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Reversibility of T_k-topological spaces and the universality problem for the collection of T_k-topological spaces

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Abstract. As a generalization of the Khalimsky line topological space, consider the topological space $(\mathbb{Z}, T_k)$ (resp. $(\mathbb{Z}, T'_k)$) on the set of integers, where the topology T_k (resp. T'_k) is generated by the set S_k (resp. S'_k) as a subbase, $k \in \mathbb{Z}$, and $S_k := \{S_{k,t} \,|\, S_{k,t} := \{2t, 2t+1, 2t+2k+1\}, t \in \mathbb{Z}\}$ (resp. $S'_k := \{S'_{k,t} \,|\, S'_{k,t} := \{2t, 2t+1, 2t+2k\}, t \in \mathbb{Z}\}$). For $k \in \mathbb{Z} \setminus \{0\}$, each of the T_k- and T'_k-topological space indeed satisfies the $T_{\frac{1}{2}}$-separation axiom. Besides, for $k \in \mathbb{Z} \setminus \{0\}$, each of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$ is an Alexandroff space and is neither a Kuratowski space nor a regular space. The paper initially proves that each of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k), k \in \mathbb{Z}$, is reversible. Next, let $\mathcal{T}$ be the collection of T_k-topological spaces $\{(\mathbb{Z}, T_k), k \in \mathbb{Z} \setminus \{0\}\}$, and $\mathcal{T}'$ be the set of T'_k-topological spaces $\{(\mathbb{Z}, T'_k), k \in \mathbb{Z} \setminus \{0\}\}$. Then the paper deals with an existence problem of a universal element in $\mathcal{T}$ and $\mathcal{T}'$.

1. Introduction

In the present paper, we follow the notations $\mathbb{N}$ and $\mathbb{Z}$ that are the sets of positive integers (i.e., natural numbers) and integers, respectively. Besides, for distinct integers $a, b \in \mathbb{Z}$ we take the notation $[a, b]_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, a \le t \le b\}$ and "$\subset$" (resp. $\sharp X$) which denotes a 'proper subset or equal' (resp. the cardinality of the set X). In addition, the notation " $:=$" is used to introduce a new term. For $k \in \mathbb{N}$ and $i \in [0, k-1]_{\mathbb{Z}}$, we take the notation $k\mathbb{Z} + i := \{kt + i \,|\, t \in \mathbb{Z}\}$. Besides, $\mathbb{N}_-$ and $\aleph_0$ indicate the set of negative integers and the first infinite cardinality, respectively.

A topological space (X, T) is called an Alexandroff space [1, 2] if each $x \in X$ has a minimal open neighborhood, i.e., there is the smallest open set containing x. Since the paper is strongly associated with the Khalimsky (K-, for brevity) topological line, let us first recall the K-topological line and its product topology on $\mathbb{Z}^n$ for the n-dimensional K-topological space, $n \in \mathbb{N}$. The K-topology κ on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is generated by the set $\{[2t-1, 2t+1]_{\mathbb{Z}}, \{2t+1\} \,|\, t \in \mathbb{Z}\}$ as a base [22]. Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called an n-dimensional K-topological space and denoted by $(\mathbb{Z}^n, \kappa^n)$ [23] and it turns out that it

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is an Alexandroff topological space. Indeed, the papers [5, 10–19] include the study of its various properties.

In order to address an open problem related to the generalization of the K-topological line [15, 16], the recent papers [16, 17] studied infinitely many types of topological structures on $\mathbb{Z}$, say $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, $k \in \mathbb{Z}$, where the topologies T_k and T'_k are, respectively, generated by the sets S_k and S'_k in (1.1) below as subbases [15].

$$\begin{cases} S_k := \{S_{k,t} \,|\, S_{k,t} := \{2t, 2t+1, 2t+2k+1\},\ t \in \mathbb{Z}\} \text{ and} \\ S'_k := \{S'_{k,t} \,|\, S'_{k,t} := \{2t, 2t+1, 2t+2k\}, t \in \mathbb{Z}\}. \end{cases} \quad (1.1)$$

Owing to the subbases of (1.1), it is clear that both $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, $k \in \mathbb{Z}$, are Alexandroff spaces [16]. Furthermore, in case $k \in \mathbb{Z} \setminus \{1, -1\}$, it turns out that each $(\mathbb{Z}, T_k)$ is not homeomorphic with the K-topological line [15] . Indeed, $(\mathbb{Z}, T_{-1})$ is equal to $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}, T_1)$ is not equal to $(\mathbb{Z}, \kappa)$ but homeomorphic with $(\mathbb{Z}, \kappa)$ [15]. Besides, it turns out that $(\mathbb{Z}, T_k)$ is connected if and only if $k \in \{-1, 1\}$ [16]. Even though $(\mathbb{Z}, T_k)$ is neither a regular space nor a Kuratowski space (see Proposition 2.5), it was proved that it satisfies the $T_{\frac{1}{2}}$-separation axiom (see Corollary 2.4 of [16]).

Hereinafter, in the space $(\mathbb{Z}, T_k)$, for $X \subset \mathbb{Z}$, $(X, (T_k)_X)$ denotes the subspace induced from $(\mathbb{Z}, T_k)$. Let us now recall the notion of a topological embedding (embedding, for brevity) [24]. Given two topological spaces X and Y, we say that an embedding of X to Y is a function $f : X \to Y$ that maps X homeomorphically to the subspace $f(X)$ in Y.

