Filomat 39:5 (2025), 1617–1629
https://doi.org/10.2298/FIL2505617M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hyperbolic Ricci solitons on K-contact manifolds and its applications in spacetimes

Gaurab Mitra[a], Tarak Mandal[a,*], Avijit Sarkar[a]

[a] *Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India*

Abstract. The aim of the present article is delve into characterizations of hyperbolic Ricci solitons which bear wave characters of K-contact metrics and associated curvatures of manifolds as self similar solutions of a hyperbolic Ricci flow. Also, we investigate gradient hyperbolic Ricci solitons on η-Einstein K-contact manifolds and construct an example to verify the deduced results. Finally we analyze hyperbolic Ricci solitons in the framework of quasi-Einstein spacetimes.

1. Introduction

Kong and Liu [24] introduced hyperbolic Ricci flow in order to illustrate the wave nature of the metrics and curvatures of manifolds with beautiful analogy between hyperbolic Ricci flow and wave equations. This is novel and very natural to analyze certain interesting phenomena in the geometry of manifolds. It possesses several interesting properties from Mathematics as well as in Physics. According to Kong [25], hyperbolic Ricci solitons as self similar solutions of hyperbolic Ricci flow are of prime importance as they are highly related with solutions of Einstein's field equations in vacuum. Soliton theory becomes further interesting by coupling with contact metric theory [5]. Solitons with contact metric was first studied by Sharma [30]. Thus, we get natural motivation to study hyperbolic Ricci solitons in the framework of K-contact manifolds.

In this connection it should be mention that in [21], Hamilton was introduced the notion of Ricci soliton on a Riemannian manifold $(\mathcal{N}^m, g)$, which is the generalization of the Einstein metrics defined by

$$\pounds_{\mathcal{P}} g + 2\mathcal{S} = 2\lambda g, \tag{1}$$

where $\pounds_{\mathcal{P}}$ denotes the Lie derivative operator along the potential vector field $\mathcal{P}$, $\mathcal{S}$ the Ricci tensor of type (0,2) and λ is a constant. If $\mathcal{P} = \text{grad}\zeta$, then the equation (1) can be rewritten as follows

$$\mathcal{S} + \text{Hess}\zeta = \lambda \text{g}, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53D25.
Keywords. Hyperbolic Ricci solitons; gradient hyperbolic Ricci solitons; K-contact manifolds; η-Einstein K-contact manifolds; super quasi-Einstein spacetimes.
Received: 29 July 2024; Revised: 27 November 2024; Accepted: 21 December 2024
Communicated by Mića Stanković
* Corresponding author: Tarak Mandal
Email addresses: gaurab24041993@gmail.com (Gaurab Mitra), mathtarak@gmail.com (Tarak Mandal), avjaj@yahoo.co.in (Avijit Sarkar)
ORCID iDs: https://orcid.org/0009-0001-6782-9724 (Gaurab Mitra), https://orcid.org/0000-0003-4808-8454 (Tarak Mandal), https://orcid.org/0000-0002-7370-1698 (Avijit Sarkar)

where Hess being the Hessian operator of the smooth function ζ. A Ricci soliton is the limit of the solutions of Ricci flow evolution equation on $(\mathcal{N}, g_0)$ given by $\frac{\partial g}{\partial t} = -2\mathcal{S}, \quad g(0) = g_0$. Ricci solitons on different kind of manifolds have been studied by several researchers [10, 12, 19, 22, 28, 31, 32]. A hyperbolic geometric flow is defined by [24]

$$\frac{\partial^2 g}{\partial t^2} = -2\mathcal{S}, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t} = \mathcal{H}_0, \tag{3}$$

where $\mathcal{H}_0$ is a symmetric (0,2)-tensor field on $\mathcal{N}$. Recently, this types of flow are studied by various authors in different perspectives [1–4, 8–11, 17, 18, 23].

A Riemannian manifold $(\mathcal{N}^m, g)$ is called a hyperbolic Ricci soliton (HRS, in short) if their exists a vector field $\mathcal{P}$ on $\mathcal{N}$ such that

$$\mathcal{S} + \tau \pounds_{\mathcal{P}} g + \frac{1}{2}\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}} g) = \lambda g, \tag{4}$$

for some real scalars τ and λ on $\mathcal{N}$ [20]. A HRS is called a Ricci Soliton if the potential vector field is 2-Killing [13–15] and $\tau = \frac{1}{2}$. A HRS is said to be shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. Also, a HRS is said to be a gradient hyperbolic Ricci soliton (GHRS, in short) if their exists a potential function ζ such that $\mathcal{P} = \text{grad}\zeta$. Thus the equation (4) takes the form

$$\mathcal{S} + 2\tau(\text{Hess}\zeta) + \pounds_{\text{grad}\zeta}(\text{Hess}\zeta) = \lambda \text{g}, \tag{5}$$

where gradζ denotes the gradient of the potential function ζ.

The present article is organized as follows: After the introduction, we give some preliminaries in the Section 2. In Section 3, we investigate certain results of HRS on (2m+1)-dimensional K-contact manifolds. In Section 4, we have studied GHRS on η-Einstein K-contact manifold of dimension (2m+1). In Section 5, we give an example to verify the deduced results. In the last section, we apply HRS in super quasi-Einstein spacetimes.

2. Preliminaries

A differentiable manifold $\mathcal{N}^{2m+1}$ is known as a contact manifold if there exists a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\mathcal{N}$. For a given contact 1-form η there exists a unique Reeb vector field θ such that $d\eta(\theta, W) = 0$ and $\eta(\theta) = 1$. Let (ϕ, θ, η, g) be a contact metric structure, where ϕ, θ, η and g are, respectively, a $(1,1)$-tensor field, a $(1,0)$ type vector field, a 1-form and an associated metric of η, such that

$$\phi^2(W) = -W + \eta(W)\theta, \quad \eta(W) = g(W, \theta), \quad d\eta(W, U) = g(W, \phi U), \tag{6}$$

$$\phi\theta = 0, \quad \eta.\phi = 0 \quad \text{rank}(\phi) = 2m,$$

for every vector field W, U on $\mathcal{N}^{2m+1}$ [6]. The tensor $h = \frac{1}{2}\pounds_\theta \phi$ is known to be self-adjoint, where £ denotes the Lie derivative operator that anti-commutes with ϕ and satisfies the conditions $tr(h) = 0$, $tr(h\phi) = 0$, where 'tr' indicates trace.

