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$*$-q-central idempotent and $*$-quasi-normal ring

Liufeng Cao[a], **Junchao Wei**[b]

[a] *Department of Mathematics, Yancheng Institute of Technology, China*
[b] *School of Mathematical Sciences, Yangzhou University, China*

Abstract. In a $*$-ring, an idempotent $e \in R$ is called $*$-*quarter-central* (or $*$-*q-central* for short) if $e^*R(1-e)Re^* = 0$. If all idempotents in a $*$-ring R are $*$-q-central, R is said to be a $*$-*quasi-normal ring*. In the paper, we prove that in a $*$-ring, each $*$-q-central idempotent must be *q-central*, and the converse of the conclusion is not true. Moreover, we give some equivalent conditions to claim when a q-central idempotent is $*$-q-central. From the viewpoint of currently prevailing generalizations of rings in the literature, $*$-q-central idempotents and $*$-quasi-normal rings are the generalizations of *q*-central idempotents and *quasi-normal rings*, respectively.

1. Introduction

In classical ring theory, idempotents play an important role. A ring R is called *abelian* if each idempotent $e \in R$ is central, i.e., $ae = eae = ea$ for any $a \in R$. In [8], Chase firstly used $ea = eae$ (e is an idempotent in R and $a \in R$) to study the generalizations of triangular matrix rings. Then, Birkenmeier in [4] defined an idempotent $e \in R$ to be *left semicentral* (resp. *right semicentral*) if $ae = eae$ (resp. $ea = eae$) for any $a \in R$. An idempotent $e \in R$ is called *semicentral* if it is either left semicentral or right semicentral. Based on previous papers, Chen defined a ring R to be *semiabelian* if every idempotent in R is semicentral [9]. For the works of semicentral idempotents and semiabelian rings, one can refer to [5, 6, 11, 13–15, 21–23]. As the generalization of left or right semicentral idempotent in rings, Lam in [16] defined an idempotent $e \in R$ to be q-central if $eR(1-e)Re = 0$, and a ring R to be *q-abelian* if all idempotents in R are q-central. Coincidentally, q-central idempotents have first appeared under the name of "*inner Peirce trivial idempotents*" in [2]. Moreover, in [21], Wei defined quasi-normal rings, which are q-abelian rings. We need to point out that in [12, 18, 20], q-central idempotents and q-abelian rings are seen as "2-central rings" and "2-Abelian rings", respectively.

A ring R is said to be an *involution ring* (or a $*$-*ring* for short) if there exists a bijection $*: R \to R$ of R such that for any $a, b \in R$,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

In this paper, we generalize the notions of q-central idempotents and q-abelian rings from a ring to a $*$-ring. An idempotent e in a $*$-ring R is said to be $*$-quarter-central (or $*$-q-central shortly) if $e^*R(1-e)Re^* = 0$.

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The first named author was supported by National Natural Science Foundation of China (Grant No. 12371041).
Email addresses: 1204719495@qq.com (Liufeng Cao), jcweiyz@126.com (Junchao Wei)
ORCID iDs: https://orcid.org/0000-0001-9758-2585 (Liufeng Cao), https://orcid.org/0000-0002-7310-1836 (Junchao Wei)

In a $*$-ring, it is turned out that any $*$-q-central idempotent is q-central. However, the converse of the conclusion is not true. We consider when a q-central idempotent is $*$-q-central. A $*$-ring is said to $*$-quasi-normal if $e^*R(1-e)Re^* = 0$ for any idempotent $e \in R$. Moreover, we give some new characterizations of $*$-quasi-normal rings. It is worth mentioning that some examples that appear in this paper are inspired by Chen's work [10].

Throughout the paper, all rings are associative and unital. In a $*$-ring, the symbols $\mathrm{E}(R)$, $\mathrm{N}(R)$, $\mathrm{Z}(R)$, $\mathrm{S}_l(R)$, $\mathrm{S}_r(R)$, $\mathrm{PE}(R)$, R^{PI}, q-idem(R) and $\mathbb{Z}_n$ stand for the set of all idempotents of R, the set of all nilpotent elements of R, the center of R, the set of all left semicentral idempotents of R, the set of all right semicentral idempotents of R, the set of all projections of R, the set of all partial isometries of R, the set of all q-central idempotents of R, and the ring of integers modulo a positive integer n, respectively.

2. $*$-q-central idempotent

In this section, we will give the definition of $*$-q-central idempotents, and give some characterizations.

Definition 2.1. *Let R be a $*$-ring. An idempotent $e \in \mathrm{E}(R)$ is called $*$-q-central if $e^*R(1-e)Re^* = 0$.*

Here, we give an example to show that there exists a $*$-ring that has non-trivial $*$-q-central idempotents.

Example 2.2. *Let $R = \mathbb{Z}_2[x]/(x^3-1)$ and define $* : a_1 + a_2x + a_3x^2 \mapsto a_1 + a_3x + a_2x^2$, where $a_1, a_2, a_3 \in \mathbb{Z}_2$. Then R is a $*$-ring. It is easy to check that $e = x + x^2 \in \mathrm{E}(R)$ and $e^*(1-e) = (x+x^2)^*(1-x-x^2) = 0$. Hence, $x + x^2$ is $*$-q-central. Similarly, one can prove that $1 + x + x^2 \in \mathrm{E}(R)$ and it is $*$-q-central.*

In a ring R, for any $x, y \in R$, let $[x, y]$ denote the additive commutator $xy - yx$. It follows from [20, Lemma 1, Claim 2] that $e[e, R] = [e, R](1-e)$ and $[e, R]e = (1-e)[e, R]$ for any idempotent $e \in R$.

