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Translationally regularly varying functions in the sense of Karamata and some asymptotic equivalence relations

Danica Fatić[a], Valentina Timotić[b], Dragan Djurčić[a,*]

[a]*University of Kragujevac, Faculty of Technical Sciences in Čačak, Serbia*
[b]*University of East Sarajevo, Faculty of Philosophy in Pale, Bosnia and Herzegovina*

Abstract. In this paper, we will analyze the relationship of some asymptotic equivalence relations for functions in terms of compositions of functions, and in terms of translationally regularly varying functions in the sense of Karamata.

1. Introduction

Jovan Karamata (1902-1967), a famous mathematician from Belgrade, in the 30s of the 20th century, in his papers [14] and [15] gave the basis of a very well-known theory in asymptotic analysis and its applications, which we call the classic Karamata theory of regular variation (see for example the monography [2]). Since its origin, this theory has had a very rapid, interesting and diverse development in fundamentals and applications. It is noticeable that its application part is more intensive than the development of theory in the basics. The application of this theory can be found for example in theories of summability, difference and differential equations. Let us especially emphasize in this paper the application of one modification of this theory in asymptotic analysis of divergent processes. Regarding the previous one, see e.g monographs [2, 8, 18, 21] and papers [16, 17, 20, 23]. It should be noted that this theory also has a place outside of mathematics, for example electrical engineering [13] and cosmology [19]. Karamata's theory of regular variation has a functional and sequence aspect (see e.g [4, 7, 9, 12]) and a large number of modifications (e.g [2, 3, 10, 11]). The direct connection between functional and sequence aspect has been made for example in the papers [2] and [9].

Definition 1.1 (Regularly varying function). A function $f : [a, \infty) \to (0, \infty)$ is a *regularly varying* function in the sense of Karamata if it is measurable and

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = r_f(\lambda) < \infty, \tag{1}$$

holds for some $a > 0$ and each $\lambda > 0$. The class of regularly varying function is denoted by RV_φ.

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* Corresponding author: Dragan Djurčić
Email addresses: danicafatic@gmail.com (Danica Fatić), valentina.timotic@ffuis.edu.ba (Valentina Timotić), dragan.djurcic@gradina.ftn.kg.ac.rs (Dragan Djurčić)
ORCID iDs: https://orcid.org/0000-0001-7802-764X (Danica Fatić), https://orcid.org/0000-0002-7141-4724 (Dragan Djurčić)

Translationally regularly varying functions and translationally regularly varying sequences were discussed in the paper [22]. Translationally regularly varying sequences are important objects in the theory of selection principles and the theory of infinite topological games (see, for example [6]).

Definition 1.2 (Translationally regularly varying function). A function $f : [a, \infty) \to (0, \infty)$ is a *translationally regularly varying* function in the sense of Karamata, if it is measurable and

$$\lim_{x\to\infty} \frac{f(\lambda + x)}{f(x)} = r_f^T(\lambda) < \infty, \tag{2}$$

holds for some $a > 0$ and each $\lambda \in \mathbb{R}$. The class of translationally regularly varying functions is denoted by $Tr(RV_\varphi)$.

The following results come from the paper [22].

Lemma 1.3. *Let a function* $f \in Tr(RV_\varphi)$.

(a) $r_f^T(\lambda) > 0$ *holds for each* $\lambda \in \mathbb{R}$.

(b) $r_f^T(\lambda) = e^{\lambda\rho}$ *holds for each* $\lambda \in \mathbb{R}$ *and some* $\rho \in \mathbb{R}$.

(c) $\lim_{x\to\infty}\left(\sup_{\lambda\in T}\left\{\frac{f(\lambda+x)}{f(x)} - r_f^T(\lambda)\right\}\right) = 0$ *holds for each compact set* $T \subsetneq \mathbb{R}$.