A contact metric structure is said to be K-contact if the characteristics vector field θ is Killing vector field. For a K-contact manifold, the following conditions hold:

$$\nabla_W \theta = -\phi W, \quad (\nabla_W \phi)U = R(\theta, W)U. \tag{7}$$

$$Q\theta = 2m\theta, \quad R(W, \theta)\theta = W - \eta(W)\theta. \tag{8}$$

$$(\nabla_W \eta)U + (\nabla_U \eta)W = 0, \quad (\nabla_W \eta)U = \phi(W, U) = g(W, \phi U), \tag{9}$$

where ∇, R, Q denote the Levi-Civita connection, curvature tensor and the Ricci operator of g respectively. The contact structure on $\mathcal{N}$ is said to be normal if the almost complex structure on $\mathcal{N} \times R$ defined by $J(W, f\frac{d}{dt}) = (\phi W - f\theta, \eta(W)\frac{d}{dt})$, where f is a real function on $\mathcal{N} \times R$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian manifolds are K-contact and 3-dimensional K-contact manifolds are Sasakian. For a Sasakian manifold, we have

$$R(W,U)\theta = \eta(U)W - \eta(W)U. \tag{10}$$

Also, it is well known that, an η-Einstein K-contact manifold of dimension $(2m+1)$ satisfies the following curvature properties:

$$\mathcal{S}(W,U) = (\frac{\mathcal{R}}{2m} - 1)g(W,U) + [(2m+1) - \frac{\mathcal{R}}{2m}]\eta(W)\eta(U), \tag{11}$$

which gives

$$\mathcal{S}(U,\theta) = 2m\eta(U). \tag{12}$$

In addition, the manifold $(\mathcal{N}^{2m+1}, g)$ is called nearly quasi-Einstein manifold [33] if $\mathcal{S} = \alpha_1 g + \alpha_2 \mathcal{E}_t$ for some functions α_1, α_2 and a non-vanishing symmetric (0,2)-tensor $\mathcal{E}_t$ on $\mathcal{N}^{2m+1}$.

Definition 2.1. *[7] On a Riemannian manifold $(\mathcal{N}^{2m+1}, g)$, a vector field $\mathcal{P}$ is said to be concircular if it satisfies*

$$\nabla_W \mathcal{P} = \zeta W, \tag{13}$$

for any vector field W and a real smooth function ζ on $\mathcal{N}$. Also, a vector field $\mathcal{P}$ is called concircular Killing vector field if it satisfies

$$(\pounds_{\mathcal{P}} g)(W,U) = 2\alpha g(W,U), \tag{14}$$

for some real smooth function α on $\mathcal{N}$.

Definition 2.2. *[19] On a Riemannian manifold $(\mathcal{N}^{2m+1}, g)$, a vector field $\mathcal{P}$ is said to be Ricci bi-conformal vector field if it satisfies*

$$(\pounds_{\mathcal{P}} g)(W,U) = f_1 g(W,U) + f_2 \mathcal{S}(W,U), \tag{15}$$

and

$$(\pounds_{\mathcal{P}} \mathcal{S})(W,U) = f_1 \mathcal{S}(W,U) + f_2 g(W,U), \tag{16}$$

for some non-zero smooth functions f_1, f_2 on $\mathcal{N}$.

3. Hyperbolic Ricci solitons (HRS) on K-contact manifolds of dimension $(2m+1)$

Theorem 3.1. *A K-contact manifold $\mathcal{N}^{2m+1}$ admitting a HRS with the potential vector field $\mathcal{P}$ which is pointwise collinear with the Reeb vector field θ is nearly quasi-Einstein manifold.*

Proof. Let $(\mathcal{N}, g, \mathcal{P}, \tau, \lambda)$ be a HRS whose potential vector field $\mathcal{P}$ such that $\mathcal{P} = \mathcal{B}\theta$, for a smooth function $\mathcal{B}$. With the help of Lie derivative operator and using (6) and (7), we get

$$\pounds_{\mathcal{P}} g(W,U) = g(W, \text{grad}\mathcal{B})\eta(U) + g(U, \text{grad}\mathcal{B})\eta(W), \tag{17}$$

for any vector field W, U and grad$\mathcal{B}$ denotes the gradient of $\mathcal{B}$.

By the definition of the Lie derivative operator and using (17), we infer

$$\begin{aligned}\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}} g(W,U)) = {} & \mathcal{P}.g(W, \text{grad}\mathcal{B})\eta(U) + g(W, \text{grad}\mathcal{B})\mathcal{P}(\eta(U)) + \mathcal{P}.g(U, \text{grad}\mathcal{B})\eta(W) \\ & - g(U, \text{grad}\mathcal{B})\mathcal{P}(\eta(W)) - g(\pounds_{\mathcal{P}} W, \text{grad}\mathcal{B})\eta(U) - g(U, \text{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}} W) \\ & - g(\pounds_{\mathcal{P}} U, \text{grad}\mathcal{B})\eta(W) - g(W, \text{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}} U).\end{aligned} \tag{18}$$

Using (17) and (18) in equation (4), one can obtain

$$\begin{aligned}\mathcal{S}(W,U) &= \lambda g(W,U) - \frac{1}{2}\mathcal{P}.g(W,\mathrm{grad}\mathcal{B})\eta(U) - \frac{1}{2}g(W,\mathrm{grad}\mathcal{B})\mathcal{P}(\eta(U)) \\ &- \frac{1}{2}\mathcal{P}.g(U,\mathrm{grad}\mathcal{B})\eta(W) - \frac{1}{2}g(U,\mathrm{grad}\mathcal{B})\mathcal{P}(\eta(W)) + \frac{1}{2}g(\pounds_{\mathcal{P}}W,\mathrm{grad}\mathcal{B})\eta(U) \\ &+ \frac{1}{2}g(\pounds_{\mathcal{P}}U,\mathrm{grad}\mathcal{B})\eta(W) + \frac{1}{2}g(W,\mathrm{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}}U) + \frac{1}{2}g(U,\mathrm{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}}W) \\ &- \tau g(W,\mathrm{grad}\mathcal{B})\eta(U) - \tau g(U,\mathrm{grad}\mathcal{B})\eta(W). \end{aligned} \tag{19}$$