Indeed, owing to (1.1), the space $(\mathbb{Z}, T_0)$ has $\aleph_0$ connected components $P_{2m} := \{2m, 2m+1\}, m \in \mathbb{Z}$, which are both closed and open in $(\mathbb{Z}, T_0)$. Hence $(\mathbb{Z}, T_0)$ cannot be embedded into any $(\mathbb{Z}, T_k), k \neq 0$, because $(\mathbb{Z}, T_k), k \neq 0$ does not have any subspace which is homeomorphic with $(\mathbb{Z}, T_0)$. Then we may raise an interesting query related to both the reversibility of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$, and the existence problem of a universal element in the collection of T_k-topological spaces, $k \in \mathbb{Z} \setminus \{0\}$. First of all, for $k \in \mathbb{Z} \setminus \{0\}$, based on the topological structures of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, the present paper initially establishes the following maps $h_{(p,q)}$ and $h'_{(p,q)}$, where $p, q \in \mathbb{Z} \setminus \{0\}$ and $|q| = |p| + 1$, as embeddings.

$$\begin{cases} (1)\, h_{(p,q)} : (\mathbb{Z}, T_p) \to (\mathbb{Z}, T_q) \text{ and} \\ (2)\, h'_{(p,q)} : (\mathbb{Z}, T'_p) \to (\mathbb{Z}, T'_q). \end{cases} \quad (1.2)$$

Then, based on (1.2), we will prove that

$$\begin{cases} (\mathbb{Z}, \kappa) \text{ is homeomorphic with each of the subspaces} \\ (1)\, (\mathbb{Z} \setminus Im(h_{(p,q)}), (T_q)_{\mathbb{Z} \setminus Im(h_{(p,q)})}) \text{ induced from } (\mathbb{Z}, T_q) \text{ and} \\ (2)\, (\mathbb{Z} \setminus Im(h'_{(p,q)}), (T'_q)_{\mathbb{Z} \setminus Im(h'_{(p,q)})}) \text{ induced from } (\mathbb{Z}, T'_q), \end{cases} \quad (1.3)$$

where "Im" indicates the image of a given map.

Using the embedding of (1.2) and some properties of (1.3), we deal with the universality problem in the sets $\mathcal{T} := \{(\mathbb{Z}, T_k) \,|\, k \in \mathbb{Z} \setminus \{0\}\}$ and $\mathcal{T}' := \{(\mathbb{Z}, T'_k) \,|\, k \in \mathbb{Z} \setminus \{0\}\}$.

This paper is organized as follows: Section 2 provides some basic notions associated with T_k- or T'_k-topological spaces, $k \in \mathbb{Z}$. Section 3 studies the reversibility of the topological spaces $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, respectively, where $k \in \mathbb{Z}$. Section 4 studies the existence problem of a universal element in $\mathcal{T}$ and $\mathcal{T}'$, where $\mathcal{T}$ is the collection of T_k-topological spaces $(\mathbb{Z}, T_k)$, and $\mathcal{T}'$ is the set of T'_k-topological spaces $(\mathbb{Z}, T'_k)$, $k \in \mathbb{Z} \setminus \{0\}$. Section 5 concludes the paper with summary and further work.

2. Some properties of the T_k-topological spaces

Owing to the subbases in (1.1) for the topologies T_k and T'_k, the following are obtained [15, 16].

Remark 2.1. ([16]) *For $k \in \mathbb{Z} \setminus \{0\}$,*

$$\mathcal{B} := \{S_{k,t} \mid t \in \mathbb{Z}\} \cup \{\{2t+1\} \mid t \in \mathbb{Z}\} \tag{2.1}$$

is a base for $(\mathbb{Z}, T_k)$.

For $A \subset \mathbb{Z}$, owing to the Alexandorff topological structures of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, we consider the smallest open set of a point x in $(\mathbb{Z}, T_k)$ (resp. $(\mathbb{Z}, T'_k)$) or a subspace $(A, (T_k)_A)$ (resp. $(A, (T'_k)_A)$). Then we take only the notation $SN_k(x)$ (resp. $SN'_k(x)$) if there is no danger of ambiguity. Similarly, for $B \subset A \subset \mathbb{Z}$, since we also consider the closure of a subset B in $(\mathbb{Z}, T_k)$ (resp. $(\mathbb{Z}, T'_k)$) or $(A, (T_k)_A)$ (resp. $(A, (T'_k)_A)$), we take only the notation $Cl_k(B)$ (resp. $Cl'_k(B)$) if there is no danger of confusion (see Lemma 2.2 below). By Remark 2.1, the following properties are obtained.

Lemma 2.2. ([16]) *(1) For $k \in \mathbb{Z} \setminus \{0\}$, under $(\mathbb{Z}, T_k)$, for each $t \in \mathbb{Z}$, we have*

$$\begin{cases} SN_k(2t) = S_{k,t}, SN_k(2t+1) = \{2t+1\} \text{ and} \\ Cl_k(\{2t+1\}) = \{2t, 2t+1, 2t-2k\} = S'_{-k,t}, \; Cl_k(\{2t\}) = \{2t\}. \end{cases}$$

(2) With $(\mathbb{Z}, T_0)$, for each $t \in \mathbb{Z}$, $SN_0(2t) = SN_0(2t+1) = \{2t, 2t+1\} = Cl_0(\{2t\}) = Cl_0(\{2t+1\})$.

By Lemma 2.2(2), it is clear that $(\mathbb{Z}, T_0)$ is not a Kolmogoroff space.

Lemma 2.3. ([16]) *For $k \in \mathbb{Z} \setminus \{0\}$, the spaces $(\mathbb{Z}, T_k)$, $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_{-k})$, and $(\mathbb{Z}, T'_{-k})$ are homeomorphic.*

We say that a topological space (X, T) satisfies the $T_{\frac{1}{2}}$-separation axiom ($T_{\frac{1}{2}}$-space, for brevity) if each singleton is either a closed or an open set in (X, T) [6].

Corollary 2.4. *(1) For $k \in \mathbb{Z} \setminus \{0\}$, the space $(\mathbb{Z}, T_k)$ satisfies the $T_{\frac{1}{2}}$-separation axiom [15].*
(2) The product space $(\mathbb{Z}^n, (T_k)^n)$, $n \geq 2$, does not satisfy the $T_{\frac{1}{2}}$-separation axiom (see Lemma 5.1 of [12]).

