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Topologically ergodic C_0-semigroups and ergodicity of their operators

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Abstract. This paper introduces topologically ergodic C_0-semigroups. It demonstrates that the set of topologically ergodic C_0-semigroups is a proper subset of the set of recurrent C_0-semigroups. It also proves that these C_0-semigroups exist in any infinite-dimensional space. In addition, the article establishes that the topological ergodicity of any operator of a C_0-semigroup implies the topological ergodicity of the C_0-semigroup and vice versa. Moreover, the research shows that the topological ergodicity of a C_0-semigroup is equivalent to the topological ergodicity of the direct sum of the C_0-semigroup with itself. Lastly, the research states some criteria for topological ergodicity that are in accordance with open sets and generalized kernels.

1. Introduction

Let us assume that X is a separable Banach space and $B(X)$ is the set of bounded and linear operators on X. Hypercyclicity and related topics are among the widely studied subjects in dynamical systems. An operator $T \in B(X)$ is called hypercyclic if there exists $x \in X$ so that $orb(T, x)$ is dense in X or equivalently for arbitrary nonempty open subsets U and V of X, there exists an integer $n \geq 0$ such that $T^n(U) \cap V \neq \phi$[2, Theorem 1.2]. Some new results about hyercyclic operators can be found in [9] and [10].

The set of all non-negative integers n with this property that $T^n(U) \cap V \neq \phi$ is denoted by $N_T(U, V)$ which is named the return set from U to V [8, Definition 1.49]. If T is hypercyclic, then $N_T(U, V)$ is infinite for any open nonempty sets U and V. If $N_T(U, V)$ is syndetic for any open sets U and V, then T would be a topologically ergodic operator[8, p.28]. As mentioned in [14] a subset A of nonnegative integers is syndetic if the strictly increasing sequence (a_n) that forms it, is syndetic; In other words:

$$\sup_{n\geq 1}(a_{n+1} - a_n) < \infty.$$

In addition, [14] introduces subspace-ergodic operators.

An interesting superset of the set of hypercyclic operators is the set of recurrent operators. If for any open nonempty set U, some $n > 0$ exists such that $T^n(U) \cap U \neq \phi$, then T would be recurrent[5]. By

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definition, any hypercyclic operator is recurrent. For more about this type of operators one can refer to [7] and [12].

C_0-semigroup is another structure that is considered in dynamical systems. A C_0-semigroup $(T_t)_{t\geq 0}$ on X, would mean a family of operators with these properties:

(i) $T_0 = I$,

(ii) $T_{s+t} = T_s T_t$,

(iii) $\lim_{s\to t} T_s x = T_t x$, for any $x \in X$.

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is hypercyclic if there exists $x \in X$ so that $orb((T_t), x) = \{T_t x : t \geq 0\}$ is dense in X or equivalently for any open nonempty sets U and V of X, there exists $t \geq 0$ such that $T_t(U) \cap V \neq \phi$[3]. In [1] other types of C_0-semigroups are presented. If $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic, then $(T_t)_{t\geq 0}$ is weakly mixing[8, Definition 7.8(b)]. Recall that if $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroups on Banach spaces X and Y respectively, then for any $x \in X$ and $y \in Y$,

$$(T_t \oplus S_t)(x, y) = T_t x \oplus S_t y.$$

As [6] mentions, a semigroup $(T_t)_{t\geq 0}$ on X satisfies the hypercyclicity criterion(HCC) if and only if for any two non-empty, open sets U and V of X and any neighborhood W of zero, $t > 0$ can be found such that

$$T_t(U) \cap W \neq \phi \quad \textit{and} \quad T_t(W) \cap V \neq \phi.$$

Satisfying the HCC is equivalent to weakly mixing[8, Theorem 7.28]. Also, a C_0-semigroup is mixing if, for any nonempty open sets U and V, there exists $t_0 \geq 0$ so that $T_t(U) \cap V \neq \phi$ for any $t > t_0$[8, Definition 7.8(a)]. Refer to [16] for more. [4] and [15] explore the behavior of orbits and operators of a hypercyclic C_0-semigroup.