We define a non-vanishing $(0,2)$-tensor $\mathcal{E}_t$ by

$$\begin{aligned}\mathcal{E}_t(W,U) &= -\frac{1}{2}\mathcal{P}.g(W,\mathrm{grad}\mathcal{B})\eta(U) - \frac{1}{2}g(W,\mathrm{grad}\mathcal{B})\mathcal{P}(\eta(U)) - \frac{1}{2}\mathcal{P}.g(U,\mathrm{grad}\mathcal{B})\eta(W) \\ &- \frac{1}{2}g(U,\mathrm{grad}\mathcal{B})\mathcal{P}(\eta(W)) + \frac{1}{2}g(\pounds_{\mathcal{P}}W,\mathrm{grad}\mathcal{B})\eta(U) + \frac{1}{2}g(\pounds_{\mathcal{P}}U,\mathrm{grad}\mathcal{B})\eta(W) \\ &+ \frac{1}{2}g(W,\mathrm{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}}U) + \frac{1}{2}g(U,\mathrm{grad}\mathcal{B})\eta(\pounds_{\mathcal{P}}W) - \tau g(W,\mathrm{grad}\mathcal{B})\eta(U) \\ &- \tau g(U,\mathrm{grad}\mathcal{B})\eta(W). \end{aligned} \tag{20}$$

Using (20) in (19), we obtain

$$\mathcal{S}(W,U) = \lambda g(W,U) + \mathcal{E}_t(W,U), \tag{21}$$

which shows that $\mathcal{N}$ is nearly quasi-Einstein manifold.
This completes the proof. □

Theorem 3.2. *If $\mathcal{N}^{2m+1}$ is a K-contact manifold admitting HRS with potential vector field as the Reeb vector field θ, then the soliton is shrinking.*

Proof. With the help of (7), we obtain

$$(\pounds_{\theta}g)(W,U) = 0. \tag{22}$$

Also, from the definition of the Lie derivative operator, we obtain

$$\pounds_{\theta}(\pounds_{\theta}g(W,U)) = \theta.\pounds_{\theta}g(W,U) - \pounds_{\theta}g(\pounds_{\theta}W,U) - \pounds_{\theta}g(W,\pounds_{\theta}U) \tag{23}$$

Substituting $W = U = \theta$ in the above and using (6), we get

$$\pounds_{\theta}(\pounds_{\theta}g(\theta,\theta)) = 0. \tag{24}$$

Again, putting $W = U = \theta$ in (4) and using (24), we get

$$\mathcal{S}(\theta,\theta) = \lambda. \tag{25}$$

Also, from (8), we get

$$\mathcal{S}(\theta,\theta) = 2m. \tag{26}$$

Comparing the last two equations, we get

$$\lambda = 2m,$$

clearly, λ is positive.
This proves the theorem. □

Corollary 3.3. *If an η-Einstein K-contact manifold admits HRS with potential vector field as the Reeb vector field, then the soliton is shrinking.*

Theorem 3.4. *Let a $(2m+1)$ dimensional K-contact manifold $\mathcal{N}$ admit a HRS with potential vector field $\mathcal{P}$. If $\mathcal{P}$ is a concircular vector field, then*

$$\lambda = 2m + \mathcal{P}.\zeta + 2\zeta^2 + 2\tau\zeta.$$

Proof. With the help of (13), one can obtain

$$\begin{aligned}(\pounds_{\mathcal{P}} g)(W,U) &= g(\nabla_W \mathcal{P}, U) + g(W, \nabla_U \mathcal{P}) \\ &= 2\zeta g(W,U). \end{aligned} \tag{27}$$

Also, from the definition of Lie derivative operator, we get

$$\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}} g(W,U)) = 2\mathcal{P}.\zeta g(W,U) + 4\zeta^2 g(W,U). \tag{28}$$

Using (27) and (28) in (4), we obtain

$$\mathcal{S}(W,U) + 2\tau\zeta g(W,U) + \mathcal{P}.\zeta g(W,U) + 2\zeta^2 g(W,U) = \lambda g(W,U). \tag{29}$$

Substituting $W = U = \theta$ in (29) and comparing with (8), we get

$$\lambda = 2m + \mathcal{P}.\zeta + 2\zeta^2 + 2\tau\zeta. \tag{30}$$

This completes the proof. □

Theorem 3.5. *If an η-Einstein K-contact manifolds $\mathcal{N}^{2m+1}$ admits HRS with solenoidal vector field $\mathcal{P}$, then the soliton is shrinking and $\lambda = 2m$.*

Proof. from (4), we get

$$\begin{aligned}\mathcal{S}(W,U) &= \lambda g(W,U) - \tau[g(\nabla_W \mathcal{P}, U) + g(W, \nabla_U \mathcal{P})] \\ &\quad - \frac{1}{2}[\mathcal{P}.\pounds_{\mathcal{P}} g(W,U) - \pounds_{\mathcal{P}} g(\pounds_{\mathcal{P}} W, U) - \pounds_{\mathcal{P}} g(W, \pounds_{\mathcal{P}} U)]. \end{aligned} \tag{31}$$

Contracting W and U in (31), one can obtain

$$\mathcal{R} = \lambda(2m+1) - 2\tau Div\mathcal{P}. \tag{32}$$

Since $\mathcal{P}$ is solenoidal, $Div\mathcal{P} = 0$.
From (12) and (32), we get

$$\lambda = 2m.$$

This proves the theorem. □

Theorem 3.6. *Let an η-Einstein K-contact manifold $\mathcal{N}^{2m+1}$ admit HRS whose potential vector field $\mathcal{P}$ is concircular and orthogonal to the Reeb vector field. Then a HRS is shrinking if and only if the manifold $\mathcal{N}$ is locally isometric to $\mathcal{E}^{m+1} \times S^4$, for any $m > 1$.*

Proof. With the help of Lie derivative operator and using (6) and (7), we get

$$(\pounds_{\mathcal{P}} g)(\theta, \theta) = 0. \tag{33}$$

Since $\mathcal{P}$ is a concircular vector field on $\mathcal{N}$, so for any vector field W and a real smooth function ζ, we have

$$\nabla_W \mathcal{P} = \zeta W.$$

Using this, we get

$$(\pounds_{\mathcal{P}}g)(W,U) = 2\zeta g(W,U).$$

Substituting $W = U = \theta$ in the foregoing equation, we get

$$(\pounds_{\mathcal{P}}g)(\theta,\theta) = 2\zeta. \tag{34}$$

From (33) and (34), we get

$$\zeta = 0.$$

Which implies that $\mathcal{P}$ is Killing ($\zeta = 0$) and hence the manifold $\mathcal{N}$ is locally isometric to $\mathcal{E}^{m+1} \times S^4$ in dimension > 3. Also we have

$$\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}}g(W,U)) = \mathcal{P}.\pounds_{\mathcal{P}}g(W,U) - \pounds_{\mathcal{P}}g(\pounds_{\mathcal{P}}W,U) - \pounds_{\mathcal{P}}g(W,\pounds_{\mathcal{P}}U).$$