Since Corollary 2.4 plays an important role in studying the universality problem in Section 4, we just recall the non-satisfaction of the $T_{\frac{1}{2}}$-separation axiom of $(\mathbb{Z}^2, (T_k)^2)$. To be specific, with $(\mathbb{Z}^2, (T_k)^2)$, for $t_1, t_2 \in \mathbb{Z}$, consider the set

$$\{2t_1, 2t_1+1, 2t_1+2k+1\} \times \{2t_2, 2t_2+1, 2t_2+2k+1\}$$

which is the smallest open set containing the point $(2t_1, 2t_2)$. Then take the point $p := (2t_1, 2t_2+1)$ in $(\mathbb{Z}^2, (T_k)^2)$, owing to Lemma 2.2, it is clear that the singleton $\{p\}$ is neither a closed nor an open set in $(\mathbb{Z}^2, (T_k)^2)$.

As for the regularity [24] of $(\mathbb{Z}, T_k)$, we obtain the following

Proposition 2.5. *For $k \in \mathbb{Z} \setminus \{0\}$, each of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$ is neither a Kuratowski space nor a regular space.*

Proof. For each even integer x, since $SN_k(x)$ should include the odd integer $x+1$ (see Lemma 2.2(1)), $(\mathbb{Z}, T_k)$ is not a Kuratowski space.
Next, with $(\mathbb{Z}, T_k)$, since the singleton $\{x\}, x \in 2\mathbb{Z}$, is a closed set (see Lemma 2.2), take a point $x+1 \notin \{x\}$. Then there are not disjoint open sets U and V in $(\mathbb{Z}, T_k)$ such that $x+1 \in U$ and $\{x\} \subset V$ (see Lemma 2.2(1)). Hence $(\mathbb{Z}, T_k)$ is neither a Kuratowski space nor a regular space. By Lemma 2.3, the proof is completed. □

3. Reversibility of T_k-topological spaces and dense subsets of T_k-topological subspaces

It is well known that a finite topological space or a discrete topological space or any space with cofinite topology is reversible [25]. Hence, when studying the reversibility of a topological space, we need to focus on the infinite topological spaces without discrete topology and a space without cofinite topology. This section proves that each of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, $k \in \mathbb{Z}$, is reversible.

Definition 3.1. ([25]) We say that a topological space is reversible if every continuous self-bijection is a homeomorphism.

Theorem 3.2. *For $k \in \mathbb{Z}$, $(\mathbb{Z}, T_k)$ is reversible.*

Proof. (Case 1) Assume the case $k = 0$, i.e., $(\mathbb{Z}, T_0)$. Then we obtain a partition $\{P_{2m} \,|\, m \in \mathbb{Z}, P_{2m} = \{2m, 2m+1\}\}$ of $\mathbb{Z}$ up to connectedness, i.e., $\exists\, \aleph_0$ connected components P_{2m} $m \in \mathbb{Z}$, so that each P_{2m} is both closed and open in $(\mathbb{Z}, T_0)$. With the hypothesis of a continuous self-bijection of $f : (\mathbb{Z}, T_0) \to (\mathbb{Z}, T_0)$, assume $x \in P_{2m_1}$ and $f(x) \in P_{2m_2}$. Then the element $x'(\neq x) \in P_{2m_1}$ should be mapped by f into P_{2m_2} such that

$$\begin{cases} \text{if } f(x) = 2m_2, \text{ then } f(x') = 2m_2 + 1 \text{ and} \\ \text{if } f(x) = 2m_2 + 1, \text{ then } f(x') = 2m_2. \end{cases}$$

Hence we have $f(P_{2m_1}) = P_{2m_2}$ which implies that the map f is an open map so that f is a homeomorphism. (Case 2) Assume the case $k \in \mathbb{N}$, i.e., $(\mathbb{Z}, T_k)$. Let us recall that for a point x in $(\mathbb{Z}, T_k)$ we have the property: $\{\sharp SN_k(x) \,|\, x \in \mathbb{Z}\} = \{1, 3\}$ (see Lemma 2.2). Let us now consider a continuous bijection $f : (\mathbb{Z}, T_k) \to (\mathbb{Z}, T_k)$. Then, based on the map f, for $x \in \mathbb{Z}_e$, $f(x)$ should be mapped into $\mathbb{Z}_e$, where $\mathbb{Z}_e$ is the set of even integers. If not, we have a contradiction to the given continuous bijection f. More precisely, on the contrary, suppose that $f(x) \in \mathbb{Z}_o$, where $\mathbb{Z}_o$ is the set of odd integers. Then, owing to the continuity of f, we obtain $f(SN_k(x)) \subset SN_k(f(x))$ such that $\sharp SN_k(x) = 3$ and $\sharp SN_k(f(x)) = 1$, which invokes a contradiction to the bijection of f because $SN_k(f(x)) = \{f(x)\}$ is an open set in $(\mathbb{Z}, T_k)$ and $SN_k(x) = \{x, x+1, x+2k+1\}$.

Hence, for $x \in \mathbb{Z}_e$, owing to $\sharp SN_k(x) = 3 = \sharp SN_k(f(x))$, we obtain

$$\left.\begin{cases} f(SN_k(x)) = SN_k(f(x)) = \{f(x), f(x)+1, f(x)+2k+1\}, \\ \text{where } SN_k(x) = \{x, x+1, x+2k+1\} \text{ and } f(x) \in \mathbb{Z}_e. \end{cases}\right\} \tag{3.1}$$

Next, for $x \in \mathbb{Z}_o$ we obtain $f(x) \in \mathbb{Z}_o$. If not, we have a contradiction to the given continuous bijection f. More precisely, suppose that $f(x) \in \mathbb{Z}_e$, where $x \in \mathbb{Z}_o$. Then we obtain

$$f(Cl_k(\{x\})) \not\subseteq Cl_k(\{f(x)\}) \text{ since } \sharp Cl_k(\{x\}) = 3 \text{ and } \sharp Cl_k(\{f(x)\}) = 1 \text{ (see Lemma 2.2)}, \tag{3.2}$$

which leads to a contradiction.

