \wedge \Omega^2(e)$ and denoted by $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}$.

Definition 2.7. ([1, 25]) The complement of a BFS-set $\Omega_A \in (BF^U)^E$ is shown by $(\Omega_A)^c = (U_E - \Omega_A) = \Omega^c_{A_1}$ where $\Omega^c : E \to BF^U$ is a mapping defined by $\delta^+_{\Omega^c(e)}(u) = 1 - \delta^+_{\Omega(e)}(u)$ and $\delta^-_{\Omega^c(e)}(u) = -1 - \delta^-_{\Omega(e)}(u)$ for all $e \in E$ and $u \in U$.

Theorem 2.8. ([25]) *Let $\Omega^1_{A_1}$, $\Omega^2_{A_2}$ be two BFS-sets over U.*

(*i*) $((\Omega^1_{A_1})^c)^c = \Omega^1_{A_1}$.

(*ii*) *If* $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, *then* $(\Omega^2_{A_2})^c \tilde{\subseteq} (\Omega^1_{A_1})^c$.

(*iii*) $(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cup} (\Omega^2_{A_2})^c$.

(*iv*) $(\Omega^1_{A_1} \tilde{\cup} \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cap} (\Omega^2_{A_2})^c$.

Definition 2.9. ([36]) Let $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in (BF^U)^E$. Then, $\Omega^1_{A_1}$ is a BFS-subset of $\Omega^2_{A_2}$ if $\delta^+_{\Omega^1(e)}(u) \leq \delta^+_{\Omega^2(e)}(u)$ and $\delta^-_{\Omega^1(e)}(u) \geq \delta^-_{\Omega^2(e)}(u)$, which is shown by $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$.

Definition 2.10. ([12]) Let $\Omega_A \in (BF^U)^E$ with $A = \{e\}$. If there is a $u \in U$ such that $\delta^+_{\Omega(e)}(u) \neq 0$ or $\delta^-_{\Omega(e)}(u) \neq 0$ and $\delta^+_{\Omega(e)}(u') = \delta^-_{\Omega(e)}(u') = 0$ for all $u' \in U \backslash \{u\}$, then Ω_A is called a BFS-point in U. It is denoted by $e^{(p,n)}_u$.

Definition 2.11. ([12]) The BFS-point $e^{(p,n)}_u$ is said to belongs to a BFS-set Ω_A, denoted by $e^{(p,n)}_u \tilde{\in} \Omega_A$, if $p \leq \delta^+_{\Omega(e)}(u)$ and $n \geq \delta^-_{\Omega(e)}(u)$.

Definition 2.12. ([28]) Let $(BF^U)^E$ and $(BF^V)^K$ be two the families of all bipolar fuzzy soft sets on U and V with parameters from E and K, respectively. Assume that $\mathfrak{u} : U \to V$ and $\mathfrak{g} : E \to K$ be two mappings. Then, the mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^K$ is called a BFS-mapping from U to V, defined as the following :

(i) Let $\Omega_A \in (BF^U)^E$. Then, $\mathfrak{f}(\Omega_A) = (\mathfrak{f}(\Omega))_{A_1}$ is the BFS-set over V with parameters from K given by $\mathfrak{f}(\Omega_A) = \{\langle k, \mathfrak{f}(\Omega)(k)\rangle : k \in K\}$ such that $\mathfrak{f}(\Omega)(k) = \{(v, \delta^+_{\mathfrak{f}(\Omega)(k)}(v), \delta^-_{\mathfrak{f}(\Omega)(k)}(v)) : v \in V\}$, where

$$\delta^+_{\mathfrak{f}(\Omega)(k)}(v) = \begin{cases} \sup\{\delta^+_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(k) \cap A \neq \emptyset, \\ 0, & \text{if } \textit{otherwise}, \end{cases}$$

$$\delta^-_{\mathfrak{f}(\Omega)(k)}(v) = \begin{cases} \inf\{\delta^-_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(k) \cap A \neq \emptyset, \\ 0, & \text{if } \textit{otherwise}. \end{cases}$$

Then, $\mathfrak{f}(\Omega_A)$ is called BFS-image of BFS-set Ω_A under $\mathfrak{f}$.

(ii) Let $\Omega^1_{A_1} \in (BF^V)^K$. Then, $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = (\mathfrak{f}^{-1}(\Omega^1))_A$ is the BFS-set over U with parameters from E given by $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = \{\langle e, \mathfrak{f}^{-1}(\Omega^1)(e)\rangle : e \in E\}$ such that $\mathfrak{f}^{-1}(\Omega^1)(e) = \{(u, \delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u), \delta^-_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u)) : u \in U\}$, where

$$\delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^+_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{for } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if } \textit{otherwise}, \end{cases}$$

$$\delta^{-}_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^{-}_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{for } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if } \textit{otherwise}. \end{cases}$$

Then, $\mathfrak{f}^{-1}(\Omega^1_{A_1})$ is called BFS inverse image of BFS-set $\Omega^1_{A_1}$.

Theorem 2.13. ([28]) *Let* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^K$ *be a BFS-mapping. Then, for* $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$ *and* $\Gamma^1_{B_1}, \Gamma^2_{B_2} \in (BF^V)^K$, *the following properties are satisfied.*

(*i*) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cup} \Omega^2_{A_2}) = \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cup} \mathfrak{f}(\Omega^2_{A_2})$.

(*ii*) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cup} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cup} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

(*iii*) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}) \tilde{\subseteq} \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap} \mathfrak{f}(\Omega^2_{A_2})$.

(*iv*) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cap} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cap} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

(*v*) $\Omega^1_{A_1} \tilde{\subseteq} \mathfrak{f}^{-1}(\mathfrak{f}(\Omega^1_{A_1}))$, $\mathfrak{f}(\mathfrak{f}^{-1}(\Gamma^1_{B_1})) \tilde{\subseteq} \Gamma^1_{B_1}$.

(*vi*) *If* $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, *then* $\mathfrak{f}(\Omega^1_{A_1}) \tilde{\subseteq} \mathfrak{f}(\Omega^2_{A_2})$.

(*vii*) *If* $\Gamma^1_{B_1} \tilde{\subseteq} \Gamma^2_{B_2}$, *then* $\mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

Definition 2.14. ([29]) A family τ of BFS-sets over U is said to be a BFS-topology on U if it satisfies the following properties:
$(bfst_1)$ U_E and ϕ_A are members of τ.
$(bfst_2)$ If $\Omega^i_{A_i} \in \tau$ for all $i \in I$, an index set, then $\tilde{\bigcup}_{i\in\Lambda}\Omega^i_{A_i} \in \tau$.
$(bfst_3)$ If $\Omega^1_{A_1}, \Omega^2_{A_2} \in \tau$, then $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \tau$.
We say (U, τ, E) is a BFS-topological space. A member in τ is called a BFS-open set and its complement is called a BFS-closed set.

Definition 2.15. ([29]) Let (U, τ, E) be a BFS-topological space and $\Omega_A \in (BF^U)^E$. The BFS-interior of Ω_A is the union of all BFS-open sets contained in Ω_A, denoted by $(\Omega_A)^o$. From $(bfst_2)$ it is clear that $(\Omega_A)^o$ is a BFS-open set. This set is largest BFS-open set contained in Ω_A.

Definition 2.16. ([29]) Let (U, τ, E) be a BFS-topological space and $\Omega_A \in (BF^U)^E$. The closure of Ω_A is the intersection of all BFS-closed sets containing Ω_A; this set is denoted by $\overline{\Omega_A}$. It is easily seen that $\overline{\Omega_A}$ is the smallest closed set containing Ω_A.

Definition 2.17. ([13]) Let (U, τ_1, E), (V, τ_2, K) be two BFS-topological spaces and $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, K)$ be a BFS-mapping. The BFS-mapping $\mathfrak{f}$ is said to be BFS-continuous if $\mathfrak{f}^{-1}(\Gamma_B) \in \tau_1$ for any $\Gamma_B \in \tau_2$.

3. BFS-proximity structure

In this section, we give the concept of a BFS-proximity structure and discuss its related properties. With a BFS-closure operator, we generate a BFS-topology from a given BFS-proximity. Moreover, we offer the different interpretation of this structure, known as BFS-ρ-neighborhood. Then, we present the definition of a BFS-proximity mapping and compare this notion with the BFS-continuous mappings.

Definition 3.1. A binary relation $\rho \subseteq (BF^U)^E \times (BF^U)^E$ is a BFS-proximity on U_E if ρ satisfies the following axioms:
$(bfsp_1)$ $\phi_A \; \overline{\rho} \; \Omega_A$.
$(bfsp_2)$ If $\Omega_A \tilde{\cap} \Gamma_B \neq \phi_A$, then $\Omega_A \; \rho \; \Gamma_B$.