Then

$$\pounds_{\mathcal{P}}g = \pounds_{\mathcal{P}}(\pounds_{\mathcal{P}}g) = 0.$$

Using the above in (4) and substituting $W = U = \theta$ in (12) and (4), we get

$$\lambda = 2m.$$

This completes the proof. □

4. Gradient hyperbolic Ricci solitons on (2m + 1)-dimensional η-Einstein K-contact manifolds

Theorem 4.1. *If an η-Einstein K-contact manifolds $\mathcal{N}^{2m+1}$ admitting GHRS, then the Laplacian of the potential vector field is constant.*

Proof. From (5), we get

$$\mathcal{S}(W,U) = \lambda g(W,U) - 2\tau g(\nabla_W \text{grad}\zeta, U) - \pounds_{\text{grad}\zeta}.g(\nabla_W \text{grad}\zeta, U). \tag{35}$$

Contracting along the vector field W and U in (35), we obtain

$$\mathcal{R} = \lambda(2m+1) - 2\tau Div(\text{grad}\zeta) - \pounds_{\text{grad}\zeta}(\Delta\zeta)), \tag{36}$$

Δ being Laplacian operator. Since in a GHRS, the potential vector field is gradient of the smooth function ζ, so by the definition of the Lie derivative, we get

$$(\pounds_{\text{grad}\zeta}g)(W,U) = g(\nabla_W \text{grad}\zeta, U) + g(W, \nabla_U \text{grad}\zeta). \tag{37}$$

Putting $W = U = \epsilon_i$ and summing over i, we get

$$\sum_{i=1}^{2m+1} (\pounds_{\text{grad}\zeta}g)(\epsilon_i,\epsilon_i) = 2\Delta\zeta, \tag{38}$$

where (ϵ_i), $i = 1, 2, \ldots\ldots, (2m+1)$ is the orthonormal basis of the tangent space at each point of the manifold. Taking Lie derivative operator on the equation (38), we obtain

$$2\pounds_{\text{grad}\zeta}(\Delta\zeta) = \sum_{i=1}^{2m+1} [\text{grad}\zeta.\pounds_{\text{grad}\zeta}g(\epsilon_i,\epsilon_i) - 2\pounds_{\text{grad}\zeta}g(\pounds_{\text{grad}\zeta}\epsilon_i,\epsilon_i)].$$

Hence

$$\pounds_{\text{grad}\zeta}(\Delta\zeta) = 0. \tag{39}$$

Using (39) in (36), we get

$$\mathcal{R} = \lambda(2m+1) - 2\tau\Delta\zeta. \tag{40}$$

Combining (40) and (12), we obtain

$$\Delta\zeta = \frac{(2m+1)(\lambda - 2m)}{2\zeta}, \tag{41}$$

which is a constant. □

5. Example

Let $\mathcal{N} = \{(x, y, z, u, v) \in \mathbb{R}^5 : z \neq 0\}$ be a five-dimensional manifold, (x, y, z, u, v) being the standard co-ordinates in $\mathbb{R}^5$, whose basis vector fields are given by

$$\epsilon_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad \epsilon_2 = \frac{\partial}{\partial y}, \quad \epsilon_3 = \frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \quad \epsilon_4 = \frac{\partial}{\partial v}, \quad \epsilon_5 = \frac{\partial}{\partial z}.$$

By direct computation, one can find

$$[\epsilon_1, \epsilon_2] = -2\epsilon_5, \quad [\epsilon_1, \epsilon_3] = 0, \quad [\epsilon_1, \epsilon_4] = 0, \quad [\epsilon_1, \epsilon_5] = 0,$$

$$[\epsilon_2, \epsilon_3] = 0, \quad [\epsilon_2, \epsilon_4] = 0, \quad [\epsilon_2, \epsilon_5] = 0,$$

$$[\epsilon_3, \epsilon_4] = -2\epsilon_5, \quad [\epsilon_3, \epsilon_5] = 0, \quad [\epsilon_4, \epsilon_5] = 0.$$

Let the metric tensor g be defined by

$$g(\epsilon_i, \epsilon_j) = \{1, for \quad i = j \; and \quad 0, for \quad i \neq j\}, \; where \; i, j = 1 \; to \; 5.$$

The 1-form η and the $(1, 1)$ tensor field ϕ are, respectively, defined by $\eta(U_1) = g(U_1, \epsilon_5)$ for every vector field U_1 on the manifold and

$$\phi\epsilon_1 = -\epsilon_2, \quad \phi\epsilon_2 = \epsilon_1, \quad \phi\epsilon_3 = -\epsilon_4, \quad \phi\epsilon_4 = \epsilon_3, \quad \phi\epsilon_5 = 0.$$

Then we find that

$$\eta(\epsilon_5) = 1, \quad \phi^2 W = -W + \eta(W)\epsilon_3,$$

$$g(\phi W, \phi U) = g(W, U) - \eta(W)\eta(U),$$

$$d\eta(W, U) = g(W, \phi U)$$

for every vector fields W, U on the manifold. Thus $(\phi, \epsilon_5, \eta, g)$ defines a contact metric structure.

By Koszul's formula, we can find

$$\nabla_{\epsilon_i}\epsilon_j = \begin{pmatrix} 0 & -\epsilon_5 & 0 & 0 & \epsilon_2 \\ \epsilon_5 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & 0 & 0 & -\epsilon_5 & \epsilon_4 \\ 0 & 0 & \epsilon_5 & 0 & -\epsilon_3 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3 & 0 \end{pmatrix}$$

where $i, j = 1, 2, 3, 4, 5$ and ∇ denotes the Levi-Civita connection. Thus, we see that $\nabla_W \epsilon_5 = -\phi W$ for every vector field W. Hence $\mathcal{N}$ is a K-contact manifold. The components of the Riemannian curvature tensor are given by

$$R(\epsilon_1,\epsilon_2)\epsilon_2 = -3\epsilon_1, \quad R(\epsilon_1,\epsilon_2)\epsilon_1 = 3\epsilon_2, \quad R(\epsilon_1,\epsilon_2)\epsilon_3 = 2\epsilon_4, \quad R(\epsilon_1,\epsilon_2)\epsilon_4 = -2\epsilon_3,$$

$$R(\epsilon_1,\epsilon_2)\epsilon_5 = 0, \quad R(\epsilon_1,\epsilon_3)\epsilon_1 = 0, \quad R(\epsilon_1,\epsilon_3)\epsilon_2 = \epsilon_4, \quad R(\epsilon_1,\epsilon_3)\epsilon_3 = 0,$$