Proposition 2.3. *Let R be a $*$-ring and $e \in \mathrm{E}(R)$ be a $*$-q-central idempotent. Then*
(1) $e = ee^*e$.
(2) $e^*e[e, R][R, e]e^* = 0$.
(3) $e^*[e, R][R, e]ee^* = 0$.
(4) $e \in q$-idem(R), that is, e is q-central.

Proof. (1) By assumption, $e^*(1-e)e^* = 0$, i.e., $e^* = e^*ee^*$, which gives $e = ee^*e$.
(2) By $e[e, R] = [e, R](1-e)$ for any $e \in \mathrm{E}(R)$, we have

$$e^*e[e, R][R, e]e^* = e^*[e, R](1-e)[R, e]e^* \subseteq e^*R(1-e)Re^* = 0.$$

Hence, $e^*e[e, R][R, e]e^* = 0$.
(3) By $[e, R]e = (1-e)[e, R]$ for any $e \in \mathrm{E}(R)$, we have

$$e^*[e, R][R, e]ee^* = e^*[e, R][e, R]ee^* = e^*[e, R](1-e)[e, R]e^* \subseteq e^*R(1-e)Re^* = 0.$$

(4) By (2), we have $e^*ex(1-e)yee^* = 0$ for any $x, y \in R$. Then by (1), $ex(1-e)ye = e(e^*ex(1-e)yee^*)e = 0$. It follows that $eR(1-e)Re = 0$, and so e is q-central. □

The following conclusion arises from Proposition 2.3 immediately.

Corollary 2.4. *Let R be a $*$-ring and $e \in \mathrm{E}(R)$ be $*$-q-central. Then*
(1) $e^*R(1-e)$ and $eR(1-e)^*$ are both right ideals of R.
(2) $(1-e)Re^*$ and $(1-e)^*Re$ are both left ideals of R.

In Proposition 2.3, we show that a $*$-q-central idempotent must be q-central, the following example will show that the converse of the statement is not true.

Example 2.5. *Let $R = \mathbb{Z}[x]/(x^2 + x)$ and the involution $*$ defined by $(a_1 + a_2x)^* = a_1 - a_2 - a_2x$, where $a_1, a_2 \in \mathbb{Z}$ and $\{1, x\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[x]/(x^2 + x)$. Then R is a $*$-ring and $\mathrm{E}(R) = \{0, 1, -x, 1 + x\}$ [7]. Taking $e = -x \in \mathrm{E}(R)$, then $eR(1-e)Re = 0$. This is because $-x(1+x)(-x) = x^2(1+x) = -x(1+x) = -(x^2+x) = 0$. Hence, $-x$ is q-central. However, $eR(1-e)^*Re$ cannot be equal to 0. This is since $-x(1+x)^*(-x) = x^2(-x) = -x^3 = -x \neq 0$. It follows that $-x$ is not $*$-q-central.*

The following conclusion is evident.

Corollary 2.6. *Let R be a $*$-ring and $e \in \mathrm{E}(R)$. Then the following statements are equivalent:*
(1) e is $*$-q-central.
(2) $e^*xye^* = e^*xeye^*$ for any $x, y \in R$.
(3) $exye = exe^*ye$ for any $x, y \in R$.

In the following, we denote the set of all $*$-q-central idempotents of a $*$-ring R by q^*-idem(R) simply. Let R be a ring and $e \in q$-idem(R), consider the map $\varphi : R \to eRe$ determined by $\varphi(r) = ere$. Then $\varphi(\mathrm{N}(R)) \subseteq \mathrm{N}(R)$, $\varphi(\mathrm{E}(R)) \subseteq \mathrm{E}(R)$, $\varphi(\mathrm{S}_l(R)) \subseteq \mathrm{S}_l(R)$, $\varphi(\mathrm{S}_r(R)) \subseteq \mathrm{S}_r(R)$ and $\varphi(q\text{-idem}(R)) \subseteq q\text{-idem}(R)$, see [2] or [16].

Proposition 2.7. *Let R be a $*$-ring and $e \in q^*$-idem(R). Then $\varphi(q^*\text{-idem}(R)) \subseteq q^*\text{-idem}(R)$.*

Proof. On one hand, it follows from Proposition 2.3 (4) that $exye = exeye$ for any $x, y \in R$. Assume that $f \in q^*$-idem(R), then $(efe)^2 = efeefe = efefe = ef^2e = efe$, which implies $efe \in \mathrm{E}(R)$. On the other hand, for any $x, y \in R$, we have

$$e^*xye^* = e^*xeye^*.$$

Similarly,

$$f^*xyf^* = f^*xfyf^*, \ \forall x, y \in R.$$

We need to show $(efe)^*R(1 - efe)R(efe)^* = 0$, i.e., $e^*f^*e^*xye^*f^*e^* = e^*f^*e^*xefeye^*f^*e^*$ for any $x, y \in R$. In fact,

$$\begin{aligned} e^*f^*(e^*xye^*)f^*e^* &= e^*(f^*e^*xeye^*f^*)e^* \\ &= e^*(f^*(e^*xe)(eye^*)f^*)e^* \\ &= e^*f^*e^*xefeye^*f^*e^*. \end{aligned}$$

This gives the desired result. □

Example 2.5 states that a q-central idempotent may not be $*$-q-central. The following result shows that when a q-central idempotent is $*$-q-central, which is immediate from Proposition 2.3.