($bfsp_3$) If $\Omega_A \; \rho \; \Gamma_B$, then $\Gamma_B \; \rho \; \Omega_A$.
($bfsp_4$) $\Omega_A \; \rho \; (\Gamma_B \tilde{\cup} \Lambda_C)$ if and only if $\Omega_A \; \rho \; \Gamma_B$ or $\Omega_A \; \rho \; \Lambda_C$.
($bfsp_5$) If $\Omega_A \; \overline{\rho} \; \Gamma_B$, then there exists a $\Lambda_C \in (BF^U)^E$ such that $\Omega_A \; \overline{\rho} \; \Lambda_C$ and $\Gamma_B \; \overline{\rho} \; (U_E - \Lambda_C)$,

where $\overline{\rho}$ indicates negation of ρ. The triplet (U, ρ, E) is called a BFS-proximity space.

Remark 3.2. The idea of a BFS-proximity is the generalization of prevalent proximities such as fuzzy proximities [17], soft proximities [16], and fuzzy soft proximities [9] as stated below:

(i) If we use just one parameter and ignore the the negative membership degree, then the BFS-proximity will coincide with the fuzzy proximity.

(ii) If we ignore the negative membership degree and the fuzzy value set of each parameter becomes a crisp set, then the BFS-proximity will coincide with the soft proximity.

(iii) If we ignore the negative membership degree, then the BFS-proximity will coincide with the fuzzy soft proximity.

Example 3.3. A binary relation ρ on U_E defined as follows is a BFS-proximity:

$$\Omega_A \; \rho \; \Gamma_B \Leftrightarrow \Omega_A \neq \phi_A \text{ and } \Gamma_B \neq \phi_A.$$

Theorem 3.4. *Let $\mathfrak{C} : (BF^U)^E \to (BF^U)^E$ be an operator fulfilling the following condisitions:*
(bfo_1) $\Omega_A \tilde{\subseteq} \mathfrak{C}(\Omega_A)$.
(bfo_2) $\mathfrak{C}(\mathfrak{C}(\Omega_A)) = \mathfrak{C}(\Omega_A)$.
(bfo_3) $\mathfrak{C}(\Omega_A \tilde{\cup} \Gamma_B) = \mathfrak{C}(\Omega_A) \tilde{\cup} \mathfrak{C}(\Gamma_B)$.
(bfo_4) $\phi_A = \mathfrak{C}(\phi_A)$.
Then, we obtain a BFS-topology as below:

$$\tau = \{\Omega_A \in (BF^U)^E : \mathfrak{C}((\Omega_A)^c) = (\Omega_A)^c\}.$$

Also, considering this BFS-topology, we establish that $\overline{\Omega_A} = \mathfrak{C}(\Omega_A)$ for every $\Omega_A \in (BF^U)^E$.
We refer to the operator $\mathfrak{C}$ as the BFS-closure operator.

Proof. ($bfst_1$) From the condition (bfo_1), we have $(\phi_A)^c \tilde{\subseteq} \mathfrak{C}((\phi_A)^c)$, which indicates that $(\phi_A)^c = \mathfrak{C}((\phi_A)^c)$. Therefore, $\phi_A \in \tau$. Also, by (bfo_4), we have $(U_E)^c = \mathfrak{C}((U_E)^c)$. Hence, $U_E \in \tau$.
($bfst_2$) Let $\Omega_A, \Gamma_B \in \tau$. According to the definition of τ, we obtain $\mathfrak{C}((\Omega_A)^c) = (\Omega_A)^c$ and $\mathfrak{C}((\Gamma_B)^c) = (\Gamma_B)^c$. From the condition of (bfo_3) and Theorem 2.8 (iii)-(iv),

$$\mathfrak{C}((\Omega_A \tilde{\cap} \Gamma_B)^c) = \mathfrak{C}((\Omega_A)^c \tilde{\cup} (\Gamma_B)^c) = \mathfrak{C}((\Omega_A)^c) \tilde{\cup} \mathfrak{C}((\Gamma_B)^c) = (\Omega_A)^c \tilde{\cup} (\Gamma_B)^c = (\Omega_A \tilde{\cap} \Gamma_B)^c.$$

Thus, $\Omega_A \tilde{\cap} \Gamma_B \in \tau$.
($bfst_3$) Let $\{\Omega^i_{A_i} : i \in I\} \subseteq \tau$. The definition of BFS-intersection gives $\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c \tilde{\subseteq} (\Omega^i_{A_i})^c$ for all $i \in I$. By (bfo_3), we obtain easily that $\mathfrak{C}$ is order preserving. Thereby, we get $\mathfrak{C}(\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c) \tilde{\subseteq} \mathfrak{C}((\Omega^i_{A_i})^c) = (\Omega^i_{A_i})^c$. Thus, $\mathfrak{C}(\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c) \tilde{\subseteq} \widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c$. On the other hand, using (bfo_1), $\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c \tilde{\subseteq} \mathfrak{C}(\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c)$. Hence, from the fact that $\mathfrak{C}((\widetilde{\bigcup}_{i\in I} \Omega^i_{A_i})^c) = \mathfrak{C}(\widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c) = \widetilde{\bigcap}_{i\in I} (\Omega^i_{A_i})^c = (\widetilde{\bigcup}_{i\in I} \Omega^i_{A_i})^c$ it follows that $\widetilde{\bigcup}_{i\in I} \Omega^i_{A_i} \in \tau$.

We shall now demonstrate that for every $\Omega_A \in (BF^U)^E$, $\overline{\Omega_A} = \mathfrak{C}(\Omega_A)$ with respect to BFS-topology τ. Due to $(\overline{\Omega_A})^c \in \tau$, we obtain $\mathfrak{C}(\overline{\Omega_A}) = \overline{\Omega_A}$. Because $\mathfrak{C}$ is order preserving, we have $\mathfrak{C}(\Omega_A) \tilde{\subseteq} \mathfrak{C}(\overline{\Omega_A}) = \overline{\Omega_A}$. On the other hand, by (bfo_2), we get $(\mathfrak{C}(\Omega_A))^c \in \tau$. Thus, since $\Omega_A \tilde{\subseteq} \mathfrak{C}(\Omega_A)$ and $\overline{\Omega_A}$ is the smallest closed set containing Ω_A, we obtain $\overline{\Omega_A} \tilde{\subseteq} \mathfrak{C}(\Omega_A)$. □

The above theorem reveals that a BFS-closure operator and a BFS topology have a close relationship.

The following lemma plays a vital role in proving our theorems.

Lemma 3.5. *Consider a BFS-proximity space (U, ρ, E). Then, the following conditions are fulfilled:*

(*i*) *If $\Omega_A \ \rho\ \Gamma_B$ and $\Delta_D \tilde{\supseteq} \Omega_A$, $\Lambda_C \tilde{\supseteq} \Gamma_B$, then $\Delta_D \ \rho\ \Lambda_C$.*

(*ii*) *$\Omega_A \ \rho\ \Omega_A$ for each $\Omega_A \neq \phi_A$.*

(*iii*) *$\Omega_A \ \rho\ U_E$ if and only if $\Omega_A \neq \phi_A$.*

Proof. It is easily proven. □

Now, in order to generate a BFS-topology from a BFS-proximity space (U, ρ, E), we take a BFS-set as described below:

$$\overline{\Omega_A} = U_E - \tilde{\cup}\{\Gamma_B \in (BF^U)^E : \Omega_A \ \overline{\rho}\ \Gamma_B\}$$

for each $\Omega_A \in (BF^U)^E$. Next, we obtain the subsequent theorem.

Theorem 3.6. *Let (U, ρ, E) be a BFS-proximity space. Then, the mapping $\Omega_A \to \overline{\Omega_A}$ has the properties (bfo_1)–(bfo_4). So, the family*

$$\tau(\rho) = \{\Omega_A \in (BF^U)^E : \overline{(\Omega_A)^c} = (\Omega_A)^c\}$$

is a BFS-topology on U_E.