(d) $f(x) = e^{\rho x} \cdot c(x) \cdot \exp\left(\int_a^x \frac{\varepsilon(\log u)}{u \log u} du\right)$ *for some* $a > 0$*, for each* $x \geqslant a$*, some* $\rho \in \mathbb{R}$ *and measurable functions* c *and* ε *for which* $c(x) \to c \in (0, \infty)$*, as* $x \to \infty$*, and* $\varepsilon(u) \to 0$*, as* $u \to \infty$ *holds.*

Definition 1.4 (Strongly asymptotic equivalent functions). Let $\mathcal{F} = \{x(t) \mid t \geqslant a,\ a > 0,\ x(t) \to \infty \text{ as } t \to \infty\}$ be the class of positive functions. We say that two functions $x \in \mathcal{F}$ and $y \in \mathcal{F}$ are *multiplicatively strongly asymptotic equivalent (equal)*, if asymptotic relation

$$\rho_1 : \lim_{t\to\infty} \frac{x(t)}{y(t)} = 1 \tag{3}$$

holds, we write $x(t)\rho_1 y(t)$, and *additively strongly asymptotic equivalent (equal)*, if asymptotic relation

$$\rho_2 : \lim_{t\to\infty}(x(t) - y(t)) = 0 \tag{4}$$

holds, we write $x(t)\rho_2 y(t)$.

In the section Main results, we will prove within the analysis of divergent processes that asymptotic relation (4) implies the asymptotic relation

$$\rho_1^f : \lim_{t\to\infty} \frac{f(x(t))}{f(y(t))} = 1 \tag{5}$$

(we will write $f(x(t))\rho_1^f f(y(t))$), if $f \in Tr(RV_\varphi)$, see [5], where a similar problem was analyzed and solved.

Also, we will prove that the previous implication satisfy only those functions f which belong to the modified class of the class $Tr(RV_\varphi)$ (whose proper subclass is the class $Tr(RV_\varphi)$).

Definition 1.5 (O-regularly varying function). A function $f : [a, \infty) \to (0, \infty)$ is *O-regularly varying* function in the sense of Karamata, if it is measurable and

$$\lim_{t\to\infty}\left(\sup_{x\geqslant t} \frac{f(\lambda x)}{f(x)}\right) = k_f(\lambda) < \infty, \tag{6}$$

holds for some $a > 0$ and each $\lambda \in \mathbb{R}$. This class of functions is denoted with ORV_φ. For $f \in ORV_\varphi$ the function $k_f(\lambda)$, $\lambda > 0$, is called main index function.

Definition 1.6 (Translationally O-regularly varying function). A function $f : [a,\infty) \to (0,\infty)$ is *translationally O-regularly varying* function in the sense of Karamata, if it is measurable and

$$\lim_{t\to\infty}\left(\sup_{x\geqslant t}\frac{f(\lambda+x)}{f(x)}\right) = k_f^T(\lambda) < \infty, \tag{7}$$

holds for some $a > 0$ and each $\lambda \in \mathbb{R}$. This class of functions is denoted with $Tr(ORV_\varphi)$. For $f \in Tr(ORV_\varphi)$ the function $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is called the main index function.

Remark 1.7. The main index function $k_f^T(\lambda)$ can be continuous or discontinuous for $\lambda = 0$. Related to the class ORV_φ and $Tr(ORV_\varphi)$ see [2] and [22].

2. Main results

In the following claim we will prove some properties of functions from the class $Tr(RV_\varphi)$.

Lemma 2.1. *Let $f \in Tr(RV_\varphi)$. Then*

$$\lim_{x\to\infty,\lambda\to 0}\frac{f(\lambda+x)}{f(x)} = 1. \tag{8}$$

Proposition 2.2. *Let $f \in Tr(RV_\varphi)$. If*

$$x(t)\rho_2 y(t), \tag{9}$$

then

$$f(x(t))\rho_1^f f(y(t)) \tag{10}$$

for each $x \in \mathcal{F}$ and $y \in \mathcal{F}$.

Problem 2.3. *If for each $x \in \mathcal{F}$ and $y \in \mathcal{F}$ equations* (9) *and* (10) *hold, does it imply $f \in Tr(RV_\varphi)$?*

Example 2.4. Let us observe $a > 0$ and the function

$$f(x) = \exp\left(\int_a^x \frac{2+\sin(\log(\log u))}{u\log u}du\right),\ \text{for } x \geqslant a. \tag{11}$$

Then $f \notin Tr(RV_\varphi)$, and function f satisfies Proposition 2.2.