Hence, for $x \in \mathbb{Z}_o$, since $SN_k(x) = \{x\}$, owing to $\sharp SN_k(x) = 1 = \sharp SN_k(f(x))$, we have

$$(f(SN_k(x)) \subset SN_k(f(x)) = \{f(x)\}) \Rightarrow (f(SN_k(x)) = SN_k(f(x))). \tag{3.3}$$

Based on (3.1) and (3.3), for any x in $(\mathbb{Z}, T_k)$ we eventually have $f(SN_k(x)) = SN_k(f(x))$, which implies that f is an open map, i.e., f is a homeomorphism.
(Case 3) Assume the case $k \in \mathbb{N}_{-1}$. By using a method similar to (Case 2) above, a given continuous bijection f is also an open map, which implies that the map f is a homeomorphism. □

Corollary 3.3. *For $k \in \mathbb{Z}$, $(\mathbb{Z}, T'_k)$ is reversible.*

Proof. Since reversibility is a topological property [25], by Lemma 2.3, and Theorem 3.2, the proof is completed. □

Since $(\mathbb{Z}, T_{-1})$ is equal to $(\mathbb{Z}, \kappa)$, by Theorem 3.2, the following is obtained.

Corollary 3.4. ([5]) *$(\mathbb{Z}, \kappa)$ is reversible.*

The following fact plays an important role in studying T_k-topological spaces.

Proposition 3.5. ([17])
With the topological space $(\mathbb{Z}, T_k), k \in \mathbb{N}$, a partition $\{P_i \,|\, i \in [1,k]_{\mathbb{Z}}\}$ of $\mathbb{Z}$ exists up to connectedness, i.e., each P_i is a component of $(\mathbb{Z}, T_k)$, where $P_i = \bigcup_{t \in 2k\mathbb{Z}+2(i-1)} \{t, t+1, t+2k+1\}$. Then, in $(\mathbb{Z}, T_k), k \in \mathbb{N}$, the following are obtained,

$$\begin{cases} (1)\, (P_i, (T_k)_{P_i}) \text{ is homeomorphic with } (P_j, (T_k)_{P_j}), i, j \in [1,k]_{\mathbb{Z}}. \\ (2)\, \text{For any } i \in [1,k]_{\mathbb{Z}}, (P_i, (T_k)_{P_i}) \text{ is homeomorphic with } (\mathbb{Z}, \kappa). \end{cases}$$

By Proposition 3.5(2), since topological features of $(\mathbb{Z}, T_k)$ are totally influenced by the corresponding properties of the subspaces $(P_i, (T_k)_{P_i}), i \in [1,k]_{\mathbb{Z}}$, we need to characterize topological features of $(P_i, (T_k)_{P_i})$, as follows:

Theorem 3.6. *With Proposition 3.5, the following are obtained.*
(1) Each $(P_i, (T_k)_{P_i}), i \in [1,k]_{\mathbb{Z}}$, has the dense subset $2k\mathbb{Z} + 2i - 1$.
(2) The set $2k\mathbb{Z} + 2(i-1)$ is a nowhere dense subset of $(P_i, (T_k)_{P_i})$.

Proof. (1) With the topological space $(P_i, (T_k)_{P_i})$, for each element $x \in 2k\mathbb{Z} + 2(i-1) \subset P_i$, $SN_k(x)$ contains the odd integer $x+1$, i.e., $SN_k(x) \cap (2k\mathbb{Z} + 2i - 1) \neq \emptyset$, which implies that $Cl_k(2k\mathbb{Z} + 2i - 1) = P_i$.
(2) Since $Cl_k(2k\mathbb{Z} + 2(i-1)) = 2k\mathbb{Z} + 2(i-1)$, each nonempty subset of $2k\mathbb{Z} + 2(i-1)$ is not an open subset of $(P_i, (T_k)_{P_i})$. Namely, $Int_k(Cl_k(2k\mathbb{Z} + 2(i-1))) = \emptyset$. □

Example 3.7. (1) The sets $6\mathbb{Z}+1$, $6\mathbb{Z}+3$, and $6\mathbb{Z}+5$ are, respectively, dense subsets of $(P_1, (T_3)_{P_1})$, $(P_2, (T_3)_{P_2})$, and $(P_3, (T_3)_{P_3})$.
(2) The sets $6\mathbb{Z}$, $6\mathbb{Z}+2$, and $6\mathbb{Z}+4$ are, respectively, nowhere dense subsets of $(P_1, (T_3)_{P_1})$, $(P_2, (T_3)_{P_2})$, and $(P_3, (T_3)_{P_3})$.

4. The universality problem for a collection of T_k-topological spaces, $k \in \mathbb{Z} \setminus \{0\}$

The study of the universality problem has been studied in the set of regular or almost regular spaces [3, 4, 7–9, 20, 21]. Since the regularity is independent of the $T_{\frac{1}{2}}$-separation axiom, it is very meaningful to study the universality problem in the set $\mathcal{T}$: the collection of T_k-topological spaces, $k \in \mathbb{Z} \setminus \{0\}$ (see Proposition 2.5), which can play an important role in pure and applied topology. This section investigates some conditions that for $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$, $(\mathbb{Z}, T_{|k_1|})$ is topologically embedded into $(\mathbb{Z}, T_{|k_2|})$. Besides, it also develops various properties associated with this embedding.

Definition 4.1. ([20, 21]) Let C be a class of topological spaces. Then we say that a topological space (X, T) is universal in C if the following conditions are satisfied.
(1) $(X, T) \in C$ and
(2) for each $(Y, T') \in C$, there exists an embedding of (Y, T') into (X, T).