Proof. Let us show that properties (bfo_1) – (bfo_4) of the mapping $\Omega_A \to \overline{\Omega_A}$ hold.
(bfo_1) It is clear that $\Omega_A = \phi_A$. Let $\Omega_A \neq \phi_A$. Choose all $\Gamma_B \in (BF^U)^E$ with $\Omega_A \ \overline{\rho}\ \Gamma_B$. Since, by $(bfsp_2)$, we have $\Omega_A \tilde{\cap} \Gamma_B = \phi_A$. From this, it follows that $\delta^+_{\Omega(e)}(u) = 0$ or $\delta^+_{\Gamma(e)}(u) = 0$ and likewise, $\delta^-_{\Omega(e)}(u) = 0$ or $\delta^-_{\Gamma(e)}(u) = 0$ for any $e \in E$ and any $u \in U$. Then, we have

$$\delta^+_{\bigvee\{\Gamma(e):\ \Omega_A\ \overline{\rho}\ \Gamma_B\}}(u) \leq 1 - \delta^+_{\Omega(e)}(u) \quad (1)$$

$$\delta^-_{\bigvee\{\Gamma(e):\ \Omega_A\ \overline{\rho}\ \Gamma_B\}}(u) \geq -1 - \delta^-_{\Omega(e)}(u) \quad (2)$$

because $\delta^+_{\Gamma(e)}(u) \leq 1 - \delta^+_{\Omega(e)}(u)$ and $\delta^-_{\Gamma(e)}(u) \geq -1 - \delta^-_{\Omega(e)}(u)$. Thus, by (1) and (2), we get $\Omega_A \tilde{\subseteq} \overline{\Omega_A}$.

(bfo_2) It suffices to demonstrate that $\Gamma_B \ \rho\ \Omega_A$ if and only if $\Gamma_B \ \rho\ \overline{\Omega_A}$. Necessity is obvious by the Lemma 3.5 (*i*). For sufficiency, let $\Gamma_B \ \rho\ \overline{\Omega_A}$. Suppose that $\Gamma_B \ \overline{\rho}\ \Omega_A$. Using the property $(bfsp_5)$, there exists a $\Lambda_C \in (BF^U)^E$ such that $\Gamma_B \ \overline{\rho}\ \Lambda_C$ and $\Omega_A \ \overline{\rho}\ (U_E - \Lambda_C)$. Since $\overline{\Omega_A} \tilde{\nsubseteq} \Lambda_C$, there are an $e \in E$ and a $u \in U$ with $\delta^+_{\Lambda(e)}(u) < \delta^+_{\overline{\Omega}(e)}(u)$ or $\delta^-_{\Lambda(e)}(u) > \delta^-_{\overline{\Omega}(e)}(u)$. We now take the numbers α, β satisfying $\delta^+_{\Lambda(e)}(u) < \alpha < \delta^+_{\overline{\Omega}(e)}(u)$ or $\delta^-_{\Lambda(e)}(u) > \beta > \delta^-_{\overline{\Omega}(e)}(u)$. Taking into account the first term, we can choose a BFS-point $e_u^{(1-\alpha,0)} \tilde{\in} U_E$. Due to $1 - \alpha \leq 1 - \delta^+_{\Lambda(e)}(u)$, we get $e_u^{(1-\alpha,0)} \tilde{\subseteq}\ U_E - \Lambda_C$. Furthermore, we obtain $e_u^{(1-\alpha,0)} \rho\ \Omega_A$ because, otherwise, we would have $\delta^+_{\overline{\Omega}(e)}(u) \leq 1 - (1 - \alpha) = \alpha$. A situation that is untenable. Given that $e_u^{(1-\alpha,0)} \rho\ \Omega_A$ and $e_u^{(1-\alpha,0)} \tilde{\subseteq}\ U_E - \Lambda_C$. It follows that $\Omega_A \ \rho\ (U_E - \Lambda_C)$. This contradicts the fact that $\Omega_A \ \overline{\rho}\ (U_E - \Lambda_C)$.

(bfo_3) It is simple to confirm that $\overline{\Omega_A \tilde{\cup} \Gamma_B} \tilde{\supseteq} \overline{\Omega_A} \tilde{\cup} \overline{\Gamma_B}$. On the other hand, assume that there are an $e \in E$ and a $u \in U$ such that $\delta^+_{\overline{\Omega \vee \Gamma}(e)}(u) > \delta^+_{(\overline{\Omega} \vee \overline{\Gamma})(e)}(u)$ or $\delta^-_{\overline{\Omega \vee \Gamma}(e)}(u) < \delta^-_{(\overline{\Omega} \vee \overline{\Gamma})(e)}(u)$. Considering the first term, we select an ϵ where

$$\alpha = \delta^+_{\overline{(\Omega \vee \Gamma)}(e)}(u) > \delta^+_{(\overline{\Omega} \vee \overline{\Gamma})(e)}(u) + \epsilon.$$

Let $\delta^{+}_{\overline{\Omega(e)}}(u) \geq \delta^{+}_{\overline{\Gamma(e)}}(u)$. Since $\delta^{+}_{\overline{\Omega(e)}}(u) = 1 - \delta^{+}_{\bigvee\{\Lambda(e):\, \Omega_A\, \overline{\rho}\, \Lambda_C\}}(u) < \alpha - \epsilon$, there occurs a $\Lambda_C \in (BF^U)^E$ with $1 - \delta^{+}_{\Lambda(e)}(u) < \alpha - \epsilon$ and $\Omega_A\, \overline{\rho}\, \Lambda_C$. Also, by

$$1 - \delta^{+}_{\Lambda(e)}(u) \geq \delta^{+}_{\overline{\Omega(e)}}(u) \geq \delta^{+}_{\overline{\Gamma(e)}}(u) > \delta^{+}_{\overline{\Gamma(e)}}(u) - \frac{\epsilon}{2}.$$

Therefore, we obtain $1-\delta^{+}_{\Lambda(e)}(u)+\frac{\epsilon}{2} > \delta^{+}_{\overline{\Gamma(e)}}(u)$. Given that $\delta^{+}_{\overline{\Gamma(e)}}(u) = 1-\delta^{+}_{\bigvee\{\Delta(e):\, \Gamma_B\, \overline{\rho}\, \Delta_D\}}(u)$, there is a $\Delta_D \in (BF^U)^E$ such that $\Delta_D\, \overline{\rho}\, \Gamma_B$ and $\delta^{+}_{\Lambda(e)}(u)-\frac{\epsilon}{2} < \delta^{+}_{\Delta(e)}(u)$. Due to Lemma 3.5 (i), we have $(\Lambda_C \tilde{\cap} \Delta_D)\, \overline{\rho}\, \Omega_A$ and $(\Lambda_C \tilde{\cap} \Delta_D)\, \overline{\rho}\, \Gamma_B$. So, we get $(\Lambda_C \tilde{\cap} \Delta_D)\, \overline{\rho}\, (\overline{\Omega_A \tilde{\cup} \Gamma_B})$. According to $(bfsp_2)$, we know that $\delta^{+}_{\overline{\Omega \vee \Gamma(e)}}(u) \leq 1-\delta^{+}_{(\Lambda \wedge \Gamma)(e)}(u)$. Moreover, we obtain $\delta^{+}_{\Lambda(e)}(u) - \frac{\epsilon}{2} < \delta^{+}_{(\Delta \wedge \Lambda)(e)}(u)$. Hence,

$$\alpha = \delta^{+}_{\overline{\Omega \vee \Gamma(e)}}(u) \leq 1 - \delta^{+}_{(\Lambda \wedge \Gamma)(e)}(u) < 1 - \delta^{+}_{\Lambda(e)}(u) + \frac{\epsilon}{2} < \alpha - \epsilon + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2},$$

which results in a contradiction. The case of the second term is similar.
(bfo_4) Since $\phi_A\, \overline{\rho}\, U_E$, we easily obtain $\overline{\phi_A} = \phi_A$. □

Definition 3.7. Consider a BFS-proximity space (U, ρ, E) and let $\Omega_A, \Gamma_B \in (BF^U)^E$. If $\Omega_A\, \overline{\rho}\, (U_E - \Gamma_B)$, then the BFS-set Γ_B is called a BFS-ρ-neighborhood of Ω_A. This can be expressed symbolically as $\Omega_A \Subset \Gamma_B$. We demonstrate the negation of $\Subset$ with $\not\Subset$.

Theorem 3.8. *Consider a BFS-proximity space (U, ρ, E). The relation $\Subset$ fulfills the following properties:*
$(bfspn_1)$ $\phi_A \Subset \Omega_A$.
$(bfspn_2)$ If $\Omega_A \Subset \Gamma_B$, then $(U_E - \Gamma_B) \Subset (U_E - \Omega_A)$.
$(bfspn_3)$ If $\Omega_A \Subset \Gamma_B$, then $\Omega_A \tilde{\cap} (\Gamma_B)^c = \phi_A$.
$(bfspn_4)$ $\Omega_A \Subset (\Gamma_B \tilde{\cap} \Lambda_C)$ if and only if $\Omega_A \Subset \Gamma_B$ and $\Omega_A \Subset \Lambda_C$.
$(bfspn_5)$ If $\Omega_A \tilde{\subseteq} \Gamma_B \Subset \Lambda_C \tilde{\subseteq} \Delta_D$, then $\Omega_A \Subset \Delta_D$.
$(bfspn_6)$ If $\Omega_A \Subset \Gamma_B$, then there is a $\Lambda_C \in (BF^U)^E$ with $\Omega_A \Subset \Lambda_C \Subset \Gamma_B$.