$$R(\epsilon_1,\epsilon_3)\epsilon_4 = -\epsilon_2, \quad R(\epsilon_1,\epsilon_3)\epsilon_5 = 0, \quad R(\epsilon_1,\epsilon_4)\epsilon_1 = 0, \quad R(\epsilon_1,\epsilon_4)\epsilon_2 = -\epsilon_3,$$

$$R(\epsilon_1,\epsilon_4)\epsilon_3 = \epsilon_2, \quad R(\epsilon_1,\epsilon_4)\epsilon_4 = 0, \quad R(\epsilon_1,\epsilon_4)\epsilon_5 = 0, \quad R(\epsilon_1,\epsilon_5)\epsilon_1 = -\epsilon_5,$$

$$R(\epsilon_1,\epsilon_5)\epsilon_2 = 0, \quad R(\epsilon_1,\epsilon_5)\epsilon_3 = 0, \quad R(\epsilon_1,\epsilon_5)\epsilon_4 = 0, \quad R(\epsilon_1,\epsilon_5)\epsilon_5 = \epsilon_1,$$

$$R(\epsilon_2,\epsilon_1)\epsilon_1 = -3\epsilon_2, \quad R(\epsilon_2,\epsilon_3)\epsilon_3 = 0, \quad R(\epsilon_2,\epsilon_4)\epsilon_4 = 0, \quad R(\epsilon_2,\epsilon_5)\epsilon_5 = \epsilon_2,$$

$$R(\epsilon_3,\epsilon_1)\epsilon_1 = 0, \quad R(\epsilon_3,\epsilon_3)\epsilon_3 = 0, \quad R(\epsilon_3,\epsilon_2)\epsilon_2 = 0, \quad R(\epsilon_3,\epsilon_4)\epsilon_4 = -3\epsilon_3,$$

$$R(\epsilon_3,\epsilon_5)\epsilon_5 = \epsilon_3, \quad R(\epsilon_3,\epsilon_4)\epsilon_1 = 2\epsilon_2, \quad R(\epsilon_4,\epsilon_1)\epsilon_1 = 0, \quad R(\epsilon_4,\epsilon_3)\epsilon_3 = -3\epsilon_4,$$

$$R(\epsilon_4,\epsilon_2)\epsilon_2 = 0, \quad R(\epsilon_4,\epsilon_4)\epsilon_4 = 0, \quad R(\epsilon_4,\epsilon_5)\epsilon_5 = \epsilon_4, \quad R(\epsilon_5,\epsilon_1)\epsilon_1 = \epsilon_5,$$

$$R(\epsilon_5,\epsilon_3)\epsilon_3 = \epsilon_5, \quad R(\epsilon_5,\epsilon_2)\epsilon_2 = \epsilon_5, \quad R(\epsilon_5,\epsilon_4)\epsilon_4 = \epsilon_5.$$

The non-zero components of the Ricci tensor are given by

$$\mathcal{S}(\epsilon_1,\epsilon_1) = -2, \quad \mathcal{S}(\epsilon_2,\epsilon_2) = -2, \quad \mathcal{S}(\epsilon_3,\epsilon_3) = -2, \quad \mathcal{S}(\epsilon_4,\epsilon_4) = -2, \quad \mathcal{S}(\epsilon_5,\epsilon_5) = 4,$$

and $\mathcal{S}(\epsilon_i,\epsilon_j) = 0$, for all $i \neq j$; $i, j = 1, 2, 3, 4, 5$. Thus we see that, the scalar curvature $\mathcal{R}$ is -4.

Let the potential vector field be the Reeb vector field ϵ_5. Then, from (4), we see that $\lambda = 4$. Thus the soliton is shrinking. Hence Theorem 3.2 is verified.

Let $\zeta = ae^z + b$ satisfies the GHRS, where a, b are real constants. Then $\mathrm{grad}\zeta = (\zeta - b)\epsilon_5$. Thus

$$\nabla_{\epsilon_1}\mathrm{grad}\zeta = (\zeta - b)\epsilon_2,$$
$$\nabla_{\epsilon_2}\mathrm{grad}\zeta = -(\zeta - b)\epsilon_1,$$
$$\nabla_{\epsilon_3}\mathrm{grad}\zeta = (\zeta - b)\epsilon_4,$$
$$\nabla_{\epsilon_4}\mathrm{grad}\zeta = -(\zeta - b)\epsilon_3,$$
$$\nabla_{\epsilon_5}\mathrm{grad}\zeta = (\zeta - b)\epsilon_5.$$

Therefore

$$\Delta\zeta = \sum_{i=1}^{5} g(\nabla_{\epsilon_i}\mathrm{grad}\zeta, \epsilon_i) = (\zeta - b).$$

In view of (35), we obtain the following

$$\lambda = -2,$$
$$\lambda - 2\tau(\zeta - b) = 4.$$

Again, from (36), we have

$$5\lambda - 2\tau(\zeta - b) = -4.$$

Solving the last two equations, we obtain $\lambda = -2$ and $\zeta - b = -\frac{3}{\tau}$, a constant. Thus, the Laplacian of the potential vector field is constant.

6. Hyperbolic Ricci solitons on super quasi-Einstein spacetimes

The notion of a generalized quasi-Einstein manifolds was introduced in the paper [16]. According to the author, a non-flat Riemannian manifold $(\mathcal{N}^m, g)$ $(m \geq 3)$ is called generalized quasi-Einstein if its Ricci tensor $\mathcal{S}$ of the type $(0,2)$ is not identically zero and satisfies the condition

$$\mathcal{S}(W,U) = a_1 g(W,U) + b_1\eta_1(W)\eta_1(U) + c_1(\eta_1(W)\eta_2(U) + \eta_1(U)\eta_2(W)), \tag{42}$$

where a_1, b_1, c_1 are associated scalars of which $b_1 \neq 0$ and $c_1 \neq 0$ and η_1, η_2 are two non-zero 1-forms such that

$$\eta_1(W) = g(W,\mu_1) \text{ and } \quad \eta_2(W) = g(W,\mu_2),$$

and μ_1, μ_2 are two unit vector fields perpendicular to each other. This paper also deals with super quasi-Einstein manifolds. A Lorentzian manifold $(\mathcal{N}^m, g)$ $(m \geq 4)$ is called super quasi-Einstein spacetime if its non-zero Ricci tensor $\mathcal{S}$ satisfies