Remark 2.5.
1. If $f \in Tr(RV_\varphi)$ then $f \in Tr(ORV_\varphi)$ and then $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is continuous for $\lambda = 0$.
2. For the function f from the previous example ($f \notin Tr(RV_\varphi)$) it holds that $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is continuous for $\lambda = 0$.
3. For the function $f(x) = 2 + \sin(e^x)$, $x \geqslant 1$, it holds $f \in Tr(ORV_\varphi)$ and $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is discontinuous for $\lambda = 0$.

Lemma 2.6. *Let $f \in Tr(ORV_\varphi)$ and let $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ be its main index function, then $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is continuous function for $\lambda = 0$ if and only if $k_f^T(\lambda)$, $\lambda \in \mathbb{R}$ is continuous function for each $\lambda \in \mathbb{R}$.*

Let us denote with $Tr(IRV_\varphi)$ the class of functions from $Tr(ORV_\varphi)$ for which the main index function is continuous. It holds

$$Tr(RV_\varphi) \subsetneqq Tr(IRV_\varphi) \subsetneqq Tr(ORV_\varphi). \tag{12}$$

In the next proposition we will answer the question from the previous discussion.

Proposition 2.7. *Let $x \in \mathcal{F}$ and $y \in \mathcal{F}$. Also, let $f(x)$, $x \geqslant a$ be positive and measurable.*

(a) *If $x(t)\rho_2 y(t)$ and if $f \in Tr(IRV_\varphi)$, then $f(x(t))\rho_1^f f(y(t))$.*

(b) *If $f(x(t))\rho_1^f f(y(t))$ whenever $x(t)\rho_2 y(t)$, then $f \in Tr(IRV_\varphi)$.*

3. Proofs

3.1. Proof of Lemma 2.1

Let $f \in Tr(RV_\varphi)$. According to (b) from Lemma 1.3 follows

$$r_f^T(\lambda) = e^{\lambda\rho} = \lim_{x\to\infty} \frac{f(\lambda + x)}{f(x)}$$

for some $\rho \in \mathbb{R}$ and each $\lambda \in \mathbb{R}$. According to (c) from Lemma 1.3, for $\varepsilon > 1$, and for $\varepsilon_1 > 0$ follows

$$-\varepsilon_1 + \frac{1}{\varepsilon} \leqslant \frac{f(\lambda + x)}{f(x)} \leqslant \varepsilon + \varepsilon_1,$$

for some $x_0 > 0$ and each $x \geqslant x_0$ and each λ for which $m \leqslant \lambda \leqslant M$, where $m = \min\{-\frac{\log\varepsilon}{\rho}, \frac{\log\varepsilon}{\rho}\}$ and $M = \max\{-\frac{\log\varepsilon}{\rho}, \frac{\log\varepsilon}{\rho}\}$. Thus,

$$\lim_{x\to\infty,\ \lambda\to 0} \frac{f(\lambda + x)}{f(x)} = 1.$$

3.2. Proof of Proposition 2.2

Let $f \in Tr(RV_\varphi)$ and let $x \in \mathcal{F}$ and $y \in \mathcal{F}$. Also, let $x(t)\rho_2 y(t)$. Then, according to Lemma 1.3 follows

$$\lim_{t\to\infty} \frac{f(x(t))}{f(y(t))} = \lim_{t\to\infty} \frac{f(x(t) - y(t)) + y(t))}{f(y(t))} = 1.$$

3.3. Proof of Lemma 2.6

Let the main index function $k^T(\lambda)$, $\lambda \in \mathbb{R}$ be continuous for every $\lambda \in \mathbb{R}$. Then it is continuous for $\lambda = 0$. Let us prove the opposite direction. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $\triangle\lambda \in \mathbb{R} \setminus \{0\}$. Then it holds that

$$\begin{aligned} k_f^T(\lambda + \triangle\lambda) &= \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\lambda + \triangle\lambda + x)}{f(x)}\right) \leq \\ &\leq \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\lambda + \triangle\lambda + x)}{f(\triangle\lambda + x)}\right) \cdot \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\triangle\lambda + x)}{f(x)}\right) \\ &= k_f^T(\lambda) \cdot k_f^T(\triangle\lambda), \end{aligned}$$

for observed λ and $\triangle\lambda$.