Proof. $(bfspn_1)$ is clear.

$(bfspn_2)$ Consider $\Omega_A \Subset \Gamma_B$. Then, we have $\Omega_A\, \overline{\rho}\, (U_E - \Gamma_B)$. From $(bfsp_3)$, we obtain $(U_E - \Gamma_B)\, \overline{\rho}\, \Omega_A$; in other words, $(U_E - \Gamma_B) \Subset (U_E - \Omega_A)$.

$(bfspn_3)$ If $\Omega_A \Subset \Gamma_B$, from $(bfsp_2)$, then we have $\Omega_A \tilde{\cap} (\Gamma_B)^c = \phi_A$.

$(bfspn_4)$ The criterion is met since

$$\begin{aligned}
\Omega_A \Subset (\Gamma_B \tilde{\cap} \Lambda_C) &\Leftrightarrow \Omega_A\, \overline{\rho}\, (\Gamma_B \tilde{\cap} \Lambda_C)^c \\
&\Leftrightarrow \Omega_A\, \overline{\rho}\, (\Gamma_B)^c \tilde{\cap} (\Lambda_C)^c \\
&\Leftrightarrow \Omega_A\, \overline{\rho}\, (\Gamma_B)^c \ \ and\ \ \Omega_A\, \overline{\rho}\, (\Lambda_C)^c \\
&\Leftrightarrow \Omega_A \Subset \Gamma_B \ \ and\ \ \Omega_A \Subset \Lambda_C.
\end{aligned}$$

$(bfspn_5)$ Suppose that $\Omega_A \not\Subset \Delta_D$. So, we have $\Omega_A\, \rho\, (U_E - \Delta_D)$. Given that $\Omega_A \tilde{\subseteq} \Gamma_B$ and $(U_E - \Delta_D) \tilde{\subseteq} (U_E - \Lambda_C)$, we get $\Gamma_B\, \rho\, (U_E - \Lambda_C)$. Thus, $\Gamma_B \not\Subset \Lambda_C$, leading to a contradiction.

$(bfspn_6)$ If $\Omega_A \Subset \Gamma_B$, then $\Omega_A\, \overline{\rho}\, (U_E - \Gamma_B)$. From $(bfsp_5)$, there exists a $\Delta_C \in (BF^U)^E$ such that $\Omega_A\, \overline{\rho}\, (U_E - \Lambda_C)$ and $\Lambda_C\, \overline{\rho}\, (U_E - \Gamma_B)$. Hence, we obtain $\Omega_A \Subset \Lambda_C \Subset \Gamma_B$. □

Theorem 3.9. *Consider (U, ρ, E) as a BFS-proximity space and let $\Omega_A, \Gamma_B \in (BF^U)^E$. Then, the following properties are valid:*

(*i*) *$\Omega_A \Subset \Gamma_B$ if and only if $\overline{\Omega_A} \Subset \Gamma_B$.*

(*ii*) *If $\Omega_A \Subset \Gamma_B$, then there is a $\Delta_D \in \tau(\rho)$ such that $\Omega_A \tilde{\subseteq} \Delta_D \tilde{\subseteq} \Gamma_B$.*

(*iii*) *If $\Omega_A\, \overline{\rho}\, \Gamma_B$, then there exist the BFS-sets Λ_C and Δ_D where $\Omega_A \Subset \Lambda_C$, $\Gamma_B \Subset \Delta_D$ and $\Lambda_C\, \overline{\rho}\, \Delta_D$.*

Proof. (i) It is clear from the fact $\Gamma_B \,\overline{\rho}\, \Omega_A \Leftrightarrow \Gamma_B \,\overline{\rho}\, \overline{\Omega_A}$ that we proved in the Theorem 3.6.
(ii) If $\Omega_A \Subset \Gamma_B$, then $\Omega_A \,\overline{\rho}\, (U_E - \Gamma_B)$ and we have

$$\overline{(U_E - \Gamma_B)} = U_E - \tilde{\cup}\{\Lambda_C \in (BF^U)^E : \Lambda_C \,\overline{\rho}\, U_E - \Gamma_B\} \tilde{\subseteq} U_E - \Omega_A. \tag{3}$$

Let $\Delta_D = U_E - (\overline{U_E - \Gamma_B})$. It is immediately seen that

$$\overline{(U_E - \Delta_D)} = \overline{\overline{(U_E - \Gamma_B)}} = \overline{(U_E - \Gamma_B)} = (U_E - \Delta_D). \tag{4}$$

When (3) and (4) are combined, we get a $\Delta_D \in \tau(\rho)$ with $\Omega_A \tilde{\subseteq} \Delta_D \tilde{\subseteq} \Gamma_B$.
(iii) In the event that $\Omega_A \,\overline{\rho}\, \Gamma_B$, by $(bfsp_5)$, there exists an $\Delta_D \in (BF^U)^E$ such that $\Omega_A \,\overline{\rho}\, \Delta_D$ and $\Gamma_B \,\overline{\rho}\, (U_E - \Delta_D)$. Since $\Delta_D \,\overline{\rho}\, \Omega_A$, there is a BFS-set Λ_C such that $\Delta_D \,\overline{\rho}\, \Lambda_C$ and $\Omega_A \,\overline{\rho}\, (U_E - \Lambda_C)$. Hence, there occur the BFS-sets Λ_C and Δ_D satisfying $\Omega_A \Subset \Lambda_C$, $\Gamma_B \Subset \Delta_D$ and $\Lambda_C \,\overline{\rho}\, \Delta_D$. □

Theorem 3.10. *Consider the relation $\Subset$ on $(BF^U)^E$ satisfying the conditions $(bfspn_1) - (bfspn_6)$. According to the following formula, ρ is a BFS-proximity on U_E :*

$$\Omega_A \,\overline{\rho}\, \Gamma_B \Leftrightarrow \Omega_A \Subset (U_E - \Gamma_B).$$

Also, depending on the BFS-proximity defined above, Γ_B is a BFS-ρ-neighborhood of Ω_A if and only if $\Omega_A \Subset \Gamma_B$.

Proof. Our first step is to validate the axioms $(bfsp_1) - (bfsp_5)$.
$(bfsp_1)$ If $\Omega_A \in (BF^U)^E$, then we have $\phi_A \Subset U_E - \Omega_A$ by $(bfspn_1)$. From here, we get $\phi_A \,\overline{\rho}\, \Omega_A$.
$(bfsp_2)$ If $\Omega_A \,\overline{\rho}\, \Gamma_B$, then from formula above, we obtain $\Omega_A \,\overline{\rho}\, (U_E - \Gamma_B)$. By $(bfspn_3)$, $\Omega_A \tilde{\cap} \Gamma_B = \Omega_A \tilde{\cap} ((\Gamma_B)^c)^c = \phi_A$.
$(bfsp_3)$ Let $\Omega_A \,\overline{\rho}\, \Gamma_B$. Then, we get $\Omega_A \Subset (\Gamma_B)^c$. From $(bfspn_2)$, $\Gamma_B \Subset (\Omega_A)^c$ and so that $\Gamma_B \,\overline{\rho}\, \Omega_A$.
$(bfsp_4)$ Take $\Omega_A \,\overline{\rho}\, (\Gamma_B \tilde{\cup} \Lambda_C)$. Therefore, $\Omega_A \Subset (U_E - (\Gamma_B \tilde{\cup} \Lambda_C))$. Using $(bfspn_4)$, we obtain $\Omega_A \Subset (U_E - \Gamma_B)$ and $\Omega_A \Subset (U_E - \Lambda_C)$. Hence, $\Omega_A \,\overline{\rho}\, \Gamma_B$ and $\Omega_A \,\overline{\rho}\, \Lambda_C$.
$(bfsp_5)$ Consider $\Omega_A \,\overline{\rho}\, \Gamma_B$. So, we have $\Omega_A \Subset (U_E - \Gamma_B)$. Hence, with the help of $(bfspn_6)$, there is a BFS-set Λ_C such that $\Omega_A \Subset \Lambda_C \Subset (U_E - \Gamma_B)$. Thus, $\Omega_A \,\overline{\rho}\, (U_E - \Lambda_C)$ and $\Lambda_C \,\overline{\rho}\, \Gamma_B$. □

Theorem 3.11. *Let (U, ρ, E) be a BFS-proximity space and $\Omega_A \in (BF^U)^E$. Then,*

$$\overline{\Omega_A} = \tilde{\cap}\{\Gamma_B \in (BF^U)^E : \Omega_A \Subset \Gamma_B\}.$$