$$\mathcal{S}(W,U) = a_1 g(W,U) + b_1\eta_1(W)\eta_2(U) + c_1(\eta_1(W)\eta_2(U) + \eta_1(U)\eta_2(W)) + d_1\mathcal{Y}(W,U), \tag{43}$$

where a_1, b_1, c_1, d_1 are non-vanishing smooth functions and η_1, η_2 are two non-zero 1-forms such that

$$\eta_1(W) = g(W,\mu_1) \text{ and } \quad \eta_2(W) = g(W,\mu_2),$$

for any vector field W. Here μ_1 and μ_2 are unit timelike vector field and unit spacelike vector field respectively such that $g(\mu_1,\mu_2) = 0$ and $\mathcal{Y}$ is a symmetric $(0,2)$-tensor with zero trace such that $\mathcal{Y}(W,\mu_1) = 0$. In the absence of cosmological constant, the Einstein's field equation is given by

$$\mathcal{S}(W,U) = \kappa\mathcal{E}_m(W,U) + \frac{\mathcal{Y}}{2} g(W,U), \tag{44}$$

where κ, $\mathcal{E}_m$ and $\mathcal{R}$ are respectively gravitational constant, energy momentum tensor and scalar curvature. Further the energy momentum tensor $\mathcal{E}_m$ is given by

$$\mathcal{E}_m(W,U) = \rho g(W,U) + (\rho+\gamma)\eta_1(W)\eta_1(U) + \eta_1(W)\eta_2(U) + \eta_1(U)\eta_2(W) + \mathcal{Y}(W,U), \tag{45}$$

for some isotropic pressure ρ and energy density γ. From the equation (44) and (45), we have the following [16]:

$$\mathcal{R} = -\frac{2\kappa}{m-2}((m-1)\rho - \gamma) \tag{46}$$

and

$$\begin{aligned}\mathcal{S}(W,U) = {} & \kappa\frac{(\gamma-\rho)}{(m-2)} g(W,U) + \kappa(\rho+\gamma)\eta_1(W)\eta_1(U) + \kappa(\eta_1(W)\eta_2(U) \\ & + \eta_1(U)\eta_2(W) + \mathcal{Y}(W,U)).\end{aligned} \tag{47}$$

For more details about super quasi-Einstein spacetimes, please see the paper [16, 26, 27, 29].

Theorem 6.1. *If a Ricci recurrent super quasi-Einstein spacetime of dimension $m \geq 4$ with a covariantly constant symmetric (0,2)-tensor $\mathcal{Y}$ and a closed non-vanishing 1-form η_1 admits HRS whose potential vector field is the unit timelike vector field μ_1, then the sum of isotropic pressure and energy density is zero.*

Proof. Taking covariant differentiation of (52) with respect to any vector field $\mathcal{P}$, we get

$$\begin{aligned}(\nabla_{\mathcal{P}}\mathcal{S})(W,U) &= \nabla_{\mathcal{P}}\mathcal{S}(W,U) - \mathcal{S}(\nabla_{\mathcal{P}}W,U) - \mathcal{S}(W,\nabla_{\mathcal{P}}U)\\ &= \frac{\kappa}{m-2}(\nabla_{\mathcal{P}}\gamma - \nabla_{\mathcal{P}}\rho)g(W,U) + \frac{\kappa(\gamma-\rho)}{m-2}(\nabla_{\mathcal{P}}g)(W,U)\\ &+ \kappa[(\nabla_{\mathcal{P}}\gamma + \nabla_{\mathcal{P}}\rho))\eta_1(W)\eta_1(U) + (\rho+\gamma)[(\nabla_{\mathcal{P}}\eta_1)(W)\eta_1(U)\\ &+ (\nabla_{\mathcal{P}}\eta_1)(U)\eta_1(W)]] + \kappa[\eta_2(U)(\nabla_{\mathcal{P}}\eta_1)(W) + \eta_2(W)(\nabla_{\mathcal{P}}\eta_1)(U)\\ &+ \eta_1(W)(\nabla_{\mathcal{P}}\eta_2)(U) + \eta_1(U)(\nabla_{\mathcal{P}}\eta_2)(W) + (\nabla_{\mathcal{P}}\mathcal{Y})(W,U)],\end{aligned} \tag{48}$$

for every vector fields W, U and $\mathcal{P}$ on the manifold $\mathcal{N}$.
Inserting $W = \mu_1$ and $U = \mu_2$ in the above equation, one can obtain

$$\kappa(\rho+\gamma)g(\nabla_{\mathcal{P}}\mu_1,\mu_2) + \kappa[g(\nabla_{\mathcal{P}}\mu_1,\mu_1) - g(\nabla_{\mathcal{P}}\mu_2,\mu_2)]. \tag{49}$$

Since $\mathcal{S}$ is Ricci recurrent, μ_1 is orthogonal to μ_2 and $(\nabla_{\mathcal{P}}\mathcal{Y})(W,U) = 0$.
Again since, $(\nabla_{\mathcal{P}}\eta_1)\mu_2 = g(\nabla_{\mathcal{P}}\mu_1,\mu_2)$, $(\nabla_{\mathcal{P}}\eta_1)\mu_1 = g(\nabla_{\mathcal{P}}\mu_1,\mu_1)$,
$(\nabla_{\mathcal{P}}\eta_2)\mu_2 = g(\nabla_{\mathcal{P}}\mu_2,\mu_2)$ and $g(\mu_1,\mu_1) = -1$, $g(\mu_2,\mu_2) = 1$.
Using the above to the equation (49), we get

$$\kappa(\rho+\gamma)g(\nabla_{\mathcal{P}}\mu_1,\mu_2) = 0.$$

If we consider $\mathcal{P}$ as potential vector field, then the above equation becomes

$$\kappa(\rho+\gamma)g(\nabla_{\mu_1}\mu_1,\mu_2) = 0. \tag{50}$$

Now from (4), we get

$$\begin{aligned}&\mathcal{S}(W,U) + \tau[g(\nabla_W\mu_1,U) + g(\nabla_U\mu_1,W)] + \frac{1}{2}[\mu_1.\pounds_{\mu_1}g(W,U)\\ &- \pounds_{\mu_1}g(\nabla_{\mu_1}W - \nabla_W\mu_1,U) - \pounds_{\mu_1}g(W,\nabla_{\mu_1}U - \nabla_U\mu_1)] = \lambda g(W,U).\end{aligned} \tag{51}$$