As mentioned in the previous part, since $(\mathbb{Z}, T_0)$ is not embedded into any $(\mathbb{Z}, T_k)$, $k \in \mathbb{Z} \setminus \{0\}$, let us consider the following:

Lemma 4.2. *Let us consider the collection of T_k-topological spaces $(\mathbb{Z}, T_k)$, $k \in \mathbb{Z} \setminus \{0\}$. Then each topological space $(\mathbb{Z}, T_k)$, $k \notin \{-1, 1\}$, is not embedded into $(\mathbb{Z}, T_1)$, $(\mathbb{Z}, T_{-1})$, $(\mathbb{Z}, T'_1)$, and $(\mathbb{Z}, T'_{-1})$.*

Proof. By Lemma 2.3 and the topological property of reversibility, without loss of generality, we may consider the case $k \in \mathbb{N} \setminus \{1\}$. Suppose that there is an embedding h from $(\mathbb{Z}, T_k)$, $k \neq 1$, into $(\mathbb{Z}, T_1)$. By Proposition 3.5, $(\mathbb{Z}, T_k)$ has k components such as the partition $\{P_i \,|\, i \in [1,k]_{\mathbb{Z}}\}$ of $\mathbb{Z}$ and further, the subspace $(P_i, (T_k)_{P_i})$ is homeomorphic with $(\mathbb{Z}, T_1)$ which is homeomorphic with $(\mathbb{Z}, \kappa)$. Thus, for any $i, j \in [1,k]_{\mathbb{Z}}$

with $i \neq j$, the images $h(P_i)$ and $h(P_j)$ should be nonempty, disjoint, and each of them is connected in $(\mathbb{Z}, T_1)$. However, owing to the embedding h, we have $h(P_i) = \mathbb{Z}$ so that $h(P_j) = \emptyset$, which invokes a contradiction. In a similar way, by Lemma 2.3, we prove a non-embedding from $(\mathbb{Z}, T_k)$ to $(\mathbb{Z}, T_{-1})$, $(\mathbb{Z}, T'_1)$, and $(\mathbb{Z}, T'_{-1})$, $k \notin \{-1, 1\}$. □

Since the study of the universality problem of a class of topological spaces is related to the notion of weight of a topological space, let us now recall the notion. We say that the weight of a topological space is the minimum possible cardinality of a basis for the topological space [3].

In Proposition 3.5, since the homeomorphism between $(\mathbb{Z}, \kappa)$ and $(P_i, (T_k)_{P_i})$ plays a crucial role in the present paper, let us now develop a new homeomorphism between them, which makes the approach used in [17] more advanced, as follows:

Remark 4.3. With the given partition $\{P_i \mid i \in [1, k]_{\mathbb{Z}}\}$ of $\mathbb{Z}$ of Proposition 3.5, there is a homeomorphism $\mu_i : (\mathbb{Z}, \kappa) \to (P_i, (T_k)_{P_i})$, where $P_i = \bigcup_{t \in 2k\mathbb{Z}+2(i-1)} \{t, t+1, t+2k+1\}$ (see Proposition 3.5), defined by

$$\mu_i(x) = \begin{Bmatrix} kx + 2(i-1), x \in 2\mathbb{Z}, \text{ and} \\ kx + k + 1 + 2(i-1), x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

Then it is clear that the map μ_i is a continuous bijection and an open map, which leads to a homeomorphism between them.

Example 4.4. With the topological space $(\mathbb{Z}, T_4)$, by Proposition 3.5, consider four components of $(\mathbb{Z}, T_4)$, i.e., a partition $\{P_i \mid i \in [1, 4]_{\mathbb{Z}}\}$ of $\mathbb{Z}$ up to connectedness, where $P_i = \bigcup_{t \in 8\mathbb{Z}+2(i-1)} \{t, t+1, t+9\}$. We obtain the following homeomorphism $\mu_i : (\mathbb{Z}, \kappa) \to (P_i, (T_4)_{P_i}), i \in [1, 4]_{\mathbb{Z}}$ (see Figure 1),

$$\mu_i(x) = \begin{Bmatrix} 4x + 2(i-1), x \in 2\mathbb{Z}, \text{ and} \\ 4x + 5 + 2(i-1), x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

For instance, $\mu_4 : (\mathbb{Z}, \kappa) = (\mathbb{Z}, T_{-1}) \to (P_4, (T_4)_{P_4})$ (see Figure 1), defined by

$$\mu_4(x) = \begin{Bmatrix} 4x + 6, x \in 2\mathbb{Z}, \text{ and} \\ 4x + 11, x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

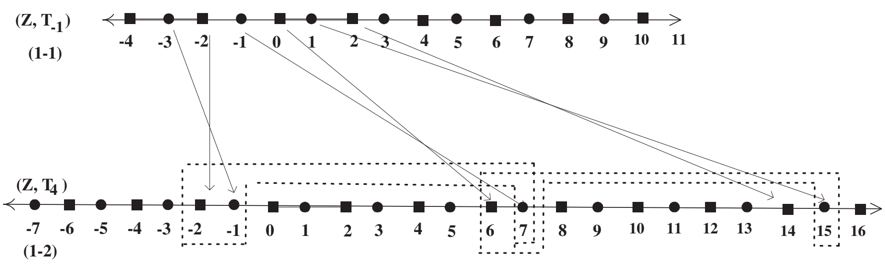

Figure 1: Configuration of the homeomorphism $\mu_4 : (\mathbb{Z}, \kappa) = (\mathbb{Z}, T_{-1}) \to (P_4, (T_4)_{P_4})$ stated in Example 4.4. The black dots and squares mean that the singleton consisting of a black dot (resp. a square) is an open (resp. closed) set in the given topological space.

By Lemma 2.3 and Proposition 3.5, since $(\mathbb{Z}, T_0)$ cannot be embedded into $(\mathbb{Z}, T_k)$, $k \in \mathbb{Z} \setminus \{0\}$, we need to focus on the other cases. In [14], we proved that for $k_1, k_2 \in \mathbb{N}$, there is an embedding of $(\mathbb{Z}, T_{k_1})$ into $(\mathbb{Z}, T_{k_2})$ if and only if $k_1 \leq k_2$. As a generalization of this approach, the following is obtained.