Corollary 2.8. *Let R be a $*$-ring and $e \in \mathrm{E}(R)$. Then $e \in q^*$-idem(R) if and only if $e \in q$-idem(R) and $e = ee^*e$.*

The following example states that Corollary 2.8 can be used to check whether an idempotent $e \in \mathrm{E}(R)$ in a $*$-ring R is $*$-q-central or not.

Example 2.9. *Let $R = T_2(\mathbb{C})$. Then it is easy to check that $\mathrm{E}(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} | x \in \mathbb{C}\}$. Define $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$, then R is a $*$-ring. Taking $e = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$, where $x \in \mathbb{C}$. It is easy to compute $ee^*e = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = 0 \neq e$. Hence, e is not $*$-q-central by Corollary 2.8. Similarly for $\begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$, where $x \in \mathbb{C}$.*

In a $*$-ring R and $e \in q^*$-idem(R), we have the following conclusion.

Lemma 2.10. *Let R be a $*$-ring and $e \in q^*$-idem(R). Then*
(1) $e^*R(1-ee^*)Re^* = 0$.
(2) $e^*R(1-e^*e)Re^* = 0$.

Proof. By Corollary 2.8, we have $e = ee^*e$. Hence,
(1) Since $e^*xye^* = e^*xeye^*$ and $exye = exeye$ for any $x, y \in R$, we have

$$e^*xye^* = e^*xeye^* = e^*(xee^*)eye^* = e^*xee^*ye^*.$$

This infers $e^*R(1-ee^*)Re^* = 0$.
(2) For any $x, y \in R$, we have $e^*xye^* = e^*xeye^*$ and $exye = exeye$. So,

$$e^*xye^* = e^*xe(e^*ey)e^* = e^*xe^*eye^*,$$

which implies $e^*R(1-ee^*)Re^* = 0$. □

Proposition 2.11. *Let R be a $*$-ring and $e \in q^*$-idem(R). Then $ee^*, e^*e \in q^*$-idem(R).*

Proof. On one hand, by $e = ee^*e$, we have $(ee^*)^2 = (ee^*e)e^* = ee^*$, and so $ee^* \in \mathrm{E}(R)$. On the other hand, by Lemma 2.10 (1), $e^*R(1-ee^*)Ree^* = e^*R(1-ee^*)Re1e^* = e^*R(1-ee^*)Re^* = 0$, which shows that $e(e^*R(1-ee^*)Ree^* = 0$. This infers $ee^* \in q^*$-idem(R). Similarly, we can prove that $e^*e \in q^*$-idem(R). □

Conversely, if $ee^* \in q^*$-idem(R) or $e^*e \in q^*$-idem(R), we cannot obtain $e \in q^*$-idem(R).

Example 2.12. *Let $R = M_2(\mathbb{Z}_2)$ and $*$ be the transposition of a matrix. Taking $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $e \in \mathrm{E}(R)$. Notice that $ee^* = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$, so $ee^* \in q^*$-idem(R). However, $ee^*e = 0 \neq e$. Hence, $e \notin q^*$-idem(R) by Corollary 2.8. Similarly, taking $e' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, then we can check that $(e')^*e' = 0 \in q^*$-idem(R) and $e' \notin q^*$-idem(R).*

Proposition 2.13. *Let R be a $*$-ring and $e \in q^*$-idem(R). Then $e - ee^*, e - e^*e \in \mathrm{N}(R)$.*

Proof. Since $e = ee^*e$, $(e-ee^*)^2 = (e-ee^*)(e-ee^*) = e - ee^* - ee^*e + ee^*ee^* = 0$. Hence, $e - ee^* \in \mathrm{N}(R)$. Similarly, one can show $e - e^*e \in \mathrm{N}(R)$. □

In general, the converse of Proposition 2.13 is not true.

Example 2.14. *Let $R = M_2(\mathbb{Z}_4) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}_4\}$ and the involution $*$ be the transposition of a matrix. Taking $e = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, then $e \in \mathrm{E}(R)$. Notice that $e - ee^* = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, so $e - ee^* \in \mathrm{N}(R)$. However, $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$, which implies that e is not q-central. Hence, by Proposition 2.3 (4), e is not q^*-central. Similarly, taking $e' = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, then $e' - (e')^*e' = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in \mathrm{N}(R)$. Moreover, $e' \notin q^*$-idem(R).*

It is well-known that $\mathrm{PE}(R) \subseteq R^{\mathrm{PI}}$, so Example 2.14 shows that $e - ee^* \in \mathrm{N}(R)$ or $e - e^*e \in \mathrm{N}(R)$ cannot yield $e \in \mathrm{PE}(R)$. Here, we give a new example to show this statement.