Substituting $W = \mu_1$ and $U = \mu_2$ in the above equation, we obtain

$$\begin{aligned}&\mathcal{S}(\mu_1,\mu_2) + \tau[g(\nabla_{\mu_1}\mu_1,\mu_2) + g(\nabla_{\mu_2}\mu_1,\mu_1)]\\ &+ \frac{1}{2}[\pounds_{\mu_1}g(\nabla_{\mu_1}\mu_1,\mu_2) + \pounds_{\mu_1}g(\nabla_{\mu_2}\mu_1,\mu_1)] = 0.\end{aligned} \tag{52}$$

Assuming that $g(\nabla_{\mu_1}\mu_1,\mu_2) = 0$ and $g(\nabla_{\mu_2}\mu_1,\mu_1) = 0$, we get

$$\mathcal{S}(\mu_1,\mu_2) = 0. \tag{53}$$

Again, Inserting $W = \mu_1$ and $U = \mu_2$ in (47), we get

$$\mathcal{S}(\mu_1,\mu_2) = -\kappa. \tag{54}$$

Comparing (53) and (54), we obtain

$$\kappa = 0. \tag{55}$$

Thus, we arrive at a contradiction, since the gravitational constant $\kappa > 0$. Therefore, our assumption is wrong. Hence from (50), we get

$$\rho + \gamma = 0. \tag{56}$$

This completes the proof. □

Theorem 6.2. *If a super quasi-Einstein spacetime of dimension $m \geq 4$ admits HRS with conformal Killing vector field, then the manifold is Einstein and the gravitational constant κ is given by*

$$\kappa = \frac{m(m-2)(2\alpha^2 + 2\zeta\alpha + \mathcal{P}\alpha - \lambda)}{2[(m-1)\rho - \gamma]}. \tag{57}$$

Proof. Taking Lie derivative operator on (14),we get

$$\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}} g(W,U)) = [2\mathcal{P}(\alpha) + 4\alpha^2]g(W,U). \tag{58}$$

Using (14) and (58) in (4), we obtain

$$\mathcal{S}(W,U) = [\lambda - 2\zeta\alpha - \mathcal{P}(\alpha) - 2\alpha^2]g(W,U). \tag{59}$$

Contracting (59) along the vector fields W and U, we get

$$\mathcal{R} = [\lambda - 2\zeta\alpha - \mathcal{P}(\alpha) - 2\alpha^2]m. \tag{60}$$

Comparing (46) and (60), we obtain

$$\kappa = \frac{m(m-2)(2\alpha^2 + 2\zeta\alpha + \mathcal{P}\alpha - \lambda)}{2[(m-1)\rho - \gamma]}, \tag{61}$$

which gives

$$\lambda - 2\zeta\alpha - \mathcal{P}(\alpha) - 2\alpha^2 = C, \tag{62}$$

where $C = -\frac{2\kappa[(m-1)\rho-\gamma]}{m(m-2)}$, a constant.
Combining (59) and (62), we get

$$\mathcal{S}(W,U) = Cg(W,U). \tag{63}$$

This proves the theorem. □

Theorem 6.3. *If a super quasi-Einstein spacetime of dimension $m \geq 4$ admits HRS with Ricci bi-conformal vector field, then the manifold is Einstein and*

$$\kappa = \frac{(m-2)[2f_1\tau + \mathcal{P}(f_1) + {f_1}^2 + {f_2}^2 - 2\lambda]}{[(m-1)\rho + (m-3)\gamma][2 + 2f_1f_2 + 2\tau f_2 + \mathcal{P}(f_2)]}. \tag{64}$$

Proof. Using (58) and (15), we obtain

$$\pounds_{\mathcal{P}}(\pounds_{\mathcal{P}} g(W,U)) = [\mathcal{P}(f_1) + {f_1}^2 + {f_2}^2]g(W,U) + [\mathcal{P}(f_2) + 2f_1f_2]\mathcal{S}(W,U). \tag{65}$$

Inserting the value of (15) and (65) in the equation (4), we get

$$[1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2]\mathcal{S}(W,U) + [f_1\tau + \frac{1}{2}\mathcal{P}(f_1) + \frac{1}{2}({f_1}^2 + {f_2}^2) - \lambda]g(W,U) = 0, \tag{66}$$

which gives

$$\mathcal{S}(W,U) = -\frac{[f_1\tau + \frac{1}{2}\mathcal{P}(f_1) + \frac{1}{2}({f_1}^2 + {f_2}^2) - \lambda]}{[1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2]}g(W,U). \tag{67}$$

Substituting $W = U = \mu_1$ in (67) and (47), we obtain

$$\mathcal{S}(\mu_1,\mu_1) = \frac{f_1\tau + \frac{1}{2}\mathcal{P}(f_1) + \frac{1}{2}({f_1}^2 + {f_2}^2) - \lambda}{1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2}, \tag{68}$$

and

$$\mathcal{S}(\mu_1, \mu_1) = \frac{\kappa}{(m-2)}[(m-1)\rho + (m-3)\gamma], \tag{69}$$

since, $g(\mu_1, \mu_1) = -1$ and $\eta_2(\mu_1) = 0$. Comparing (68) and (69), we get

$$\kappa = \frac{(m-2)[2f_1\tau + \mathcal{P}(f_1) + {f_1}^2 + {f_2}^2 - 2\lambda]}{[(m-1)\rho + (m-3)\gamma][2 + 2f_1f_2 + 2\tau f_2 + \mathcal{P}(f_2)]}. \tag{70}$$

Which gives

$$-\frac{f_1\tau + \frac{1}{2}\mathcal{P}(f_1) + \frac{1}{2}({f_1}^2 + {f_2}^2) - \lambda}{1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2} = \tilde{C}, \tag{71}$$

where $\tilde{C} = -\frac{\kappa}{(m-2)}[(m-1)\rho + (m-3)\gamma]$, a constant.
Combining (67) and (71), we get

$$\mathcal{S}(W, U) = \tilde{C}g(W, U). \tag{72}$$

Hence, we have the theorem. □

Corollary 6.4. *If a super quasi-Einstein spacetime of dimension $m \geq 4$ admits HRS with Ricci bi-conformal vector field, then*

$$\tau = -\frac{1}{f_2}[1 + f_1f_2 + \frac{1}{2}\mathcal{P}(f_2)]. \tag{73}$$

Proof. Substituting $W = \mu_1$, $U = \mu_2$ in (66) and (47), we get

$$[1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2]\mathcal{S}(\mu_1, \mu_2) = 0, \tag{74}$$

and

$$\mathcal{S}(\mu_1, \mu_2) = -\kappa. \tag{75}$$

Combining (74) and (75), we obtain

$$[1 + \tau f_2 + \frac{1}{2}\mathcal{P}(f_2) + f_1f_2]\kappa = 0. \tag{76}$$

The above equation gives

$$\tau = -\frac{1}{f_2}[1 + f_1f_2 + \frac{1}{2}\mathcal{P}(f_2)], \tag{77}$$

as the gravitational constant $\kappa > 0$.
Thus we obtain the result. □

Acknowledgement: The authors are very much thankful to the reviewers and the editor for their constructive and valuable suggestions towards the improvement of the paper.