Proof. Firstly, we will demonstrate that $\overline{\Omega_A} \tilde{\subseteq} \tilde{\cap}\{\Gamma_B \in (BF^U)^E : \Omega_A \Subset \Gamma_B\}$. Consider a BFS-set Γ_B such that $\Omega_A \Subset \Gamma_B$. According to Theorem 3.9 and $(bfspn_3)$, we have $\overline{\Omega_A} \tilde{\subseteq} \Gamma_B$. So, it follows that $\overline{\Omega_A} \tilde{\subseteq} \tilde{\cap}\{\Gamma_B \in (BF^U)^E : \Omega_A \Subset \Gamma_B\}$. To finish the proof, we need to show that $\overline{\Omega_A} \tilde{\supseteq} \tilde{\cap}\{\Gamma_B \in (BF^U)^E : \Omega_A \Subset \Gamma_B\}$. Suppose that there exist an $e \in E$ and a $u \in U$ such that $\delta^+_{\bigwedge\{\Gamma(e):\, \Omega_A \Subset \Gamma_B\}}(u) > \delta^+_{\overline{\Omega(e)}}(u)$. In that case, there is an $\epsilon > 0$ such that

$$\delta^+_{\overline{\Omega(e)}}(u) = 1 - \delta^+_{\bigvee\{\Lambda(e):\, \Omega_A \,\overline{\rho}\, \Lambda_C\}}(u) < \delta^+_{\bigwedge\{\Gamma(e):\, \Omega_A \Subset \Gamma_B\}}(u) - \epsilon.$$

Hence, the inequality

$$1 - \delta^+_{\Delta(e)}(u) < \delta^+_{\bigwedge\{\Gamma(e):\, \Omega_A \Subset \Gamma_B\}}(u) - \epsilon$$

is satisfied, where $\Omega_A \,\overline{\rho}\, \Delta_D$ for some BFS-set Δ_D. Since $\Omega_A \,\overline{\rho}\, \Delta_D$, we obtain $\Omega_A \Subset (U_E - \Delta_D)$, and so that $\tilde{\cap}\{\Gamma_B \in (BF^U)^E : \Omega_A \Subset \Gamma_B\} \tilde{\subseteq} (U_E - \Delta_D)$. For that reason,

$$\delta^+_{\bigwedge\{\Gamma(e):\, \Omega_A \Subset \Gamma_B\}}(u) \leq 1 - \delta^+_{\Delta(e)}(u) < \delta^+_{\bigwedge\{\Gamma(e):\, \Omega_A \Subset \Gamma_B\}}(u) - \epsilon,$$

leading to a contradiction. □

Definition 3.12. Consider two BFS-proximity spaces (U, ρ_1, E) and (V, ρ_2, K). If a BFS-mapping $\mathfrak{f} : (U, \rho_1, E) \to (V, \rho_2, K)$ holds the following condition

$$\Omega_A \; \rho_1 \; \Gamma_B \Rightarrow \mathfrak{f}(\Omega_A) \; \rho_2 \; \mathfrak{f}(\Gamma_B)$$

for any $\Omega_A, \Gamma_B \in (BF^U)^E$, then it is called a BFS-proximity mapping.

The following propositions are simply proven by using the definition given above.

Proposition 3.13. *Given two BFS-proximity spaces (U, ρ_1, E) and (V, ρ_2, D). A BFS-mapping $\mathfrak{f} : (U, \rho_1, E) \to (V, \rho_2, D)$ is a BFS-proximity mapping if and only if*

$$\Lambda_C \; \overline{\rho_2} \; \Delta_D \Rightarrow \mathfrak{f}^{-1}(\Lambda_C) \; \overline{\rho_1} \; \mathfrak{f}^{-1}(\Delta_D)$$

or equivalently

$$\Lambda_C \Subset_2 \Delta_D \Rightarrow \mathfrak{f}^{-1}(\Lambda_C) \Subset_1 \mathfrak{f}^{-1}(\Delta_D).$$

Proposition 3.14. *A composition of two BFS-proximity mappings forms a BFS-proximity mapping.*

Theorem 3.15. *A BFS-proximity mapping $\mathfrak{f} : (U, \rho_1, E) \to (V, \rho_2, K)$ is BFS-continuous according to the BFS-topologies $\tau(\rho_1)$ and $\tau(\rho_2)$ on U_E and V_K, respectively.*

Proof. If we can demonstrate that $\mathfrak{f}^{-1}(\Omega_A) \in \tau(\rho_1)$ when $\Omega_A \in \tau(\rho_2)$, the proof ends. Let $\Lambda_C \in (BF^U)^E$ be any BFS-set that $\Lambda_C \; \overline{\rho_2} \; (V_K - \Omega_A)$. By Proposition 3.13, we get $\mathfrak{f}^{-1}(\Lambda_C) \; \overline{\rho_1} \; (U_E - \mathfrak{f}^{-1}(\Omega_A))$. According to $(bfsp_2)$, it can be deduced that $\overline{U_E - \mathfrak{f}^{-1}(\Omega_A)} \tilde{\subseteq} U_E - \mathfrak{f}^{-1}(\Lambda_C)$. Then, we have

$$\delta^+_{\overline{(U_E - \mathfrak{f}^{-1}(\Omega))}(e)}(u) \leq \delta^+_{(U_E - \mathfrak{f}^{-1}(\Lambda))(e)}(u) = 1 - \delta^+_{\mathfrak{f}^{-1}(\Lambda)(e)}(u) = 1 - \delta^+_{\Lambda(\mathfrak{g}(e))}\mathfrak{u}(u) = \delta^+_{(V_K - \Lambda)(\mathfrak{g}(e))}\mathfrak{u}(u)$$

for all $e \in E$ and all $u \in U$. Thus,

$$\begin{aligned}
\delta^+_{\overline{(U_E - \mathfrak{f}^{-1}(\Omega))}(e)}(u) &\leq \delta^+_{\bigwedge\{(V_K - \Lambda)(\mathfrak{g}(e)) :\, (V_K - \Omega_A) \; \overline{\rho_2} \; \Lambda_C\}}(u) \\
&= \delta^+_{\overline{(V_K - \Omega)}\mathfrak{g}(e)}\mathfrak{u}(u) \\
&= \delta^+_{(V_K - \Omega)\mathfrak{g}(e)}\mathfrak{u}(u) \\
&= 1 - \delta^+_{\Omega(\mathfrak{g}(e))}\mathfrak{u}(u) \\
&= 1 - \delta^+_{\mathfrak{f}^{-1}(\Omega)(e)}(u) \\
&= \delta^+_{(U_E - \mathfrak{f}^{-1}(\Omega))(e)}(u).
\end{aligned}$$

Hence, since $\overline{U_E - \mathfrak{f}^{-1}(\Omega_A)} \tilde{\subseteq} U_E - \mathfrak{f}^{-1}(\Omega_A)$, we get $\mathfrak{f}^{-1}(\Omega_A) \in \tau(\rho_1)$. □

4. Initial BFS-proximity structure

Here, we present the concept of an initial BFS-proximity structure. Next, we construct the existence of this structure and establish a characterization of it.

Firstly, let us give the idea of comparing BFS-proximity structures: Consider two BFS-proximities ρ_1 and ρ_2 on U_E. The relationship between ρ_1 and ρ_2, as defined below, is expressed by saying that ρ_2 is finer than ρ_1 or ρ_1 is coarser than ρ_2:

$$\rho_1 < \rho_2 \Leftrightarrow \Omega_A \; \rho_2 \; \Gamma_B \text{ implies } \Omega_A \; \rho_1 \; \Gamma_B.$$

Definition 4.1. Let $\{(U_\alpha, \rho_\alpha, E_\alpha) : \alpha \in I\}$ be a collection of the BFS-proximity spaces and let $\mathfrak{f}_\alpha : (BF^U)^E \to (U_\alpha, \rho_\alpha, E_\alpha)$ be a BFS-mapping for each $\alpha \in I$. The coarsest BFS-proximity ρ on U_E for which all BFS-mappings $\mathfrak{f}_\alpha : (U, \rho, E) \to (U_\alpha, \rho_\alpha, E_\alpha)$ $(\alpha \in I)$ are BFS-proximity mapping is called the initial BFS-proximity.