Example 2.15. *Let* $R = M_3(\mathbb{Z}_2)$ *and* $*$ *be the transposition of* R. *Taking* $e = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *then* $f = e - ee^* = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *One can easily compute* $f^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, *which infers* $f \in \mathrm{N}(R)$. *However,* $e^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq e$. *Similarly, if we set* $e' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, *then* $f' = e' - (e')^*e' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{N}(R)$ *and* $(e')^* \neq e'$.

Proposition 2.16. *Let* R *be a* $*$*-ring and* $e \in \mathrm{E}(R)$. *Then*
(1) $e \in q\text{-idem}(R)$ *if and only if* $e^* \in q\text{-idem}(R)$.
(2) $e \in q^*\text{-idem}(R)$ *if and only if* $e^* \in q^*\text{-idem}(R)$.

Proof. It follows from a straightforward verification. □

By Proposition 2.16, Lemma 2.10 can be replaced by the following formula.

Theorem 2.17. *Let* R *be a* $*$*-ring and* $e \in \mathrm{E}(R)$. *Then* $e \in q^*\text{-idem}(R)$ *if and only if one of the following two conditions hold:*
(1) $e \in q\text{-idem}(R)$ *and* $e^*R(1 - ee^*)Re^* = 0$.
(2) $e \in q\text{-idem}(R)$ *and* $e^*R(1 - e^*e)Re^* = 0$.

Proof. (1) (⇒) $e \in q\text{-idem}(R)$ and $e^*R(1 - ee^*)Re^* = 0$ follows from Proposition 2.3 (4) and Lemma 2.10 (1), respectively.

(⇐) On one hand, by $e \in q\text{-idem}(R)$ and Proposition 2.16 (1), we have $e^*xye^* = e^*xe^*ye^*$ for any $x, y \in R$. On the other hand, $e^*R(1 - ee^*)Re^* = 0$ implies $e^*xye^* = e^*xee^*ye^*$. Hence,

$$e^*xye^* = e^*(xe)e^*ye^* = e^*xeye^*.$$

It follows that $e \in q^*\text{-idem}(R)$.

(2) Similarly, we can prove this conclusion. □

The following result is a generalization of [16, Proposition 2.13].

Proposition 2.18. *Let* R *be a* $*$*-ring and* $e, f \in q^*\text{-idem}(R)$. *Then*
(1) If $ef \in \mathrm{E}(R)$ *(resp.* $fe \in \mathrm{E}(R)$*), then* $ef \in q^*\text{-idem}(R)$ *(resp.* $fe \in q^*\text{-idem}(R)$*).*
(2) $(ef)^2 = (ef)^n \in q^*\text{-idem}(R)$ *(*$(fe)^2 = (fe)^n \in q^*\text{-idem}(R)$*) for any* $3 \leq n \in \mathbb{Z}$.

Proof. (1) We need to show $(ef)^*xy(ef)^* = (ef)^*xefy(ef)^*$, that is, $f^*e^*xyf^*e^* = f^*e^*xefyf^*e^*$. In fact,

$$f^*(e^*xe)fyf^*e^* = f^*(e^*x)e(yf^*)e^* = f^*e^*xyf^*e^*.$$

This gives the desired result.

(2) By $exye = exeye$, $fxyf = fxfyf$, $e^*xye^* = e^*xeye^*$ and $f^*xyf^* = f^*xfyf^*$, one can easily prove this statement. □

Theorem 2.19. *Let* R *be a* $*$*-ring and* $e \in \mathrm{E}(R)$. *Then* $e \in q^*\text{-idem}(R)$ *if and only if* $(ex - xe^*)(e^*y - ye)e = 0$ *for any* $x, y \in R$.

Proof. (⇒) Since $e \in q^*\text{-idem}(R)$, $exye = exe^*ye$ for any $x, y \in R$. Then $(ex - xe^*)(e^*y - ye)e = exe^*ye - exye - xe^*ye + xe^*ye = 0$.

(⇐) By a direct computation, for any $x, y \in R$, we have $(ex - xe^*)(e^*y - ye)e = exe^*ye - exye - xe^*ye + xe^*ye = exe^*ye - exye = 0$, which implies $exe^*ye = exye$. Hence, by Corollary 2.6 (3), $e \in q^*\text{-idem}(R)$. □

Corollary 2.20. *Let* R *be a* $*$*-ring and* $e \in \mathrm{E}(R)$. *Then* $e \in q^*\text{-idem}(R)$ *if and only if* $(e^*x - xe)(ey - ye^*)e^* = 0$.

3. New characterizations of quasi-normal rings

The aim of this section is to give some new characterizations of quasi-normal rings (or q-abelian rings).

Theorem 3.1. *Let R be a ring. Then R is quasi-normal if and only if $e\mathrm{N}(R)(1-e)Re = 0$.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) For any $a \in R$, we have $ea(1-e) \in \mathrm{N}(R)$. By hypothesis, $e(ea(1-e))(1-e)Re = 0$, i.e., $ea(1-e)Re = 0$, this gives $eR(1-e)Re = 0$. □

Let R be a ring and I be an ideal of R, denote $T_2(R, I) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, c \in R, b \in I\}$. Then it is easy to check that $T_2(R, I)$ is a ring. The following result is inspired by [20, Theorem 4].