References

[1] S. Azami, *Harmonic-hyperbolic geometric flow*, Electron. J. Differential Equations, (2017), Paper No. 165, 9 pp.
[2] S. Azami, *Hyperbolic Ricci-Bourguignon flow*, Comput. Methods Differ. Equ., **9** (2) (2021), 399-409.
[3] S. Azami, M. Jafari, N. Jamal, A. Haseeb, *Hyperbolic Ricci solitons on perfect fluid spacetimes*, AIMS Mathematics, **9** (7), 18929-18943.
[4] S. Azami, G. Fasihi-Ramandi, *Hyperbolic Ricci soliton on warped product manifolds*, Filomat, **37** (20) (2023), 6843-6853.
[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Birkhäuser, New York, **203** (2010).
[6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer Verlag, Berlin-Newyork, **509** (1976).
[7] D. E. Blair, J. S. Kim, M. M. Tripathi, *On the Concircular Curvature Tensor of a Contact Metric Manifold*, J. Korean Math. Soc., **42** (5) (2005), 883-992.
[8] A. M. Blaga and C. Özgür, *Results of hyperbolic Ricci solitons*, Symmetry, **15** (8) (2023), 1548. https://doi.org/10.3390/sym15081548.
[9] A. M. Blaga, *On trivial gradient hyperbolic Ricci and gradient hyperbolic Yamabe soliton solitons*, Journal of Geometry, **115** (2024), 26. https://doi.org/10.1007/s00022-024-00725-6
[10] A. M. Blaga, C. Özgür, *On Ricci and hyperbolic Ricci soliton submanifolds of almost contact metric manifolds*, Proceeding of the International Conference Riemannian Geometry and Applications-RIGA 2023, Romanian Journal of mathematics and Computer Science, **13** (2) (2023), 26-34.
[11] A. M. Blaga, C. Özgür, *Some properties of hyperbolic Yamabe soliton*, https://doi.org/10.48550/arXiv.2310.15814v2
[12] A. M. Blaga, C. Özgür, *Almost η-Ricci and η-Yamabe solitons with torse-forming potential vector field*, Quaest. Math., **45** (1) (2022),143-163.
[13] A. M. Blaga, C. Özgür, *Killing and* 2*-Killing vector fields on doubly warped products*, Mathematics, **11** (2023), 4983.
[14] A. M. Blaga, C. Özgür, 2*-Killing vector fields on multiply warped product manifolds*, Chaos Solitons Fractals, **180** (2024), Paper No. 114561, 7pp.
[15] A. M. Blaga, C. Özgür, *On* 2*-Killing vector fields in almost contact metric geometry*, Period. Math. Hung., (2024).
[16] M. C. Chaki, *On super quasi-Einstein manifolds*, Publ. Math. Debrecen, **64** (2004), 481–488.
[17] S. Chaubey, M. Siddiqi and D. G. Prakasha, *Invariant submanifolds of hyperbolic Sasakian manifolds and η-Ricci-Bourguignon solitons*, Filomat, **36** (2) (2022), 409-421.
[18] W. R. Dai, D. X. Kong, K. Liu, *Hyperbolic gometric flow (I): short-time existence and nonlinear stability*, Pure and applied mathematics quarterly, **6** (2010), 331-359.
[19] U. C. De, A. Sardar, A. Sarkar, *Some conformal vector fields and conformal Ricci solitons on $N(\kappa)$-contact metric manifolds*, AUT J. Math. Comput., **2** (2021), 61–71. https://doi.org/10.22060/ajmc.2021.19220.1043.
[20] H. Faraji, S. Azami, G. Fasihi- Ramandi, *Three dimensional homogeneous hyperbolic Ricci solitons*, J. Nonlinear Math. Phys., **30** (1) (2023), 135-155.
[21] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom., **17** (2) (1982), 255-306.
[22] R. S. Hamilton, *The Ricci flow on surfaces, in Mathematics and General Relativity*, Contemporary Mathematics, Vol. **71** (American Mathematical Society, 1988), 237–262.
[23] D. A. Kaya and C. Özgür, *Hyperbolic Ricci solitons on sequential warped product manifolds*, Filomat, **38** (2024), 1023-1032.
[24] D. Kong and K. Liu, *Wave character of metrics and hyperbolic geometric flow*, J. Math. Phys., **48** (2007), 103508-1-103508-14.
[25] D. Kong, Q. Liu and C. Song, *Classical solution to dissipative hyperbolic geometric flow in two space variables*, J. Hyperbolic. Diff. Equ., **16** (2019), 223-243.
[26] S. Mallick, *Super quasi-Einstein manifolds with applications to general relativity*, Kyungpook Math. J., **58** (2018), 361-375.
[27] G. Mitra, T. Mandal, A. Sarkar, *Nearly vacuum static equations on K-contact manifolds and its applications in spacetimes*, Eur. Phys. J. plus., **139**:182 (2024). https://doi.org/10.1140/epjp/s13360-024-04964-z.
[28] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J. Geom. Anal., **26** (2016), 1280–1345.
[29] A. Sarkar and U. Biswas, *(m, ρ)-quasi-Einstein solitons on 3-dimensional trans-Sasakian manifolds and its applications in spacetimes*, Int. J. Geom. Methods in Modern Physics, (2024), 2450002 (12 pages).
[30] R. Sharma, *Certain results on K-contact and (κ, μ)-contact manifolds*, J. Geom., **89** (2008), 138-147.
[31] W. Wang, *Almost cosymplectic (κ, μ)-metrics as η-Ricci solitons*, J. Nonlinear Math. Phys., (2021). https://doi.org/10.1007/s44198-021-00019-4.
[32] Y. Wang, *Ricci solitons on 3-dimensional cosymplectic manifolds*, Math. Slovaca, **67** (2017), 979–984.
[33] H. I. Yoldas, S. E. Meric and E. Yasar, *Some characteristics of α-cosymplectic manifolds admitting Yamabe soliton*, Palestine Journal of Mathematics, **10** (1) (2021), 234-241.