Theorem 4.5. *For $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$, there is an embedding of $(\mathbb{Z}, T_{k_1})$ into $(\mathbb{Z}, T_{k_2})$ if and only if $|k_1| \leq |k_2|$.*

Before proving the assertion, if Theorem 4.5 is affirmative, by Lemma 2.3, we obtain the following embeddings for the numbers $p, q \in \mathbb{Z} \setminus \{0\}$, where $|q| = |p| + 1$

$$\begin{cases} (1)\, h_{(p,q)} : (\mathbb{Z}, T_p) \to (\mathbb{Z}, T_q) \text{ and} \\ (2)\, h'_{(p,q)} : (\mathbb{Z}, T'_p) \to (\mathbb{Z}, T'_q). \end{cases} \quad (4.1)$$

Namely, by Lemma 2.3, we obtain the series of embeddings which support the assertion.

$$\begin{cases} (1)\, T_1 \xrightarrow{h_{(1,2)}} T_2 \xrightarrow{h_{(2,3)}} \cdots \longrightarrow T_{k-1} \xrightarrow{h_{(k-1,k)}} T_k \xrightarrow{h_{(k,k+1)}} T_{k+1} \xrightarrow{h_{(k+1,k+2)}} \cdots, \\ (2)\, T_{-1} \xrightarrow{h_{(-1,-2)}} T_{-2} \xrightarrow{h_{(-2,-3)}} \cdots \longrightarrow T_{-k} \xrightarrow{h_{(-k,-k-1)}} T_{-k-1} \xrightarrow{h_{(-k-1,-k-2)}} \cdots, \\ (3)\, T'_1 \xrightarrow{h'_{(1,2)}} T'_2 \xrightarrow{h'_{(2,3)}} \cdots \longrightarrow T'_{k-1} \xrightarrow{h'_{(k-1,k)}} T'_k \xrightarrow{h'_{(k,k+1)}} T'_{k+1} \xrightarrow{h'_{(k+1,k+2)}} \cdots, \\ (4)\, T'_{-1} \xrightarrow{h'_{(-1,-2)}} T'_{-2} \xrightarrow{h'_{(-2,-3)}} \cdots \longrightarrow T'_{-k} \xrightarrow{h'_{(-k,-k-1)}} T'_{-k-1} \xrightarrow{h'_{(-k-1,-k-2)}} \cdots, \\ \text{and so on} \end{cases} \quad (4.2)$$

Motivated by the maps in (4.1) and (4.2), we now prove the assertion.

Proof. ($\Rightarrow$) Using the contrapositive law, we prove that in case $|k_1| \gneq |k_2|$, no embedding of $(\mathbb{Z}, T_{k_1})$ into $(\mathbb{Z}, T_{k_2})$ exists. On the contrary, suppose that there is an embedding from $(\mathbb{Z}, T_{k_1})$ into $(\mathbb{Z}, T_{k_2})$, where $|k_1| \gneq |k_2|$. By Lemma 2.3, it suffices to prove the case $k_1 \gneq k_2$, where $k_1, k_2 \in \mathbb{N}$. Suppose that an embedding of $(\mathbb{Z}, T_{k_1})$ into $(\mathbb{Z}, T_{k_2})$ exists. As a convenience, let us denote by $h : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{k_2})$ the embedding. Besides, by Proposition 3.5, we may assume that the set $\mathbb{Z}$ of $(\mathbb{Z}, T_{k_1})$ (resp. $(\mathbb{Z}, T_{k_2})$) is partitioned by $\{P_i \,|\, i \in [1, k_1]_{\mathbb{Z}}\}$ (resp. $\{Q_i, | \, i \in [1, k_2]_{\mathbb{Z}}\}$). Then, owing to the embedding h, there is a non-empty set $M := \{i_1, i_2, \cdots, i_{k_2}\} \subsetneq [1, k_1]_{\mathbb{Z}}$ such that

$$h(\bigcup_{i \in M} P_i) = \mathbb{Z} = \bigcup_{i \in [1,k_2]_{\mathbb{Z}}} Q_i,$$

so that

$$h(\bigcup_{i \in [1,k_1]_{\mathbb{Z}} \setminus M} P_i) = \emptyset$$

which implies that the map h is not an embedding, which invokes a contradiction to the hypothesis.
($\Leftarrow$) (Case 1) In case $|k_1| = |k_2|$, by Lemma 2.3, the proof is straightforward.
(Case 2) Let us now consider the case $|k_1| \lneq |k_2|$. Then we can consider the following four cases.
(Case 2-1) In case $k_1, k_2 \in \mathbb{N}$ such that $k_1 \lneq k_2$, we obtain the following embedding [14] (motivated by the case of (1) of (4.2))

$$h_{(k_1,k_2)} : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{k_2}) \quad (4.3)$$

defined by

$$h_{(k_1,k_2)}(x) = \begin{Bmatrix} \frac{k_2}{k_1} x, x \in 2k_1\mathbb{Z}, \\ \frac{k_2}{k_1}(x-1) + 1, x \in 2k_1\mathbb{Z} + 1, \\ \frac{k_2}{k_1}(x-2) + 2, x \in 2k_1\mathbb{Z} + 2, \\ \frac{k_2}{k_1}(x-3) + 3, x \in 2k_1\mathbb{Z} + 3, \\ \cdots \\ \frac{k_2}{k_1}(x-(2k_1-1)) + 2k_1 - 1, x \in 2k_1\mathbb{Z} + 2k_1 - 1. \end{Bmatrix} \quad (4.4)$$