Theorem 3.2. *Let R be a ring and I be an ideal of R containing $\mathrm{E}(R)$. Then R is Abelian if and only if $T_2(R, I)$ is quasi-normal.*

Proof. ($\Rightarrow$) For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(R, I)$ and $E = \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix} \in \mathrm{E}(T_2(R, I))$ satisfying $AE = 0$. Then we have

$$AE = \begin{pmatrix} ae_1 & ae_2 + be_3 \\ 0 & ce_3 \end{pmatrix} = 0; \quad \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix} = E = E^2 = \begin{pmatrix} e_1^2 & e_1e_2 + e_2e_3 \\ 0 & e_3^2 \end{pmatrix}.$$

Hence, $e_1, e_3 \in \mathrm{E}(R)$ and

$$\begin{cases} ae_1 = ce_3 = 0, \\ ae_2 + be_3 = 0, \\ e_2 = e_1e_2 + e_2e_3. \end{cases}$$

We need to show $EAT_2(R, I)E = 0$. Taking $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T_2(R, I)$. By a straightforward computation,

$$\begin{aligned} EABE &= \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix} \\ &= \begin{pmatrix} e_1a & e_1b + e_2c \\ 0 & e_3c \end{pmatrix}\begin{pmatrix} xe_1 & xe_2 + ye_3 \\ 0 & xe_3 \end{pmatrix} \\ &= \begin{pmatrix} e_1axe_1 & e_1axe_2 + e_1aye_3 + e_1bxe_3 + e_2cxe_3 \\ 0 & e_3cxe_3 \end{pmatrix}. \end{aligned}$$

Since R is Abelian,

$$\begin{cases} (e_1a)xe_1 = (ae_1)xe_1 = 0, \\ (e_1a)xe_2 = (ae_1)xe_2 = 0, \\ (e_1a)ye_3 = (ae_1)ye_3 = 0, \\ e_2c(xe_3) = e_2(ce_3)x = 0, \\ e_3c(xe_3) = e_3(ce_3)x = 0. \end{cases}$$

Moreover, $e_1b(xe_3) = e_1(be_3)x = -(e_1a)e_2x = -(ae_1)e_2x = 0$. Hence, $EABE = 0$, i.e., $EAT_2(R, I)E = 0$. It follows that $T_2(R, I)$ is quasi-normal.

($\Leftarrow$) For any $e \in \mathrm{E}(R)$, we have $\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \in \mathrm{E}(T_2(R, I))$ and $\begin{pmatrix} 0 & 1-e \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = 0$. Since $T_2(R, I)$ is quasi-normal, $\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 0 & 1-e \\ 0 & 0 \end{pmatrix} T_2(R, I) \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = 0$. For any $a \in R$, we have

$$\begin{pmatrix} 0 & 1-e \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & (1-e)a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & (1-e)ae \\ 0 & e \end{pmatrix} = 0.$$

Hence, $(1-e)ae = 0$ for any $a \in R$, i.e., $ae = eae$. It follows that e is left semicentarl, and so R is Abelian. □

Recall [17], idempotents e and f of a ring R are said to be *similar* if there exist $x, y \in R$ such that $e = xy$ and $f = yx$, which is written by $e \sim f$.

Theorem 3.3. *Let R be a ring. Then R is quasi-normal if and only if for any $e, f \in \mathrm{E}(R)$, $e \sim f$ implies $fR(1-e)Rf = 0$.*

Proof. (⇒) Since $e \sim f$, there exist x, y such that $e = xy$ and $f = yx$. It is easy to compute $f = f^2 = y(xy)x = yex$. Now, $fR(1-e)Rf = y(exR(1-e)Rye)x \subseteq y(eR(1-e)Re)x = 0$.

(⇐) For any $e \in \mathrm{E}(R)$, we have $e \sim e$. So by hypothesis, $eR(1-e)Re = 0$. □

Let $e \in \mathrm{E}(R)$. Define $I_e = \{x \in R | ex(1-e)Re = 0\}$, then it is clear $0, 1, e, 1-e \in I_e$.

Theorem 3.4. *Let R be a ring. Then the following statements are equivalent:*

(1) R is quasi-normal.

(2) $I_e = R$ for any $e \in \mathrm{E}(R)$.

(3) $I_e \lhd R$ for any $e \in \mathrm{E}(R)$.

Proof. (1)⇒(2) By R is quasi-normal, we have $eR(1-e)Re = 0$. This gives (2).

(2)⇒(3) Clear.

(3)⇒(1) Notice that $e \in I_e$, we have $eR(1-e)Re = e(eR(1-e)Re) \subseteq e(I_eR(1-e)Re) \subseteq eI_e(1-e)Re = 0$. It follows that R is quasi-normal. □

Let R be a ring, and $L_2(R) = \{\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix} | a, b \in R\}$. Then it is easy to check that R is a ring.