Let us assume that U and V are nonempty open subsets of X. The return set of U and V for $(T_t)_{t\geq 0}$ is denoted with $R(U, V)$ and it is defined as follows:

$$R(U, V) = \{t \geq 0 : T_t(U) \cap V \neq \phi\}.$$

In this paper, we emphasize the return set for C_0-semigroup $(T_t)_{t\geq 0}$ as $R_{(T_t)}(U, V)$. The definition of topologically ergodic C_0-semigroups is in accordance with the concept of the syndetic return set for C_0-semigroups. If $\mathbb{R}^+ \setminus R(U, V)$ does not contain arbitrary long intervals, intervals with arbitrary lengths, then $R(U, V)$ is syndetic. For example, suppose $(T_t)_{t\geq 0}$ is a mixing C_0-semigroup, and U and V are nonempty open subsets of X. Then there exists $t_0 > 0$ such that for any $t \geq t_0$, $T_t(U) \cap V \neq \phi$. Hence,

$$\mathbb{R}^+ \setminus R(U, V) \subseteq \{t \in \mathbb{R}^+ : t < t_0\}.$$

Therefore, $\mathbb{R}^+ \setminus R(U, V)$ does not contain intervals with lenght more than t_0. Hence, in this case $R_{(T_t)}(U, V)$ is syndetic.

Definition 1.1. *(see [8, p.208]) A C_0-semigroup $(T_t)_{t\geq 0}$ on X is topologically ergodic if for every nonempty open sets U and V of X, $R(U, V)$ is syndetic.*

An autonomous discretization of a C_0-semigroup $(T_t)_{t\geq 0}$ means $(T_{t_n})_n$, where $t_n := nt_0$ for some $t_0 > 0$. If a C_0-semigroup $(T_t)_{t\geq 0}$ is topologically ergodic, then for each $t_0 > 0$, $(T_{nt_0})_n$ is topologically ergodic[8, p.208]. If for any nonempty open set U of X, $T_t(U) \cap U \neq \phi$ for some $t > 0$, then C_0-semigroup $(T_t)_{t\geq 0}$ on X would be recurrent[13]. Notably, finite-dimensional Banach spaces support these C_0-semigroups[13].

[6] notes that a C_0-semigroup $(T_t)_{t\geq 0}$ satisfies the recurrent hypercyclicity criterion or RHCC, if and only if for any open and nonempty sets U and V and any neighborhood W_1 of zero in X, $L_1 > 0$, and $L_2 > 0$ exist which for any $s \geq 0$, $s_1 \in [s, s + L_1)$ and $s_2 \in [s, s + L_2)$ can be found such that

$$T_{s_1}(U) \cap W \neq \phi \quad \textit{and} \quad T_{s_2}(W) \cap V \neq \phi.$$

[6] also proves that if $(T_t)_{t\geq 0}$ satisfies the RHCC, then $(T_t)_{t\geq 0}$ satisfies the HCC. Moreover, [6] states an equivalent condition to the RHCC as follows:

Lemma 1.2. *(see [6])Satisfiying the following three conditions is equivalent to satisfying the RHCC, where U and V are open and nonempty sets and W is a neighborhood of zero:*

(i) $\exists t \geq 0;\ S_t(U) \cap W \neq \phi$,

(ii) $\exists r \geq 0;\ S_r(W) \cap V \neq \phi$,

(iii) $\exists M > 0;\ \forall r \geq 0;\ \exists t \in [r, r+M);\ S_t(U) \cap U \neq \phi$.

This article investigates the properties of the topologically ergodic C_0-semigroups and their relations with other types of C_0-semigroups; as well as the properties of the direct sum of the topologically ergodic C_0-semigroups.

Section 2 shows that the inverse of a topologically ergodic C_0-semigroup is also topologically ergodic. It also proves that topological ergodicity is preserved under quasiconjugacy. In addition, it is established that topological ergodic C_0-semigroups exist on any infinite-dimensional space.

Section 3 proves that the topological ergodicity of any operators of a C_0-semigroup implies the topological ergodicity of the C_0-semigroup and vice versa. Moreover, it states that the topological ergodicity indicates satisfaction of the RHCC. Furthermore, it notes that the topological ergodicity of the direct sum of two C_0-semigroups implies that any of these C_0-semigroups and any of their operators are topologically ergodic. Also, it shows that the topological ergodicity of a C_0-semigroup is equivalent to the topological ergodicity of the direct sum of that C_0-semigroup with itself. Lastly, it demonstrates that the set of recurrent C_0-semigroups is a proper superset of the set of topologically ergodic C_0-semigroups.

Section 4 defines ergodic vectors and proves that a C_0-semigroup with a dense set of ergodic vectors is topologically ergodic. In addition, it states some criteria for topological ergodicity which are in accordance with open sets, neighborhoods of zero, and generalized kernels.

2. Preliminaries

The first section of this heading shows that mixing C_0-semigroups are topologically ergodic.