(Case 2-2) In case $k_1, k_2 \in \mathbb{N}_-$ such that $|k_1| \lneq |k_2|$, we have the embedding (motivated by the case of (2) of (4.2))

$$g_2 \circ h_{(-k_1,-k_2)} \circ g_1 : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{k_2})$$

in terms of the composite

$$(\mathbb{Z}, T_{k_1}) \xrightarrow{g_1} (\mathbb{Z}, T_{-k_1}) \xrightarrow{h_{(-k_1,-k_2)}} (\mathbb{Z}, T_{-k_2}) \xrightarrow{g_2} (\mathbb{Z}, T_{k_2}),$$

where $h_{(-k_1,-k_2)}$ is the map of (4.4) and

$$g_1 : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{-k_1}) \tag{4.5}$$

defined by

$$g_1(x) = \begin{Bmatrix} x, x \in 2\mathbb{Z}, \text{ and} \\ x - 2k_1, x \in 2\mathbb{Z}+1, \end{Bmatrix}$$

and

$$g_2 : (\mathbb{Z}, T_{-k_2}) \to (\mathbb{Z}, T_{k_2}) \tag{4.6}$$

given by

$$g_2(x) = \begin{Bmatrix} x, x \in 2\mathbb{Z}, \text{ and} \\ x + 2k_2, x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

(Case 2-3) In case $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}_-$ such that $|k_1| \lneq |k_2|$, we have the following composite as an embedding

$$g_3 \circ h_{(k_1,-k_2)} : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{-k_2}) \to (\mathbb{Z}, T_{k_2}),$$

where $h_{(k_1,-k_2)}$ is the map of (4.4) and

$$g_3 : (\mathbb{Z}, T_{-k_2}) \to (\mathbb{Z}, T_{k_2})$$

defined by

$$g_3(x) = \begin{Bmatrix} x, x \in 2\mathbb{Z}, \text{ and} \\ x + 2k_2, x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

(Case 2-4) In case $k_1 \in \mathbb{N}_-$ and $k_2 \in \mathbb{N}$ such that $|k_1| \lneq |k_2|$, we have the composite as an embedding.

$$h_{(-k_1,k_2)} \circ g_4 : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{-k_1}) \to (\mathbb{Z}, T_{k_2}),$$

where $h_{(-k_1,k_2)}$ is the map of (4.4) and

$$g_4 : (\mathbb{Z}, T_{k_1}) \to (\mathbb{Z}, T_{-k_1})$$

defined by

$$g_4(x) = \begin{Bmatrix} x, x \in 2\mathbb{Z}, \text{ and} \\ x - 2k_1, x \in 2\mathbb{Z}+1. \end{Bmatrix}$$

□

To qualify the embedding $h_{(-k_1,-k_2)}, k_1, k_2 \in \mathbb{Z}$, we now refer to the case of $h_{(-2,-3)}$, as follows:

Example 4.6. Based on (Case 2-2) in the proof of Theorem 4.5, the map $h_{(-2,-3)} := g_2 \circ h_{(2,3)} \circ g_1 : (\mathbb{Z}, T_{-2}) \to (\mathbb{Z}, T_{-3})$ is an embedding, where g_1 (resp. g_2 and $h_{(2,3)}$) is the map of (4.5) (resp. (4.6) and (4.4)). In detail, the embedding $h_{(-2,-3)} = g_2 \circ h_{(2,3)} \circ g_1$ maps $SN_{-2}(0) = \{0, 1, -3\}$ onto $SN_{-3}(0) = \{0, 1, -5\}$, $SN_{-2}(2) = \{2, 3, -1\}$ onto $SN_{-3}(2) = \{2, 3, -3\}$, and so forth.

By Lemma 2.3 and Theorem 4.5, we define the following.

Definition 4.7. Given two spaces $(\mathbb{Z}, T_{k_1})$ and $(\mathbb{Z}, T_{k_2}), k_1, k_2 \in \mathbb{Z} \setminus \{0\}$, we define a relation between them as follows: We say that $T_{k_1} \lessapprox T_{k_2}$ if $|k_1| \leq |k_2|$. Namely, we obtain a relation set $(\mathcal{T}, \lessapprox)$, where $\mathcal{T} = \{(\mathbb{Z}, T_k) \mid k \in \mathbb{Z} \setminus \{0\}\}$.

Remark 4.8. (1) If $T_{k_1} \lessapprox T_{k_2}$, by Theorem 4.5, there is an embedding from $(\mathbb{Z}, T_{k_1})$ to $(\mathbb{Z}, T_{k_2})$.
(2) The relation set $(\mathcal{T}, \lessapprox)$ of Definition 4.7 need not be a partially ordered set but it is a preordered set (see Theorem 4.5).

Theorem 4.9. *Not every indexed collection $\mathcal{S}$ of topological spaces satisfying the $T_{\frac{1}{2}}$-separation axiom with weight which is less than or equal to $\aleph_0$ always has a universal element.*

Proof. Consider the relation set $(\mathcal{N}, \lessapprox)$ induced from $(\mathcal{T}, \lessapprox)$, where $\mathcal{N} := \{(\mathbb{Z}, T_k), k \in \mathbb{N}\}$. Then, by Theorem 4.5, we observe that $(\mathcal{N}, \lessapprox)$ does not have a universal element. □

Theorem 4.10. *Let $\mathcal{T}_M$ be the set $\{(\mathbb{Z}, T_k) \mid k \in M \subset \mathbb{Z} \setminus \{0\}\}$. Then $\mathcal{T}_M$ has a universal element if and only if M is finite.*

Proof. (⇒) If M is infinite, as mentioned in Theorem 4.9, $\mathcal{T}_M$ does not have a universal element.
(⇐) With $\mathcal{T}_M$, rearrange all topologies T_α in $\mathcal{T}_M$ as a sequence as follows:

$$\{T_{\alpha_1}, T_{\alpha_2}, \cdots, T_{\alpha_m}, \cdots \| \alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_m| \leq \cdots\}. \tag{4.7}$$

In (4.7), it is clear that $(\mathbb{Z}, T_{\alpha_i})$ is embedded into $(\mathbb{Z}, T_{\alpha_j})$, where $|\alpha_i| \leq |\alpha_j|$. Then, by Theorems 4.5 and 4.9, in order for the set of (4.7) to have a universal element, the set of (4.7) should be finite, i.e., M is finite. □

By Lemma 2.3 and Theorem 4.10, the following is obtained.