Therefore, we have that

$$\begin{aligned} \lim_{p\to 0}\left(\sup_{\triangle\lambda\geq p} k_f^T(\lambda + \triangle\lambda)\right) &\leq k_f^T(\lambda) \cdot \lim_{p\to 0}\left(\sup_{\triangle\lambda\geq p} k_f^T(\triangle\lambda)\right) \\ &= k_f^T(\lambda) \cdot 1 = k_f^T(\lambda), \end{aligned}$$

for observed λ.

Also, holds that

$$\begin{aligned} k_f^T(\lambda) &= \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\lambda + x)}{f(x)}\right) \leq \\ &\leq \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\lambda + \triangle\lambda + x)}{f(x)}\right) \cdot \lim_{t\to\infty}\left(\sup_{x\geq t} \frac{f(\lambda + x)}{f(\lambda + \triangle\lambda + x)}\right) \\ &= k_f^T(\lambda + \triangle\lambda) \cdot k_f^T(-\triangle\lambda), \end{aligned}$$

for observed λ and $\triangle\lambda$.

Therefore, it is valid that

$$\begin{aligned}\lim_{p\to 0}\left(\inf_{\triangle\lambda\geq p} k_f^T(\lambda)\right) &\leq \lim_{p\to 0}\left(\inf_{\triangle\lambda\geq p} k_f^T(\lambda+\triangle\lambda)\right)\cdot \lim_{p\to 0}\left(\sup_{\triangle\lambda\geq p} k_f^T(-\lambda)\right)\\ &= \lim_{p\to 0}\left(\inf_{\triangle\lambda\geq p} k_f^T(\lambda+\triangle\lambda)\right)\cdot 1 =\\ &= \lim_{p\to 0}\left(\inf_{\triangle\lambda\geq p} k_f^T(\lambda+\triangle\lambda)\right).\end{aligned}$$

Hence, $\lim_{p\to 0}\left(\inf_{\triangle\lambda\geq p} k_f^T(\lambda+\triangle\lambda)\right) \geq k_f^T(\lambda)$, for observed $\lambda \in \mathbb{R}\setminus\{0\}$.

Wherefore, holds that $\lim_{\triangle\lambda\to 0} k_f^T(\lambda+\triangle\lambda) = k_f^T(\lambda)$, for observed λ, so the function $k_f^T(\lambda)$, $\lambda\in\mathbb{R}$, is continuous for every $\lambda\in\mathbb{R}$.

3.4. Proof of the Proposition 2.7

(a) Let $x\in\mathcal{F}$ and $y\in\mathcal{F}$. Also, let $f\in Tr(IRV_\varphi)$ and $x(t)\rho_2 y(t)$. Then the function

$$k_f^T(\lambda) = \lim_{t\to\infty}\left(\sup_{x\geqslant t}\frac{f(\lambda+x)}{f(x)}\right)$$

is finite and continuous for each $\lambda\in\mathbb{R}$. Therefore,

$$k_f^T(\log\alpha) = \lim_{t\to\infty}\left(\sup_{p\geqslant t}\frac{h(\alpha p)}{h(p)}\right) = \lim_{t\to\infty}\left(\sup_{\log p\geqslant t}\frac{h(\alpha p)}{h(p)}\right) = \lim_{t\to\infty}\left(\sup_{x\geqslant t}\frac{h(e^\lambda\cdot e^x)}{h(e^x)}\right) = \lim_{t\to\infty}\left(\sup_{x\geqslant t}\frac{f(\lambda+x)}{f(x)}\right) = k_f^T(\lambda),$$

where $h(e^u) = f(u)$, $u\geqslant a$ (for some a), for each $\lambda = \ln\alpha$, $\alpha>0$, $\lambda\in\mathbb{R}$, also finite and continuous function for $\alpha>0$.