Theorem 3.5. *Let R be a ring. Then R is Abelian if and only if* $\mathrm{E}(L_2(R)) = \{\begin{pmatrix} e & 0 \\ g & e-g \end{pmatrix} | e^2 = e, g = 2eg - g^2\}$.

Proof. (⇒) Let $E = \begin{pmatrix} e & 0 \\ g & e-g \end{pmatrix} \in \mathrm{E}(L_2(R))$. Then $E = E^2$, i.e.,

$$\begin{pmatrix} e & 0 \\ g & e-g \end{pmatrix} = \begin{pmatrix} e^2 & 0 \\ ge + (e-g)g & (e-g)^2 \end{pmatrix}.$$

This gives

$$\begin{cases} e = e^2, \\ g = ge + eg - g^2. \end{cases}$$

Since R is Abelian, $e \in \mathrm{Z}(R)$, and so $ge = eg$. Hence, $g = 2eg - g^2$.

(⇐) Let $f \in \mathrm{E}(R)$. Then for any $x \in R$, let $g = (1-f)xf$ and $e = 1 - f = e^2$, we have $ge = 0$, $eg = g$ and $g^2 = 0$. So,

$$\begin{pmatrix} e & 0 \\ g & e-g \end{pmatrix}^2 = \begin{pmatrix} e^2 & 0 \\ ge + (e-g)g & (e-g)^2 \end{pmatrix} = \begin{pmatrix} e & 0 \\ g & e-g \end{pmatrix}.$$

By $eg = g$ and $g^2 = 0$, we have $g = 2eg - g^2 = 2g$. This infers $g = 0$, i.e., $(1-f)xf = 0$. It follows that f is left semicentral, and hence R is Abelian. □

Similar to Theorem 2.19 and Corollary 2.20, we have the following conclusion.

Proposition 3.6. *Let R be a $*$-ring and $e \in \mathrm{E}(R)$. Then the following statements are equivalent.*
(1) $e \in q\text{-idem}(R)$.
(2) $(ex - xe)(ey - ye)e = 0$ for any $x, y \in R$.
(3) $(e^*x - xe^*)(e^*y - ye^*)e^* = 0$ for any $x, y \in R$.

The following result is a direct conclusion from Proposition 3.6 or [16, Theorem 4.5].

Corollary 3.7. *Let R be a ring. Then R is quasi-normal if and only if $(ex - xe)(ey - ye)e = 0$ for any $e \in \mathrm{E}(R)$ and $x, y \in R$.*

In [19], the authors defined a class of matrix by $S_n(R, I) = \{\begin{pmatrix} a & b_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a & b_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & 0 & b_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix} | a, a_{ij} \in R, b_{ij} \in I\}$ to study quasi-$*$-IFP in a $*$-ring, where $n \geq 3$. Here, we will use the simplest condition: $n = 3$ to describe quasi-normal rings. Moreover, we think the following result can be generalized to arbitrary $n > 3$.

Theorem 3.8. *Let R be a Abelian ring. Then $S_3(R, I)$ is quasi-normal.*

Proof. For any $A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \in S_3(R, I)$ and $E = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & e_1 & e_4 \\ 0 & 0 & e_1 \end{pmatrix} \in \mathrm{E}(S_3(R, I))$ satisfying $AE = 0$. Then

$$AE = \begin{pmatrix} ae_1 & ae_2 + be_1 & ae_3 + be_4 + ce_1 \\ 0 & ae_1 & ae_4 + de_1 \\ 0 & 0 & ae_1 \end{pmatrix} = 0; \quad \begin{pmatrix} e_1^2 & e_1e_2 + e_2e_1 & e_1e_3 + e_2e_4 + e_3e_1 \\ 0 & e_1^2 & e_1e_4 + e_4e_1 \\ 0 & 0 & e_1^2 \end{pmatrix} = E^2 = E = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & e_1 & e_4 \\ 0 & 0 & e_1 \end{pmatrix}.$$

Hence, $e_1 \in \mathrm{E}(R)$ and

$$\begin{cases} ae_1 = 0, \\ ae_2 + be_1 = 0, \\ ae_3 + be_4 + ce_1 = 0, \\ ae_4 + de_1 = 0, \\ e_1e_2 + e_2e_1 = e_2, \\ e_1e_3 + e_2e_4 + e_3e_1 = e_3, \\ e_1e_4 + e_4e_1 = e_4. \end{cases} \tag{1}$$

Since $e_1 \in \mathrm{E}(R)$ and R is Abelian,

$$\begin{cases} e_1e_2 + e_2e_1 = 2e_1e_2, \\ e_1e_3 + e_3e_1 = 2e_1e_3, \\ e_1e_4 + e_4e_1 = 2e_1e_4. \end{cases} \tag{2}$$

By (2),

$$\begin{cases} ae_2 = 2(ae_1)e_2 = 0, \\ ae_3 = 2(ae_1)e_3 = 0, \\ ae_4 = 2(ae_1)e_4 = 0. \end{cases} \tag{3}$$

By (3), (1) can be reduced as follows

$$\begin{cases} ae_1 = be_1 = de_1 + be_4 = ce_1 = 0, \\ 2e_1e_2 = e_2, \\ 2e_1e_4 = e_4, \\ 2e_1e_3 + e_2e_4 = e_3. \end{cases} \tag{4}$$

A direct computation shows

$$EA = \begin{pmatrix} e_1a & e_1b + e_2a & e_1c + e_2d + e_3a \\ 0 & e_1a & e_1d + e_4a \\ 0 & 0 & e_1a \end{pmatrix}.$$

By (4), $e_3a = (2e_1e_3 + e_2e_4)a = 0$. Hence, $EA = 0$, which implies $EABE = 0$ for any $B \in S_3(R, I)$. It follows that $EAS_3(R, I)E = 0$, and so $S_3(R, I)$ is quasi-normal.