Lemma 2.1. *If $(T_t)_{t\geq 0}$ is a mixing C_0-semigroup on X, it is topologically ergodic.*

Proof. Let us assume that G and V are arbitrary nonempty open sets. So, there exists $s > 0$ such that

$$T_t(G) \cap V \neq \phi \text{ for any } t > s.$$

Hence, $\mathbb{R}^+ \setminus R(G, V) \subseteq [0, s]$. Therefore, $\mathbb{R}^+ \setminus R(G, V)$ can not contain arbitrary long intervals. □

As mentioned earlier, if $(T_t)_{t\geq 0}$ satisfies RHCC, then it satisfies HCC. By [8, Theorem 7.29], $(T_t)_{t\geq 0}$ is mixing and hence it is topologically ergodic by Lemma 2.1.

Example 2.2. *An operator $T \in B(X)$ is a generalized backward shift if there exists a dense set $\{\alpha_n : n \in \mathbb{N}\}$ in X such that $T(\alpha_1) = 0$ and $T(\alpha_{n+1}) = \alpha_n$[11, Definition 1.5]. A C_0-semigroup generated with a generalized backward shift T would be mixiing[11, Theorem 1.6]. Hence, by Lemma 2.1, it is topologically ergodic.*

Example 2.3. *Suppose $X = C_0(\mathbb{R}^+)$, where*

$$C_0(\mathbb{R}^+) = \{f : R^+ \to \mathbb{K} : f \text{ is a continuous function and } \lim_{x\to\infty} f(x) = 0\}.$$

Consider X endowed with $\|f\| = \sup_{x\in\mathbb{R}^+} |f(x)|$. Assume $\alpha > 0$. If for any $x \in \mathbb{R}^+$, $(T_t f)(x)$ defined with

$$(T_t f)(x) = e^{\alpha t} f(x + t),$$

then $(T_t)_{t\geq 0}$ is a mixing C_0-semigroup [8, Example 7.10(c)]. Hence, by Lemma 2.1, it is topologically ergodic.

The consequence theorem shows that the inverse of an invertible topologically ergodic C_0-semigroup is also topologically ergodic.

Theorem 2.4. *If $(T_t)_{t\geq 0}$ is a topologically ergodic C_0-semigroup on X such that for any $t > 0$, T_t^{-1} is invertible, then $(T_t^{-1})_{t\geq 0}$ is topologically ergodic.*

Proof. Suppose G and V are arbitrary nonempty open sets. Note that $T_t(U)\cap V \neq \phi$ if and only if $T_t^{-1}(V)\cap U \neq \phi$. Hence,

$$\{t \geq 0 : T_t(U) \cap V \neq \phi\} = \{t \geq 0 : T_t^{-1}(V) \cap U \neq \phi\}. \tag{1}$$

Now, $R_{(T_t)}(U, V)$ is syndetic by topological ergodicity of $(T_t)_{t\geq 0}$. Hence, by (1), $R_{(T_t^{-1})}(V, U)$ is syndetic. □

Recall that a C_0-semigroup $(T_t)_{t\geq 0}$ on X is quasiconjugate to C_0-semigroup $(S_t)_{t\geq 0}$ on Y if a continuous map $\Lambda : Y \to X$ exists with dense range such that $T_t \circ \Lambda = \Lambda \circ S_t$ for any $t > 0$. Like hypercyclicity, topological ergodicity preserves under quasiconjugacy, all of which the following proves:

Theorem 2.5. *Topological ergodicity of C_0-semigroups preserves under quasiconjugacy.*

Proof. Let us assume that Y is a Banach space and $(S_t)_{t\geq 0}$ an ergodic C_0-semigroup on Y. Suppose that $(T_t)_{t\geq 0}$ is a C_0-semigroup on X that is quasiconjugate to $(S_t)_{t\geq 0}$. So there exists $\Lambda : Y \to X$ with a dense range so that $T_t \circ \Lambda = \Lambda \circ S_t$ for any $t > 0$. Suppose G and V are nonempty open sets of X. Hence, $\Lambda^{-1}(G)$ and $\Lambda^{-1}(V)$ are nonempty open sets in Y. By topological ergodicity of $(S_t)_{t\geq 0}$, if we consider

$$R_{(S_t)}(\Lambda^{-1}(G), \Lambda^{-1}(V)) = \{t \geq 0 : (S_t)(\Lambda^{-1}(G)) \cap \Lambda^{-1}(V) \neq \phi\}, \tag{2}$$