Theorem 4.2. *Let $\{(U_\alpha, \rho_\alpha, E_\alpha) : \alpha \in I\}$ be a collection of the BFS-proximity spaces and let $\mathfrak{f}_\alpha : (BF^U)^E \to (U_\alpha, E_\alpha, \rho_\alpha)$ be a BFS-mapping for each $\alpha \in I$. The initial BFS-proximity on U_E is determined by the binary relation ρ as follows: For all $\Omega_A, \Gamma_B \in (BF^U)^E$,*

$$\Omega_A \ \rho\ \Gamma_B \Leftrightarrow \textit{for any finite families } \{\Omega^i_{A_i} : i = 1, ..., n\} \textit{ and } \{\Gamma^j_{B_j} : j = 1, ..., m\} \textit{ where}$$
$$\Omega_A = \widetilde{\bigcup}_{i=1}^{n} \Omega^i_{A_i} \textit{ and } \Gamma_B = \widetilde{\bigcup}_{j=1}^{m} \Gamma^j_{B_j}, \textit{ there exist an } \Omega^i_{A_i} \textit{ and a } \Gamma^j_{B_j} \textit{ such that}$$
$$\mathfrak{f}_\alpha(\Omega^i_{A_i}) \ \rho_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j}) \textit{ for all } \alpha \in I.$$

Proof. Firstly, we show that ρ is a BFS-proximity on U_E.

($bfsp_1$) is clear.

($bfsp_2$) If $\Omega_A\ \overline{\rho}\ \Gamma_B$, then there exist finite covers $\Omega_A = \widetilde{\bigcup}_{i=1}^{n}\Omega^i_{A_i}$ and $\Gamma_B = \widetilde{\bigcup}_{j=1}^{m}\Gamma^j_{B_j}$ of Ω_A and Γ_B, respectively, such that $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for some $\alpha = s_{ij} \in I$ where $i = 1, ..., n$ and $j = 1, ..., m$. We know that each ρ_α is a BFS-proximity. From here, we get $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \tilde{\cap}\ \mathfrak{f}_\alpha(\Gamma^j_{B_j}) = \phi_A$. Thereby, it can be deduced that

$$\mathfrak{f}_\alpha\left(\widetilde{\bigcup}_{i=1}^{n}\Omega^i_{A_i}\right) \tilde{\cap}\ \mathfrak{f}_\alpha\left(\widetilde{\bigcup}_{j=1}^{m}\Gamma^j_{B_j}\right) = \mathfrak{f}_\alpha(\Omega_A)\ \tilde{\cap}\ \mathfrak{f}_\alpha(\Gamma_B) = \phi_A.$$

Hence, $\Omega_A \tilde{\cap}\ \Gamma_B = \phi_A$.

($bfsp_3$) It is evident that $\Omega_A\ \rho\ \Gamma_B$ implies $\Gamma_B\ \rho\ \Omega_A$ because all ρ_α is a BFS-proximity.

($bfsp_4$) Firstly, let us demonstrate that the sufficient condition is satisfied. It can be readily confirmed that if $\Omega_A\ \rho\ \Gamma_B$, then we have $\Omega_A\ \rho\ \Lambda_C$ for any $\Lambda_C \tilde{\supseteq} \Gamma_B$. When $\Omega_A\ \rho\ \Gamma_B$ or $\Omega_A\ \rho\ \Lambda_C$, then $\Omega_A\ \rho\ (\Gamma_B\ \tilde{\cup}\ \Lambda_C)$ is fulfilled. On the other hand, suppose that $\Omega_A\ \overline{\rho}\ \Gamma_B$ and $\Omega_A\ \overline{\rho}\ \Lambda_C$. Following that, there are finite covers $\Omega_A = \widetilde{\bigcup}_{i=1}^{n}\Omega^i_{A_i}$ and $\Gamma_B = \widetilde{\bigcup}_{j=1}^{m}\Gamma^j_{B_j}$ of Ω_A and Γ_B, respectively, such that $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for some $\alpha = s_{ij} \in I$ where $i = 1, ..., n$ and $j = 1, ..., m$. Similarly, there exist finite covers $\Omega_A = \widetilde{\bigcup}_{p=1}^{q}\Delta^p_{D_p}$ and $\Lambda_C = \widetilde{\bigcup}_{j=m+1}^{m+l}\Gamma^j_{B_j}$ of Ω_A and Λ_C, respectively, such that $\mathfrak{f}_\alpha(\Delta^p_{D_p})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for some $\alpha = t_{pj} \in I$ where $p = 1, ..., q$ and $j = m + 1, ..., m + l$. In this case, $\{\Omega^i_{A_i}\ \tilde{\cap}\ \Delta^p_{D_p} : i = 1, ..., n; p = 1, ..., q\}$ and $\{\Gamma^j_{B_j} : j = 1, ..., m + l\}$ are finite covers of Ω_A and $\Gamma_B\ \tilde{\cup}\ \Lambda_C$, respectively. Thus, by Lemma 3.5 (i), we obtain $\Omega_A\ \overline{\rho}\ (\Gamma_B\ \tilde{\cup}\ \Lambda_C)$ because of $\mathfrak{f}_\alpha(\Omega^i_{A_i}\ \tilde{\cap}\ \Delta^p_{D_p})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for $\alpha = s_{ij} \in I$ or $\alpha = t_{pj} \in I$.

($bfsp_5$) To establish the last axiom consider the set Ψ to including all pairs (Ω_A, Γ_B) where $\Omega_A\ \overline{\rho}\ \Gamma_B$ and for every $\Lambda_C \in (BF^U)^E$ we have either $\Omega_A\ \overline{\rho}\ \Lambda_C$ or $\Gamma_B\ \overline{\rho}\ (U_E - \Lambda_C)$. The goal of the proof is to determine that Ψ is an empty set. Suppose that $(\Omega_A, \Gamma_B) \in \Psi$. Then, we have $\mathfrak{f}_\alpha(\Omega_A)\ \rho_\alpha\ \mathfrak{f}_\alpha(\Gamma_B)$ for any $\alpha \in I$. In fact, consider a $\Lambda_C \in (BF^{U_\alpha})^{E_\alpha}$ and a $\Delta_D = \mathfrak{f}_\alpha^{-1}(\Lambda_C) \in (BF^U)^E$. If $\Omega_A\ \rho\ \Delta_D$, then $\mathfrak{f}_\alpha(\Omega_A)\ \rho_\alpha\ \mathfrak{f}_\alpha(\Delta_D)$. Since $\mathfrak{f}_\alpha(\Delta_D) \tilde{\subseteq} \Lambda_C$ and from Lemma 3.5 (i), we get $\mathfrak{f}_\alpha(\Omega_A)\ \rho_\alpha\ \Lambda_C$. Likewise, in the event that $\Gamma_B\ \rho\ (U_E - \Delta_D)$ we have $\mathfrak{f}_\alpha(\Gamma_B)\ \rho_\alpha\ (U_{E_\alpha} - \Lambda_C)$. Thereby, due to ρ_α being a BFS-proximity on U_α, from the axiom ($bfsp_5$), we obtain $\mathfrak{f}_\alpha(\Omega_A)\ \rho_\alpha\ \mathfrak{f}_\alpha(\Gamma_B)$. Furthermore, it is evident that for every $(\Omega_A, \Gamma_B) \in \Psi$, there are the positive integers n, m satisfiying the covers $\Omega_A = \widetilde{\bigcup}_{i=1}^{n}\Omega^i_{A_i}$ and $\Gamma_B = \widetilde{\bigcup}_{j=1}^{m}\Gamma^j_{B_j}$ such that an $\alpha \in I$ exists along with $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for all $(i, j) \in \{1, ..., n\} \times \{1, ..., m\}$. Choose $l = n + m$, which is straightforward to recognize as $l > 2$. Therefore, for every $(\Omega_A, \Gamma_B) \in \Psi$, we select an integer l like that. However, (Ω_A, Γ_B) does not uniquely guarantee l. Consider Θ to be the collection of all intergers that correspond to Ψ's members, and consider l to be Θ's smallest element. Now, let us take an $(\Omega_A, \Gamma_B) \in \Psi$ corresponding to the smallest integer $l \in \Theta$. Then, there occur the covers $\Omega_A = \widetilde{\bigcup}_{i=1}^{n}\Omega^i_{A_i}$ and $\Gamma_B = \widetilde{\bigcup}_{j=1}^{m}\Gamma^j_{B_j}$ where $l = n + m$ and for any pair $(i, j) \in \{1, ..., n\} \times \{1, ..., m\}$, there is an $\alpha \in I$ with $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \overline{\rho}_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$. At least one of the n, m has a value larger than 1. Let $n > 1$ and take

$\Omega'_{A'} = \Omega^1_{A_1} \tilde{\cup} ... \tilde{\cup}\, \Omega^{n-1}_{A_{n-1}}$. In this context, one of the following circumstances should be valid:

(*i*) For each $\Lambda_C \in (BF^U)^E$, the options are $\Omega'_{A'}\ \rho\ \Lambda_C$ or $\Gamma_B\ \rho\ (U_E - \Lambda_C)$,

(*ii*) For each $\Lambda_C \in (BF^U)^E$, the options are $\Omega^n_{A_n}\ \rho\ \Lambda_C$ or $\Gamma_B\ \rho\ (U_E - \Lambda_C)$.