4. Characterizations of ∗-quasi-normal rings

In this section, we will study ∗-quasi-normal rings. The following conclusion is proved by us in another paper.

Lemma 4.1. *Let R be a ∗-ring. Then R is ∗-quasi-normal if and only if R is quasi-normal and* $\mathrm{E}(R) \subseteq R^{\mathrm{PI}}$.

Proof. (⇒) It follows from $e = ee^*e$ and R is quasi-normal.

(⇐) For any $e \in \mathrm{E}(R)$, we have $e^*R(1-e)^*Re^* = 0$ because R is quasi-normal and $e^* \in \mathrm{E}(R)$. Noting that $e^*eR(1-e)^*Ree^* \subseteq e^*R(1-e)^*Re^*$. Then $e^*eR(1-e)^*Ree^* = 0$ and so $e(e^*eR(1-e)^*Ree^*)e = 0$. Since $\mathrm{E}(R) \subseteq R^{\mathrm{PI}}$, $e = ee^*e$, we have $eR(1-e)^*Re = 0$. It follows that R is ∗-quasi-normal. □

Theorem 4.2. *Let R be a ∗-ring. Then R is ∗-quasi-normal if and only if $eR(1-ee^*e)Re = 0$ for any $e \in \mathrm{E}(R)$.*

Proof. (⇒) Since R is ∗-quasi-normal, $e = ee^*e$ and R is quasi-normal. Hence, $eR(1-ee^*e)Re = eR(1-e)Re = 0$.

(⇐) By $eR(1-ee^*e)Re = 0$, we have $e = ee^*e$. This gives R is quasi-normal and $\mathrm{E}(R) \subseteq R^{\mathrm{PI}}$. It follows that R is ∗-quasi-normal by Lemma 4.1. □

Lemma 4.1 can be used to check whether a quasi-normal ring is ∗-quasi-normal or not. In Theorem 3.5, assume that R is commutative, define $* : L_2(R) \to L_2(R)$, $\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix} \mapsto \begin{pmatrix} a-b & 0 \\ -b & a \end{pmatrix}$. Then one can check that $L_2(R)$ is a ∗-ring. Moreover, if R is commutative, then $L_2(R)$ is commutative, and a straightforward computation shows $\mathrm{E}(L_2(R)) = \{\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix} | a = a^2, b = 2ab - b^2, a, b \in R\}$. In particular, taking $R = \mathbb{Z}$, $R = \mathbb{Q}$, $R = \mathbb{R}$, or $R = \mathbb{C}$, we have $\mathrm{E}(L_2(R)) = \{0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\}$.

Example 4.3. *Consider $L_2(\mathbb{C})$, then by the commutativity of $L_2(\mathbb{C})$, $eR(1-e)Re = 0$ if and only if $e(1-e)e = 0$, where $e \in \mathrm{E}(L_2(\mathbb{C}))$. When $e = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$, it is easy to compute $e(1-e)e = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} = 0$. Similarly, one can check that $e(1-e)e = 0$ if $e = 0$, or $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $e = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. It follows that $L_2(\mathbb{C})$ is quasi-normal. However, $L_2(\mathbb{C})$ is not ∗-quasi-normal, this is since if $e = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$, or $e = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $ee^*e = 0 \neq e$. Meanwhile, it follows that $\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are not ∗-q-central.*

Remark 4.4. *In Example 4.3, if we set* $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$*, then the conclusions all hold.*

Theorem 4.5. *Let* R *be a* $*$*-ring. Then* R *is* $*$*-quasi-normal if and only if* $eR(1 - e^*ee^*)Re = 0$.

Proof. (⇒) is clear.

(⇐) By hypothesis, $eabe = eae^*ee^*be$ for any $a, b \in R$. If $a = b = e$, then $e = ee^*ee^*e$, if $a = b = e^*$, then $ee^*e = ee^*ee^*e$. So, we have $e = ee^*ee^*e = ee^*e$. Now, $eR(1 - ee^*e)Re = eR(1 - e)^*Re = 0$. This implies R is $*$-quasi-normal. □

Theorem 4.6. *Let* R *be a* $*$ *ring and* $ae = 0$ *implies* $e^*aRe^* = 0$*. Then* R *is* $*$*-quasi-normal.*

Proof. By $(1 - e)e = 0$, we have $e^*(1 - e)Re^* = 0$. In particular, $e^* = e^*ee^*$, i.e., $e = ee^*e$. Since $x(1 - e)e = 0$ for any $x \in R$, $e^*x(1 - e)Re^* = 0$, that is, $e^*R(1 - e)Re^* = 0$. This gives R is $*$-quasi-normal. □