then $\mathbb{R}^+ \setminus R_{(S_t)}(\Lambda^{-1}(G), \Lambda^{-1}(V))$ does not contain arbitrary long intervals. On the other hand, $S_t(\Lambda^{-1}(G)) \subseteq \Lambda^{-1}(T_t(G))$. Indeed, if $x \in S_t(\Lambda^{-1}(G))$, then $x = S_t y$ for some y with $\Lambda y \in G$. Hence, $\Lambda x = \Lambda S_t y = T_t \Lambda y \in T_t(G)$ wich gives that $x \in \Lambda^{-1}(T_t(G))$. Since x was chosen arbitrary, this shows that $S_t(\Lambda^{-1}(G)) \subseteq \Lambda^{-1}(T_t(G))$. Now, by (2),

$$\begin{aligned} R_{(S_t)}(\Lambda^{-1}(G), \Lambda^{-1}(V)) &= \{t \geq 0 : (S_t)(\Lambda^{-1}(G)) \cap \Lambda^{-1}(V) \neq \phi\} \\ &\subseteq \{t \geq 0 : \Lambda^{-1}(T_t(G)) \cap \Lambda^{-1}(V) \neq \phi\} \\ &= \{t \geq 0 : \Lambda^{-1}(T_t(G) \cap V) \neq \phi\} \\ &\subseteq \{t \geq 0 : T_t(G) \cap V \neq \phi\} = R_{(T_t)}(G, V). \end{aligned} \tag{3}$$

Therefore, by (3), $\mathbb{R}^+ \setminus R_{(T_t)}(G, V) \subseteq \mathbb{R}^+ \setminus R_{(S_t)}(\Lambda^{-1}(G), \Lambda^{-1}(V))$. Hence, $\mathbb{R}^+ \setminus R_{(T_t)}(G, V)$ does not contain arbitrary long intervals which means $(S_t)_{t\geq 0}$ is also topologically ergodic. □

Notice that in the proof of Theorem it is unnecessary that Λ has dense range. So, we can state that if $(T_t)_{t\geq 0}$ is a C_0-semigroup on X ,and $(S_t)_{t\geq 0}$ a C_0-semigroup on Y, and there exists a continuous map $\Lambda : Y \to X$ so that $T_t \circ \Lambda = \Lambda \circ S_t$ for any $t > 0$, then topological ergodicity of $(T_t)_{t\geq 0}$ implies topological ergodicity of $(S_t)_{t\geq 0}$ and vice versa.

Of note here is that the existence of topologically ergodic C_0-semigroups is a significant problem. Fortunately, these C_0-semigroups exist in every infinite-dimensional Banach space.

Theorem 2.6. *Topologically ergodic C_0-semigroups exist on any infinite-dimensional Banach space.*

Proof. According to [3, Theorem 2.4], mixing C_0-semigroups are found in any Banach space with infinite dimension. In accordance with Lemma 2.1, mixing C_0-semigroups is topologically ergodic. All of which completes the proof. □

3. The Behavior of the Operators of a Topologically Ergodic C_0-semigroup

As the following theorem proves, the topological ergodicity of any operators of a C_0-semigroup implies the topological ergodicity of the C_0-semigroup and vice versa.

Theorem 3.1. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then the following are equivalent:*

(i) T_{t_0} is a topologically ergodic operator for some $t_0 > 0$.

(ii) $(T_t)_{t\geq 0}$ is topologically ergodic.

(iii) T_t, for any $t > 0$, is topologically ergodic.

Proof. $(i) \to (ii)$. Let $t_0 > 0$ exist so that T_{t_0} is a topologically ergodic operator. So, for nonempty open sets U and V of X,

$$N_{T_{t_0}}(U, V) = \{n \in \mathbb{N} : T_{t_0}^n(U) \cap V \neq \phi\}$$

is syndetic. That means $\sup_k(n_{k+1} - n_k) < \infty$, where (n_k) is the sequence that forms $N_{T_{t_0}}(U, V)$. Let $M := \sup(n_{k+1} - n_k)$. So $\mathbb{R}^+ \setminus R(U, V)$ can not contain intervals with a length more than Mt_0. Hence, $(T_t)_{t\geq 0}$ is topologically ergodic.

$(ii) \to (iii)$. Let $t_0 > 0$. By [8, p.208], any autonomous discretization of $(T_t)_{t\geq 0}$ is topologically ergodic. So $(T_{nt_0})_n$ is topologically ergodic.