Indeed, let us suppose that both (i) and (ii) are false. Then, there exist $\Lambda^1_{C_1}, \Lambda^2_{C_2} \in (BF^U)^E$ such that $\Omega'_A\ \overline{\rho}\ \Lambda^1_{C_1}, \Gamma_B\ \overline{\rho}\ (U_E - \Lambda^1_{C_1})$ and $\Omega^n_{A_n}\ \overline{\rho}\ \Lambda^2_{C_2}, \Gamma_B\ \overline{\rho}\ (U_E - \Lambda^2_{C_2})$. By utilizing $\Lambda_C = \Lambda^1_{C_1} \tilde{\cap} \Lambda^2_{C_2}$, we get $\Omega_A\ \overline{\rho}\ \Lambda_C$ and $\Gamma_B\ \overline{\rho}\ (U_E - \Lambda_C)$, which goes counter to the assertion that $(\Omega_A, \Gamma_B) \in \Psi$. Let us suppose that (*i*) is true. Since $\Omega'_{A'} \tilde{\subseteq} \Omega_A$ and $\Omega_A\ \overline{\rho}\ \Gamma_B$, we obtain $\Omega'_{A'}\ \overline{\rho}\ \Gamma_B$. According to the condition (*i*), $(\Omega'_{A'}, \Gamma_B) \in \Psi$. But, this leads to a contradiction because $(n-1) + m = l - 1 \in \Theta$. In a similar manner, we have a contradiction if (*ii*) is also true. Therefore, from the fact that the set Ψ is empty it follows that ρ is a BFS-proximity on U_E.

One can easily observe that all mappings $\mathfrak{f}_\alpha : (U, \rho, E) \to (U_\alpha, \rho_\alpha, E_\alpha)$ $(\alpha \in I)$ are BFS-proximity mappings. Consider ρ^* as an additional BFS-proximity on U_E such that any mapping $\mathfrak{f}_\alpha : (U, \rho^*, E) \to (U_\alpha, \rho_\alpha, E_\alpha)$ $(\alpha \in I)$ is a BFS-proximity mapping. Showing that $\rho < \rho^*$ completes the proof. Let $\Omega_A\ \rho^*\ \Gamma_B$ and take any covers $\Omega_A = \widetilde{\bigcup}^n_{i=1} \Omega^i_{A_i}$ and $\Gamma_B = \widetilde{\bigcup}^m_{j=1} \Gamma^j_{B_j}$ of Ω_A and Γ_B, respectively. With respect to the property ($bfsp4$), there exists an $i \in \{1, ..., n\}$ with $\Omega^i_{A_i}\ \rho^*\ \Gamma_B$. Similarly, there exists a $j \in \{1, ..., m\}$ such that $\Omega^i_{A_i}\ \rho^*\ \Gamma^j_{B_j}$. Since every mapping $\mathfrak{f}_\alpha : (U, \rho^*, E) \to (U_\alpha, \rho_\alpha, E_\alpha)$ is a BFS-proximity mapping, it can be deduced that $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \rho_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for each $\alpha \in I$. Hence, we obtain $\Omega_A\ \rho\ \Gamma_B$. □

Theorem 4.3. *A BFS-mapping $\mathfrak{f} : (V, \rho^*, K) \to (U, \rho, E)$ is a BFS-proximity mapping if and only if $\mathfrak{f}_\alpha \circ \mathfrak{f} : (V, \rho^*, K) \to (U_\alpha, \rho_\alpha, E_\alpha)$ is a BFS-proximity mapping for every $\alpha \in I$.*

Proof. The necessity is straightforward. Conversely, consider $\mathfrak{f}_\alpha \circ \mathfrak{f}$ as a BFS-proximity mapping for every $\alpha \in I$. Let $\Omega_A\ \rho^*\ \Gamma_B$ and choose $\mathfrak{f}(\Omega_A) = \widetilde{\bigcup}^n_{i=1} \Omega^i_{A_i}$ and $\mathfrak{f}(\Gamma_B) = \widetilde{\bigcup}^m_{j=1} \Gamma^j_{B_j}$. Therefore, we obtain $\Omega_A \tilde{\subseteq} \widetilde{\bigcup}^n_{i=1} \mathfrak{f}^{-1}(\Omega^i_{A_i})$ and $\Gamma_B \tilde{\subseteq} \widetilde{\bigcup}^m_{j=1} \mathfrak{f}^{-1}(\Gamma^j_{B_j})$. As $\Omega_A\ \rho^*\ \Gamma_B$, according to ($bfsp_4$), there exist an $i \in \{1, ..., n\}$ and a $j \in \{1, ..., m\}$ such that $\mathfrak{f}^{-1}(\Omega^i_{A_i})\ \rho^*\ \mathfrak{f}^{-1}(\Gamma^j_{B_j})$. Since

$$\mathfrak{f}_\alpha \circ \mathfrak{f} \circ \mathfrak{f}^{-1}(\Omega^i_{A_i}) \tilde{\subseteq} \mathfrak{f}_\alpha(\Omega^i_{A_i}),$$
$$\mathfrak{f}_\alpha \circ \mathfrak{f} \circ \mathfrak{f}^{-1}(\Omega^j_{B_j}) \tilde{\subseteq} \mathfrak{f}_\alpha(\Omega^j_{B_j}),$$

by Lemma 3.5 (i), it can be inferred that $\mathfrak{f}_\alpha(\Omega^i_{A_i})\ \rho_\alpha\ \mathfrak{f}_\alpha(\Gamma^j_{B_j})$ for every $\alpha \in I$. Hence, we get $\mathfrak{f}(\Omega_A)\ \rho\ \mathfrak{f}(\Gamma_B)$. □

5. BFS-proximity generated by proximity

This part deals with the construction of a BFS-proximity structure from a proximity structure in the classical sense. Next, we determine a characterization of the relationship between these two structures.

Definition 5.1. Consider a set U and a subset Z of U. Then, a mapping $\Upsilon^Z : E \to (BF^U)^E$ is a BFS-set on U_E defined as in the following way: For all $e \in E$,

$$\Upsilon^Z(e) = \{(u, \delta^+_{\Upsilon^Z(e)}(u), \delta^-_{\Upsilon^Z(e)}(u)) : u \in U\},$$

where

$$\delta^+_{\Upsilon^Z(e)}(u) = \begin{cases} 1, & u \in Z, \\ 0, & u \notin Z, \end{cases}$$

$$\delta^-_{\Upsilon^Z(e)}(u) = \begin{cases} -1, & u \in Z, \\ 0, & u \notin Z. \end{cases}$$

Example 5.2. Let $U = \{u_1, u_2, u_3\}$, $Z = \{u_1, u_2, \}$ and $E = \{e_1, e_2, e_3\}$. Then,

$$\Upsilon^Z = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 1, -1), (u_2, 1, -1), (u_3, 0, 0)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 1, -1), (u_2, 1, -1), (u_3, 0, 0)\}\rangle, \\ \langle e_3, \Omega(e_3) = \{(u_1, 1, -1), (u_2, 1, -1), (u_3, 0, 0)\}\rangle \end{array} \right\}$$

is a BFS-set on U_E.

Lemma 5.3. *For any subsets Z, X of U,*

(*i*) $\Upsilon^Z \tilde{\cap} \Upsilon^X = \Upsilon^{Z\cap X}$,

(*ii*) $\Upsilon^Z \tilde{\cup} \Upsilon^X = \Upsilon^{Z\cup X}$,

(*iii*) $(\Upsilon^Z)^c = \Upsilon^{Z^c}$.

Proof. As it is simple, it gets omitted. □

It is worth noting that if a binary relation ρ over the power set of a set U satisfies the following axioms, it is called a proximity on U: For any subsets Z, X, Y of U,
(p_1) $\emptyset\,\overline{\rho}\,Z$.
(p_2) If $Z \cap X \neq \emptyset$, then $Z\,\rho\,X$.
(p_3) If $Z\,\rho\,X$, then $X\,\rho\,Z$.
(p_4) $Z\,\rho\,(X \cup Y)$ if and only if $Z\,\rho\,X$ or $Z\,\rho\,Y$.
(p_5) If $Z\,\overline{\rho}\,X$, then there exists a subset Y of U such that $Z\,\overline{\rho}\,Y$ and $X\,\overline{\rho}\,(U - Y)$ [24].