Hence, the function h is O-regularly varying function in the sense of Karamata with continuous main index (see e.g. [5]), according to [1] for each compact set $T\subsetneq(0,\infty)$, the following asymptotic relation holds:

$$\lim_{t\to\infty}\left\{\sup_{\alpha\in T}\left(\sup_{u\geqslant t}\frac{h(\alpha u)}{h(u)} - k_f^T(\log\alpha)\right)\right\} = 0.$$

Therefore, for each $\varepsilon>0$ and each $\rho>1$, there is $p_0>0$, so that for each $u\geqslant u_0$ and each $\alpha\in[\frac{1}{p},p]$, it holds

$$\frac{h(\alpha u)}{h(u)} \leqslant k_f^T(\log\alpha)+\varepsilon.$$

Hence, for the previous it holds

$$\limsup_{u\to\infty,\ \alpha\to 1}\frac{h(\alpha u)}{h(u)} \leqslant 1+\varepsilon,$$

similar, it holds that

$$\liminf_{u\to\infty,\ \alpha\to 1}\frac{h(\alpha u)}{h(u)} \geqslant \frac{1}{1+\varepsilon}.$$

Hence,

$$\lim_{u\to\infty,\ \alpha\to 1}\frac{h(\alpha u)}{h(u)} = 1.$$

Therefore,

$$\lim_{t\to\infty}\frac{f(x(t))}{f(y(t))}=\lim_{t\to\infty}\frac{((x(t)-y(t))+y(t))}{f(y(t))}=\lim_{t\to\infty}\frac{h(e^{(x(t)-y(t))}\cdot e^{y(t)})}{h(e^{y(t)})}=1.$$

Hence $f(x(t))\rho_1^f f(y(t))$.

(b) Let $f(x(t))\rho_1^f f(y(t))$ whenever $x(t)\rho_2 y(t)$, where $x\in\mathcal{F}$, $y\in\mathcal{F}$ and let $f:[a,\infty)\to(0,\infty)$ be a measurable function. Let

$$x(t)=a_n,\ \text{for } [a]+n\leqslant t<[a]+n+1$$

and

$$x(t)=b_n,\ \text{for } [a]+n\leqslant t<[a]+n+1$$

and let $x(t)\rho_2 y(t)$. Therefore,

$$\lim_{n\to\infty}\frac{f(a_n)}{f(b_n)}=\lim_{t\to\infty}\frac{f(x([t]))}{f(y([t]))}=1.$$

Let (x_n), $x_n\in[a,\infty)$, for $n\in\mathbb{N}$ be arbitrary sequence, which tends to infinity, as $n\to\infty$ and let (λ_n), $\lambda_n\in\mathbb{R}$, $n\in\mathbb{N}$, such that $\lambda_n\to 0$ as $n\to\infty$. Let us assume that

$$a_n=\lambda_n+x_n,\ \text{for } n\in\mathbb{N},$$

and

$$b_n=x_n,\ \text{for } n\in\mathbb{N}.$$

We see that

$$\lim_{n\to\infty}\frac{f(\lambda_n+x_n)}{f(x_n)}=1,$$

so that

$$\lim_{t\to\infty,\lambda\to 0}\frac{f(\lambda+t)}{f(t)}=1.$$

For $\varepsilon>0$ there exist t_0 and $\lambda_0>0$, such that

$$\frac{1}{1+\varepsilon}\leqslant\frac{f(\lambda+t)}{f(x)}\leqslant 1+\varepsilon,$$

whenever $t\geqslant t_0$ and $n-\lambda_0\leqslant\lambda\leqslant\lambda_0$. Therefore $f\in Tr(ORV_\varphi)$. Since $k_f^T(\lambda)$, $\lambda\in\mathbb{R}$ is continuous function for $\lambda=0$, according to Lemma 2.6 it follows that the function $k_f^T(\lambda)$, $\lambda\in\mathbb{R}$ is continuous function on $\mathbb{R}$. Hence, $f\in Tr(IRV_\varphi)$.

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