Let U and V be nonempty open sets of X. Then, by definition of topological ergodicity, $\{n \in \mathbb{N} : T_{nt_0}(U) \cap V \neq \phi\}$ is syndetic. But

$$\{n \in \mathbb{N} : T_{nt_0}(U) \cap V \neq \phi\} = \{n \in \mathbb{N} : T_{t_0}^n(U) \cap V \neq \phi\} = N_{T_{t_0}}(U, V). \tag{4}$$

Therefore, (4) asserts that $N_{T_{t_0}}(U, V)$ is syndetic. Since U and V are arbitrary open sets, it can concluded that T_{t_0} is a topologically ergodic operator.

$(iii) \to (i)$. It is clear. □

Theorem 3.1 extends Theorem 2.4 as follows:

Theorem 3.2. *Suppose $(T_t)_{t\geq 0}$ is a topologically ergodic C_0-semigroup on X such that T_t is invertible for any $t > 0$. Then for any natural number n, $(T_t^n)_{t\geq 0}$ and $(T_t^{-n})_{t\geq 0}$ are topologically ergodic.*

Proof. $(T_t)_{t\geq 0}$ is topologically ergodic. So, any autonomous discretization of $(T_t)_{t\geq 0}$ would be topologically ergodic which means for each $t_0 > 0$, $(T_{nt_0})_n$ is topologically ergodic. Hence, $(T_{t_0}^n)_n$ is topologically ergodic since $T_{nt_0} = T_{t_0}^n$. By part(iii) of Theorem 3.1, $T_{t_0}^n$ is an ergodic operator for any n. Therefore, by Theorem 3.1, $(T_t^n)_{t\geq 0}$ is topologically ergodic.

Moreover, the topological ergodicity of $(T_t)_{t\geq 0}$ implies that $(T_t^{-1})_{t\geq 0}$ is topologically ergodic. Similar to the first part of the proof, $((T_t^{-1})^n)_{t\geq 0} = (T_t^{-n})_{t\geq 0}$ is also topologically ergodic. □

The next lemma proves that for a C_0-semigroup, topological ergodicity indicates the satisfaction of the RHCC.

Lemma 3.3. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If $(T_t)_{t\geq 0}$ is topologically ergodic, it satisfies the RHCC.*

Proof. Suppose that $(T_t)_{t\geq 0}$ is topologically ergodic. Suppose G and V are nonempty open subsets of X and W is a neighborhood of zero in X. By definition of the topological ergodicity, $R(G, W)$ and $R(W, V)$ are syndetic sets. So for any $t \geq 0$, $s_1 \in [t, t + L_1)$ and $s_2 \in [t, t + L_2)$ exist so that

$$T_{s_1}(G) \cap W \neq \phi \quad \textit{and} \quad T_{s_2}(W) \cap V \neq \phi.$$

Therefore, $(T_t)_{t\geq 0}$ satisfies the condition of the RHCC. □

The following corollary states that topological ergodic C_0-semigroups are recurrent. Also, there are recurrent C_0-semigroups that are not topologically ergodic.

Corollary 3.4. *The topologically ergodic C_0-semigroups on X are a proper subset of the set of recurrent C_0-semigroups.*

Proof. In accordance with Lemma 3.3, if $(T_t)_{t\geq 0}$ is topologically ergodic, then $(T_t)_{t\geq 0}$ would satisfy the RHCC. Hence, it is recurrent[13, Theorem 2].

On the other hand, topologically ergodic C_0-semigroups are hypercyclic and can not exist in finite-dimensional spaces[8, Theorem 7.15]. But recurrent C_0-semigroups exist on finite-dimensional spaces[13, Theorem5]. $\square$

It is proved in [8, p.208] that topological ergodicity of $(T_t \oplus S_t)_{t\geq 0}$ on $X \oplus Y$ can be concluded from either the topological ergodicity of $(T_t)_{t\geq 0}$ or the topological ergodicity of $(S_t)_{t\geq 0}$.

Employing Theorem 3.1, in the next corollary and theorem, this study extends the result and demonstrates that the converse is also true.

Corollary 3.5. *Suppose $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroups on X and Y, respectively. If T_{t_0} is topologically ergodic for some $t_0 > 0$ or S_{s_0} is topologically ergodic for some $s_0 > 0$, then $(T_t \oplus S_t)_{t\geq 0}$ would be topologically ergodic.*

Proof. In accordance with Theorem 3.1 the topological ergodicity of T_{t_0} (or S_{s_0}), implies that $(T_t)_{t\geq 0}$ (or $(S_t)_{t\geq 0}$) is topologically ergodic. In this regard, in light of [8, p.208], $(T_t \oplus S_t)_{t\geq 0}$ is topologically ergodic. $\square$

Theorem 3.6. *If $(T_t \oplus S_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus Y$, then the topological ergodicity of $(T_t \oplus S_t)_{t\geq 0}$ would imply that $(T_t)_{t\geq 0}$ is topologically ergodic on X and $(S_t)_{t\geq 0}$ is topologically ergodic on Y.*

Especially, T_t and S_t, for any $t > 0$, are topologically ergodic.