Corollary 4.11. *Let $\mathcal{T}'_M$ be the set $\{(\mathbb{Z}, T'_k) \mid k \in M \subset \mathbb{Z} \setminus \{0\}\}$. Then $\mathcal{T}'_M$ has a universal element if and only if M is finite.*

5. Summary and further work

We initially proved that each of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k), k \in \mathbb{Z}$, is reversible. Based on the satisfaction of the $T_{\frac{1}{2}}$-separation axiom of T_k- and T'_k-topological spaces, in the sets $\mathcal{T} := \{(\mathbb{Z}, T_k) \mid k \in \mathbb{Z} \setminus \{0\}\}$ and $\mathcal{T}' := \{(\mathbb{Z}, T'_k) \mid k \in \mathbb{Z} \setminus \{0\}\}$, we have studied an existence problem of a universal element in $\mathcal{T}$ and $\mathcal{T}'$.
As a futher work, we will study some another connected topological structures on $\mathbb{Z}$ which are not homeomorphic with both $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k), k \in \mathbb{Z}$. Then we will deal with the universality problem in the category of these spaces. Besides, some properties of these space regarding a cut-point space, irreducibility, embedding, and so on will be investigated.

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References

[1] P. Alexandorff, *Uber die Metrisation der im Kleinen kompakten topologischen Räume*, Mathematische Annalen **92**(3-4) (1924), 294–301.
[2] P. Alexandorff, *Diskrete Räume*, Mat. Sb. **2** (1937), 501–518.
[3] A.V. Arkhangel'skii, *Topological function spaces*, Kluwer (1991) (Translated from Russian).
[4] A. V. Arhangel'skii, *Topological homogeneity. Topological groups and their continuous images*, Russian Math. Surveys **42** (1987), 83–131.
[5] V.A. Chatyrko, S.-E. Han, Y. Hattori, *The small inductive dimension of subsets of Alexandroff spaces*, Filomat **30**(11) (2016), 3007-3014.
[6] W. Dunham, $T_{\frac{1}{2}}$*-spaces*, Kyungpook Math. J. **17** (1977), 161–169.
[7] D. N. Georgiou, A. C. Megaritis, F. Sereti, *Base dimension-like function of the type Dind and universality*, Topol. Appl. **281** (2020), 107201(11 pp).
[8] D. N. Georgiou, A. C. Megaritis, I. Naidoo, F. Sereti, *A study of universal elements in classes of bases of topological spaces*, Commentationes Mathematicae Universitatis Carolinae **62**(4) (2021), 491–506.
[9] A. C. Megaritis, F. Sereti, *Almost regular spaces and universal elements*, Houston Journal of mathematics **49**(4) (2023), 953–969.
[10] S.-E. Han, *Semi-separation axioms of the Khalimsky topological sphere*, Topol. Appl. **275** (2020), 107006.
[11] S.-E. Han, *Low-level separation axioms from the viewpoint of computational topology*, Filomat **33**(7) (2019), 1889–1901.
[12] S.-E. Han, *Adjacency relations induced by some Alexandroff topologies on* $\mathbb{Z}^n$, AIMS(Mathematics) **7**(7) (2022), 11581–11596.
[13] S.-E. Han, I.-K. Na, *Topologies associated with the one point compactification of Khalimsky topological spaces*, Topol. Appl. **241** (2018), 333–344.
[14] S.-E. Han and J.-H. Kim, *Embeddings of the* T_k*-topological spaces*, Bulletin of the Korean Mathematical Society (Submitted) (2024), 1–14.
[15] S.-E. Han, S. Jafari and J.-M. Kang, *Topologies on* $\mathbb{Z}^n$ *that are not homeomorphic to the n-dimensional Khalimsky topological space*, Mathematics **7**(1072) (2019), doi:10.3390/math711072.
[16] S.-E. Han, S. Jafari, J.-M. Kang, and S. Lee, *Remarks on topological spaces on* $\mathbb{Z}^n$ *which are related to the Khalimsky n-dimensional space*, AIMS(Mathematics) **7**(1) (2021), 1224–1240.
[17] S.-E. Han, Jewoo Lee, Wei Yao, Junhui Kim, *Existence of a proper subspace of* $(\mathbb{Z}^n, (T_k)^n)$ *which is homeomorphic to the n-dimensional Khalimsky topological space*, Topol. Appl. **344** (2024), 108812.
[18] S.-E. Han, S.Özçağ, *The semi-*T_3*-separation axiom of Khalimsky topological spaces*, Filomat **37**(8) (2023), 2539–2559.
[19] S.-E. Han, A. Šostak, *A compression of digital images derived from a Khalimsky topological structure*, Computational and Applied Mathematics **32** (2013), 521–536.
[20] S. D. Iliadis, *A construction of containing spaces*, Topol. Appl. **107**(1-2) (2000) 97—116.
[21] S. D. Iliadis, *Universal spaces and mappings*, North-Holland Mathematics Studies, 198. Elsevier Science B.V., Amsterdam, 2005.
[22] E. Khalimsky, *Topological structures in computer science*, J. Appl. Math. Simulation **1**(1) (1987), 25–40.
[23] E. Khalimsky, R. Kopperman, P.R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topol. Appl. **36**(1) (1990), 1—17.
[24] J.R. Munkres, *Topology A First Course* (second edition), Prentice-Hall, Inc., Upper Saddle, 1975.
[25] M. Rajagopalan and A. Wilansky, *Reversible topological spaces*, J. Austral. Math. Soc. **6** (1966), 129–138.