Theorem 5.4. *Consider a proximity space (U, ρ) in the classical meaning. The binary relation ρ_i defined below is a BFS-proximity on U_E: For each $\Omega_A, \Gamma_B \in (BF^U)^E$,*

$$\Omega_A\,\overline{\rho_i}\,\Gamma_B \Leftrightarrow \textit{there exist the subsets } Z, X \textit{ of } U \textit{ such that } \Omega_A \tilde{\subseteq} \Upsilon^Z,\ \Gamma_B \tilde{\subseteq} \Upsilon^X \textit{ and } Z\,\overline{\rho}\,X.$$

Proof. If we show that ρ satisfies the axioms $(bfsp_1) - (bfsp_5)$, the proof concludes.

$(bfsp_1)$ From Definition 5.1, we obtain that $\phi_A \tilde{\subseteq} \Upsilon^\emptyset$, $\Omega_A \tilde{\subseteq} \Upsilon^U$. Since (U, ρ) is a proximity space, we get $\emptyset\,\overline{\rho}\,U$. Thus, $\phi_A\,\overline{\rho_i}\,\Omega_A$.

$(bfsp_2)$ Consider $\Omega_A\,\overline{\rho_i}\,\Gamma_B$. Then, there exist the subsets Z and X of U such that $\Omega_A \tilde{\subseteq} \Upsilon^Z$, $\Gamma_B \tilde{\subseteq} \Upsilon^X$ and $Z\,\overline{\rho}\,X$. Due to $Z\,\overline{\rho}\,X$, we get $Z \cap X = \emptyset$. Then, by Lemma 5.3 (*i*), $\Upsilon^Z \tilde{\cap} \Upsilon^X = \phi_A$. Therefore, we have $\Omega_A \tilde{\cap} \Gamma_B = \phi_A$.

$(bfsp_3)$ is obvious.

$(bfsp_4)$ The sufficiency of the condition follows from the above definition. On the other hand, suppose that $\Omega_A\,\overline{\rho_i}\,\Gamma_B$ and $\Omega_A\,\overline{\rho_i}\,\Lambda_C$. Then, there are the subsets Z, X of U with $\Omega_A \tilde{\subseteq} \Upsilon^Z, \Gamma_B \tilde{\subseteq} \Upsilon^X$ and $Z\,\overline{\rho}\,X$. Similarly, there are the subsets Y, W of U such that $\Omega_A \tilde{\subseteq} \Upsilon^Y, \Lambda_C \tilde{\subseteq} \Upsilon^W$ and $Y\,\overline{\rho}\,W$. Because of

$$\Omega_A \tilde{\subseteq} \Upsilon^Z \tilde{\cap} \Upsilon^Y = \Upsilon^{Z\cap Y}, \quad \Gamma_B \tilde{\cup} \Lambda_C \tilde{\subseteq} \Upsilon^X \tilde{\cup} \Upsilon^W = \Upsilon^{X\cup W}$$

and by (p_4), from the fact that $(Z \cap Y)\,\overline{\rho}\,(X \cup W)$ it follows that $\Omega_A\,\overline{\rho_i}\,(\Gamma_B \tilde{\cup} \Lambda_C)$.

$(bfsp_5)$ Let $\Omega_A\,\overline{\rho_i}\,\Gamma_B$. Then, there exist the subsets Z, X of U with $\Omega_A \tilde{\subseteq} \Upsilon^Z, \Gamma_B \tilde{\subseteq} \Upsilon^X$ and $Z\,\overline{\rho}\,X$. In the event that $Z\,\overline{\rho}\,X$, by using (p_5), we can say that there is a subset Y of U such that $Z\,\overline{\rho}\,Y$ and $X\,\overline{\rho}\,(U - Y)$. Hence, we have $\Omega_A\,\overline{\rho}_i\,\Upsilon^Y$ and $\Gamma_B\,\overline{\rho}_i\,(U_E - \Upsilon^Y)$ for a BFS-set Υ^Y. □

Theorem 5.5. *Consider a BFS-proximity space (U, ρ^*, E).*

(*i*) *There exists a proximity relation ρ on U with $\rho^* = \rho_i$.*

(*ii*) *If $\Omega_A\,\overline{\rho^*}\,\Gamma_B$, there are subsets Z and X of U such that $\Omega_A \tilde{\subseteq} \Upsilon^Z$, $\Gamma_B \tilde{\subseteq} \Upsilon^X$ and $\Upsilon^Z\,\overline{\rho^*}\,\Upsilon^V$.*

(*iii*) *The relation ρ is a proximity on U as follows:*

$$Z \, \rho \, X \Leftrightarrow \Upsilon^Z \, \rho^* \, \Upsilon^X.$$

Then, (i) and (ii) are equivalent and they lead to (iii).

Proof. $(i) \Rightarrow (ii)$ is clear.

It suffices to demonstrate that ρ meets the axiom $(bfsp_5)$ to show that $(ii) \Rightarrow (iii)$, as the other axioms are easily confirmed. Let $Z \, \overline{\rho} \, X$ for any subsets Z, X of U. Due to $\Upsilon^Z \, \overline{\rho^*} \Upsilon^X$, by $(bfsp_5)$, there exists a BFS-set Ω_A such that $\Upsilon^Z \, \overline{\rho^*} \, \Omega_A$ and $\Upsilon^X \, \overline{\rho^*} \, (U_E - \Omega_A)$. From the condition (ii), there are subsets Y, W, T, P of U with $\Upsilon^Z \tilde{\sqsubseteq} \Upsilon^Y$, $\Omega_A \tilde{\sqsubseteq} \Upsilon^W$, $\Upsilon^Y \, \overline{\rho^*} \Upsilon^W$ and $\Upsilon^X \tilde{\sqsubseteq} \Upsilon^T$, $(U_E - \Omega_A) \tilde{\sqsubseteq} \Upsilon^P$, $\Upsilon^T \, \overline{\rho^*} \Upsilon^P$. Using Lemma 3.5 (i), we have $\Upsilon^Z \, \overline{\rho^*} \Upsilon^W$, which indicates that $Z \, \overline{\rho} \, W$. Similarly, since $\Omega_A \tilde{\sqsubseteq} \Upsilon^W$, we obtain $U_E - \Omega_A \tilde{\sqsupseteq} U_E - \Upsilon^W = \Upsilon^{U-W}$. Therefore, we get $\Upsilon^{U-W} \tilde{\sqsubseteq} \Upsilon^P$. By Lemma 3.5 (i), it follows that $\Upsilon^X \, \overline{\rho^*} \Upsilon^{(U-W)}$, which means that $V \, \overline{\rho} \, (U - W)$.

$(ii) \Rightarrow (i)$ To establish this implication, consider the binary relation ρ on U as below:

$$Z \, \rho \, X \Leftrightarrow \Upsilon^Z \, \rho^* \Upsilon^X.$$

We know that ρ is a proximity on U. If we show that $\rho^* = \rho_i$, the proof is completed. Let $\Omega_A \, \overline{\rho^*} \, \Gamma_B$. Then, from (ii), there are the subsets Z and X of U satisfying $\Omega_A \tilde{\sqsubseteq} \Upsilon^Z$, $\Gamma_B \tilde{\sqsubseteq} \Upsilon^X$ and $\Upsilon^Z \, \overline{\rho^*} \Upsilon^X$. Therefore, we get $Z \, \overline{\rho} \, X$ and this implies that $\Omega_A \, \overline{\rho_i} \, \Gamma_B$. Conversely, given that $\Omega_A \, \overline{\rho_i} \, \Gamma_B$. From the definition of ρ_i, there are subsets Z, X of U such that $\Omega_A \tilde{\sqsubseteq} \Upsilon^Z$, $\Gamma_B \tilde{\sqsubseteq} \Upsilon^X$ and $Z \, \overline{\rho} \, X$. Thus, we obtain $\Omega_A \, \overline{\rho^*} \, \Gamma_B$, which ends the proof. □

6. Conclusion

The proximity structure not only offer a clear and conceptual solution to many significant topological issues but also has wide applications in other fields including information technology and computer science. So, numerous researchers have found out and analyzed the stronger and weaker forms of this sturcture. In this study, we primarily introduce a BFS-proximity structure and establish certain characteristics of it. Then, using the obtained BFS-closure operator, we demonstrate how each BFS-proximity produces a BFS-topology. Additionally, we offer a different interpretation of BFS-proximity, known as BFS-ρ-neighborhood. Moreover, we characterize the connection of BFS-proximity with proximity given as classically. Next, we present initial BFS-proximity. Therefore, these theoretical works will serve as a basis for future research into novel BFS-proximity approaches and numerous application domains. Also, it is possible to examine how some of the concepts discussed here could be applied to real-life problems. Later, one can reconstruct our research into another perspectives such as bipolar fuzzy N-soft sets, bipolar fuzzy soft expert sets and rough fuzzy bipolar soft sets to study the proximity and its applications on these models.

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