Proof. Let us assume that U and V are nonempty open subsets of X and let W be a neighborhood of zero in Y. Let $t_0 \in R_{(T_t \oplus S_t)}(U \oplus W, V \oplus W)$. So $(T_{t_0} \oplus S_{t_0})(U \oplus W) \cap (V \oplus W) \neq \phi$ which implies $(T_{t_0}(U) \oplus S_{t_0}(W)) \cap (V \oplus W) \neq \phi$. So that $(T_{t_0}(U) \cap V) \oplus (S_{t_0}(W) \cap W) \neq \phi$. Thus,

$$T_{t_0}(U) \cap V \neq \phi \text{ and } S_{t_0}(W) \cap W \neq \phi.$$

Hence, $t_0 \in R_{(T_t)}(U, V)$. This means

$$R_{(T_t \oplus S_t)}(U \oplus W, V \oplus W) \subseteq R_{(T_t)}(U, V). \tag{5}$$

Using (5), it gains

$$\mathbb{R}^+ \setminus R_{(T_t)}(U, V) \subseteq \mathbb{R}^+ \setminus R_{(T_t \oplus S_t)}(U \oplus W, V \oplus W). \tag{6}$$

The right set of (6) does not contain arbitrary long intervals. So, $\mathbb{R}^+ \setminus R_{(T_t)}(U, V)$ does not contain arbitrary long intervals either. Since U and V are arbitrary open sets, this leads us to the topological ergodicity of $(T_t)_{t\geq 0}$.

In addition, Theorem 3.1 asserts that T_t, for any $t > 0$, is topologically ergodic. Similarly, $(S_t)_{t\geq 0}$ is a topologically ergodic C_0-semigroup on Y and Theorem 3.1 asserts that S_t, for any $t > 0$, is topologically ergodic. $\square$

We have the next corollary by using Theorem 3.6 and Theorem 3.1.

Corollary 3.7. *If $(T_t \oplus T_t)$ is a topologically ergodic C_0-semigroup on $X \oplus X$, then $(T_t)_{t\geq 0}$ is topologically ergodic on X.*

Especially, T_t, for any $t > 0$, is topologically ergodic.

Corollary 3.7 presents the following equivalent conditions between the topological ergodicity of a C_0-semigroup and the topological ergodicity of the direct sum of it with itself.

Corollary 3.8. *For a C_0-semigroup $(T_t)_{t\geq 0}$ on X, the following are equivalent:*

(i) *T_{t_0} is a topologically ergodic operator for some $t_0 > 0$.*

(ii) *T_t, for any $t > 0$, is topologically ergodic.*

(iii) *$(T_t)_{t\geq 0}$ is topologically ergodic.*

(iv) *$(T_t \oplus T_t)_{t\geq 0}$ is topologically ergodic.*

Proof. Theorem 3.1 shows that conditions (i), (ii) and (iii) are equivalent. By [8, p.208], (iii) implies (iv). Also, by Corollary 3.7, (iv) implies (i). □

4. Some Criteria for Topological Ergodicity

Employing open sets and neighborhoods of zero, this study obtains a sufficient condition for topological ergodicity as follows:

Theorem 4.1. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X with this property that for any open set U of X and any W, a neighborhood of zero, $L > 0$ can be discovered such that for any $t \geq 0$, there exists some $s \in [t, t+L)$ such that $T_s(U) \cap W \neq \phi$ and $T_s(W) \cap U \neq \phi$. Then $(T_t)_{t\geq 0}$ would be topologically ergodic.*

Proof. By hypothesis, the conditions (*i*) and (*ii*) in Lemma 1.2 are obviously satisfied since the hypothesis holds for every open subset of X.