Theorem 4.7. *Let* R *be a* $*$ *ring and* $ae = 0$ *implies* $e^*a\mathrm{N}(R)e^* = 0$*. Then* R *is* $*$*-quasi-normal.*

Proof. Since $x(1-e)e = 0$ and $(1-e)ye^*e \in \mathrm{N}(R)$, $e^*x(1-e)(1-e)ye^*ee^* = e^*x(1-e)ye^* = 0$, i.e., $e^*R(1-e)Re^* = 0$. Hence, R is $*$-quasi-normal. □

Recall that an element in a $*$-ring is *left* $*$*-cancellable* (resp. *right* $*$*-cancellable*) if $aa^*x = aa^*y$ implies $a^*x = a^*y$ (resp. $xa^*a = ya^*a$ implies $xa^* = ya^*$). An involution $*$ is called *proper* (resp. *semiproper*) if any nonzero element $a \in R$, $aa^* = 0$ (resp. $aRa^* = 0$) implies $a = 0$. Clearly, a proper involution is semiproper [1, 3].

Theorem 4.8. *Let* R *be a* $*$*-ring with* $*$ *being semiproper. Then if* $ae = 0$ *implies* $eaRe^* = 0$ *for any* $e \in \mathrm{E}(R)$*, we have that* R *is* $*$*-quasi-normal.*

Proof. Since $x(1 - e)e = 0$ for any $x \in R$, $ex(1 - e)Re^* = 0$. It follows that $eR(1 - e)Re^* = 0$. Since

$$(e - ee^*e)R(e - ee^*e)^* = (e - ee^*e)R(e^* - e^*ee^*) = e(1 - e^*e)Re^*(1 - e)1e^* \subseteq eR(1 - e)Re^* = 0,$$

$e = ee^*e$ by $*$ is semiproper. Note that for any $x \in R$, $e^*x(1 - e)e = 0$, so $ee^*x(1 - e)Re^* = 0$. This gives $e^*(ee^*x(1 - e)Re^*) = e^*x(1 - e)Re^* = 0$, i.e., $e^*R(1 - e)Re^* = 0$. It follows that R is $*$-quasi-normal. □

Theorem 4.9. *Let* R *be a* $*$*-ring with* $*$ *being proper. Then if* $ae = 0$ *implies* $ea\mathrm{N}(R)e^* = 0$ *for any* $e \in \mathrm{E}(R)$*, we have that* R *is* $*$*-quasi-normal.*

Proof. Since $e^*(1-e)e = 0$, $ee^*(1-e)\mathrm{N}(R)e^* = 0$. It is noted that $(1-e)xe \in \mathrm{N}(R)$ for any $x \in R$, so $ee^*(1-e)xee^* = 0$. Taking $x = e^*$, we have $ee^*ee^* = ee^*ee^*ee^*$. By a straightforward computation, we have

$$\begin{aligned}(ee^* - ee^*ee^*)(ee^* - ee^*ee^*)^* &= (ee^* - ee^*ee^*)(ee^* - ee^*ee^*)\\ &= ee^*ee^* - ee^*ee^*ee^* - ee^*ee^*ee^* + (ee^*ee^*ee^*)ee^*\\ &= 0.\end{aligned}$$

Note that $*$ is proper, so $ee^* = ee^*ee^*$. By the proof of Theorem 4.8, we know that $e = ee^*e$. Since $e^*x(1-e)e = 0$ and $(1-e)ye^*e \in \mathrm{N}(R)$ for any $x, y \in R$, $ee^*x(1-e)(1-e)ye^*ee^* = ee^*x(1-e)ye^* = 0$, i.e., $ee^*R(1-e)Re^* = 0$. This infers $e^*R(1-e)Re^* = e^*(ee^*R(1-e)Re^*) = 0$. It follows that R is $*$-quasi-normal. □

Theorem 4.10. *Let* R *be a* $*$*-ring with* $*$ *being proper, and* $e^*R(1 - e)^*Re = 0$*. Then* R *is* $*$*-quasi-normal.*

Proof. By $e^*R(1-e)^*Re = 0$, we have $e^*e = e^*ee^*e$. It is easy to compute

$$\begin{aligned}(e^* - e^*ee^*)(e^* - ee^*e)^* &= (e^* - e^*ee^*)(e - ee^*e)\\ &= e^*e - e^*ee^*e - e^*ee^*e + (e^*ee^*e)e^*e\\ &= 0.\end{aligned}$$

Since $*$ is proper, $e^* = e^*ee^*$, that is, $e = ee^*e$. Again, by $e^*R(1-e)^*Re = 0$, we have $e^*ex(1-e)^*Re = 0$ for any $x \in R$. This implies $e(e^*ex(1-e)^*Re) = ex(1-e)^*Re = 0$, which infers $eR(1-e)^*Re = 0$. This shows that R is $*$-quasi-normal. □

By Theorem 2.19, we get the following result.

Proposition 4.11. *Let R be a $*$-ring. Then R is $*$-quasi-normal if and only if $(ex - xe^*)(e^*y - ye)e = 0$ for any $e \in \mathrm{E}(R)$ and $x, y \in R$.*

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Conflict of Interest

The authors declared that they have no conflict of interest.

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