Suppose U is a nonempty open set of X. In accordance with [8, Lemma 2.36], an open set G and a neighborhood W of zero can be chosen so that

$$G + W \subseteq U. \tag{7}$$

So, by hypothesis, we can find some $L > 0$ such that for any $t \geq 0$ we can find some $s \in [t, t+L)$ with $T_s(G) \cap W \neq \phi$ and $T_s(W) \cap G \neq \phi$. This gives $(T_s(G) \cap W) + (T_s(W) \cap G) \neq \phi$. Further,

$$\begin{aligned}(T_s(G) \cap W) + (T_s(W) \cap G) &\subseteq (T_s(G) + T_s(W)) \cap (W + G)\\ &= T_s(G + W) \cap (W + G)\\ &\subseteq T_s(U) \cap U.\end{aligned}$$

So, $T_s(U) \cap U \neq \phi$, and the condition (*iii*) of Lemma 1.2 holds as well. Therefore, $(T_t)_{t\geq 0}$ is topologically ergodic. □

Also, another sufficient condition exists as follows:

Theorem 4.2. *Consider $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then $(T_t)_{t\geq 0}$ is topologically ergodic if and only if $(T_t)_{t\geq 0}$ is weakly-mixing and for any open set U of X, $L > 0$ exists so that for any $t \geq 0$, $s \in [t, t+L)$ can be found such that $T_s(U) \cap U \neq \phi$.*

Proof. A topologically ergodic C_0-semigroup satisfies the RHCC, and hence, satisfies the HCC. So $(T_t)_{t\geq 0}$ is weakly-mixing. Suppose U is a nonempty open set of X and suppose W is a neighborhood of zero. In accordance with [6, Lemma 5.2], there exists $L > 0$ such that for any $t \geq 0$, there exists some $s \in [t, t+L)$ such that

$$T_s(U) \cap W \neq \phi, \text{ and } \quad T_s(W) \cap U \neq \phi. \tag{8}$$

So in light of (8) and similar to proof of Theorem 4.1, $T_s(U) \cap U \neq \phi$.

To prove the converse of the theorem, since $(T_t)_{t\geq 0}$ is weakly-mixing, it satisfies the HCC. So $t \geq 0$ can be chosen with this property that

$$T_s(U) \cap W \neq \phi, \text{ and } \quad T_s(W) \cap V \neq \phi. \tag{9}$$

Condition (9) with the other property in the hypothesis shows that $(T_t)_{t\geq 0}$ satisfies in the conditions of Lemma 1.2. Hence, $(T_t)_{t\geq 0}$ is topologically ergodic. □

Now, the following defines the topologically ergodic vectors for C_0-semigroups.

Definition 4.3. *We name $z \in X$ a topologically ergodic vector for C_0-semigroup $(T_t)_{t\geq 0}$ on X if*

$$R_{(T_t)}(z, U) = \{t \geq 0 : T_t z \in U\}$$

is syndetic for every nonempty open subset U of X. That means $\mathbb{R}^+ \setminus R_{(T_t)}(x, U)$ does not contain arbitrary long intervals. The set of ergodic points for $(T_t)_{t\geq 0}$ is denoted by $ERG(T_t)$.

Theorem 4.4. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If $\overline{ERG(T_t)} = X$, then $(T_t)_{t\geq 0}$ would be topologically ergodic.*

Proof. Let us assume that U and V are nonempty open sets of X. Suppose W is a neighborhood of zero in X. Since U is open, by hypothesis, $x \in U \cap ERG(T_t)$ exists. On the other hand, W is open. So $\{t \geq 0 : T_t x \in W\}$ is syndetic. Hence,

$$\exists L > 0; \forall t \geq 0, \text{there exists } s \in [t, t + L) \text{ such that } T_s x \in W. \tag{10}$$

Since $x \in U$, (10) concludes that

$$\exists L > 0; \forall t \geq 0, \text{there exists } s \in [t, t + L) \text{ such that } T_s U \cap W \neq \phi. \tag{11}$$

Similarly, $\{t \geq 0 : T_t x \in U\}$ is syndetic. Hence,

$$\exists M > 0; \forall t \geq 0, \text{there exists } s \in [t, t + M) \text{ such that } T_s x \in U. \tag{12}$$

Since $x \in U$, (12) means

$$\exists M > 0; \forall t \geq 0, \text{there exists } s \in [t, t + M) \text{ such that } T_s U \cap U \neq \phi. \tag{13}$$

In addition, by hypothesis, there exists $z \in W \cap ERG(T_t)$. Moreover, $\{t \geq 0 : T_t z \in V\}$ is syndetic since V is open. Hence,

$$\exists P > 0; \forall t \geq 0, \text{there exists } s \in [t, t + P) \text{ such that } T_s z \in V. \tag{14}$$

Since $z \in W$, by (14),

$$\exists P > 0; \forall t \geq 0, \text{there exists } s \in [t, t + P) \text{ such that } T_s W \cap V \neq \phi. \tag{15}$$

In conclusion, (11), (13), (15), and Lemma 1.2, imply that $(T_t)_{t\geq 0}$ is topologically ergodic. □

That being said what is the structure of topologically ergodic vectors is? Are they dense when $(T_t)_{t\geq 0}$ is topologically ergodic?

Question. Do topologically ergodic C_0-semigroups have a dense set of ergodic points?

Theorem 4.5. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Assume for every W, neighborhood of zero in X and every nonempty open set U and V of X,*

(i) *$s \geq 0$ can be chosen such that $T_s(W) \cap V \neq \phi$,*

(ii) *$t \geq 0$ can be chosen such that $T_t(U) \cap W \neq \phi$,*

(iii) there exists $\alpha > 0$ *such that for any* $t \geq 0$, $T_{t+\alpha}(U) \cap U \neq \phi$.

Then $(T_t)_{t\geq 0}$ *is topologically ergodic.*

Proof. Let $L := \alpha + 1$. Then for any $t \geq 0$ consider $s := t + \alpha$, then $s \in [t, t + L)$ and $T_s(U) \cap U \neq \phi$. All of which satisfies the conditions of Lemma 1.2. □

Theorem 4.5 concludes that if a hypercyclic C_0-semigroup satisfies condition (*iii*) of Theorem 4.5, then it is topologically ergodic. Of note here that if condition (iii) of Theorem 4.5 is replaced by this condition that for any $t \geq 0$, $T_{t+1}(U) \cap U \neq \phi$, then Theorem 4.5 remains true.

Corollary 4.6. *Let* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on* X. *If*

(i) $\overline{\{x \in X : \lim_{t\to\infty} T_t x = 0\}} = X$,

(ii) for any nonempty open set G *of* X *and any* $t \geq 0$, $T_{t+1}(G) \cap G \neq \phi$,

then $(T_t)_{t\geq 0}$ *is topologically ergodic.*

Proof. Condition (*i*) and [11, Corollary 2.8] conclude that $(T_t)_{t\geq 0}$ satisfies the HCC, and hence $(T_t)_{t\geq 0}$ is weakly mixing. Therefore, if we consider $\alpha = 1$, then it satisfies the conditions of Theorem 4.5. □

In accordance with a generalized kernel of a C_0-semigroup $(T_t)_{t\geq 0}$, it means $\bigcup_{t\geq 0} ker(T_t)$. The next corollary investigates the last sufficient condition for topological ergodicity.

Corollary 4.7. *Let* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on* X. *Suppose that*

(i) $(T_t)_{t\geq 0}$ *has a dense generalized kernel in* X,

(ii) for any nonempty open set G *of* X *and any* $t \geq 0$, $T_{t+1}(G) \cap G \neq \phi$.

Then $(T_t)_{t\geq 0}$ *is topologically ergodic.*

Proof. Let $x \in \bigcup_{t\geq 0} ker(T_t)$. Then there exists $s \geq 0$ such that $x \in ker(T_s)$. For any $t > s$,

$$T_t x = T_{t-s} T_s x = T_{t-s}(0) = 0. \tag{16}$$

So by (16),

$$T_t x = 0, \text{ for any } t > s. \tag{17}$$

Hence, in accordance with (17), $\lim_{t\to\infty} T_t x = 0$. Therefore, in light of Corollary 4.6, $(T_t)_{t\geq 0}$ is topologically ergodic. □

5. Conclusion

Dynamical systems are among the structures noted in the theory of dynamical systems. This paper has introduced and investigated topologically ergodic C_0-semigroups. These C_0-semigroups are a refinement of the recurrent C_0-semigroups. As Theorem 2.6 proves, these C_0-semigroups exist in any infinite-dimensional and separable Banach spaces. This study also proved that topological ergodicity preserves under quasi-conjugacy and inverse. Moreover, the topological ergodicity of a C_0-semigroup lead to the topological ergodicity of any of its operators and vice versa. This article investigated the direct sum of topological ergodic C_0-semigroups, which demonstrated that the topological ergodicity of the direct sum of two C_0-semigroups implies the topological ergodicity of any of them. On the other hand, its converse remains an open problem, which inquires whether the topological ergodicity of two C_0-semigroups, implies the topological ergodicity of their direct sum. However, the study established that $(T_t)_{t\geq 0}$ is topologically ergodic if and only if $(T_t \oplus T_t)_{t\geq 0}$ is topologically ergodic. Lastly, the research stated some criteria for topological ergodicity which were in accordance with the topologically ergodic vectors, open sets, and generalized kernels. To conclude, the researcher suggests investigating whether the converse of Lemma 3.3 is true or not; in other words, whether satisfying RHCC implies ergodicity or not. I also suggest exploring some other sufficient conditions for the ergodicity of a C_0-semigroup.

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