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Qualitative analysis of solutions for a parabolic $m(x)$ -biharmonic equation with logarithmic nonlinearity

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Abstract. In this paper, we consider a nonlinear parabolic $m(x)$- biharmonic equation with logarithmic source terms. Applying the potential well method combined with the Nehari manifold, the global existence and blow-up of weak solutions is proved. In addition, we establish decay estimates for the global weak solutions.

1. Introduction

We consider the following $m(x)-$biharmonic equation with variable exponents

$$\begin{cases} z_t + \Delta^2 z + \Delta^2_{m(x)} z = |z|^{p(x)-2} z \ln|z|, & x \in \Omega,\ t > 0, \\ z(x,t) = \frac{\partial z}{\partial v}(x,t) = 0, & x \in \partial\Omega, \\ z(x,0) = z_0(x) \in W_0^{2,m(.)}(\Omega), & x \in \Omega, \end{cases} \tag{1}$$

here Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, v is the unit outward normal vector on $\partial\Omega$, and $z_0(x) \geq 0$. The $m(x)-$biharmonic equation $\Delta^2_{m(x)} z$ is the nonlinear differential operator defined by

$$\Delta^2_{m(x)} z = \Delta(|\Delta z|^{m(x)-2} \Delta z).$$

We assumptions on $p(\cdot)$ and $m(\cdot)$ the following,
(A1) The exponents $p(\cdot)$ and $m(\cdot)$ are measurable function satisfying
here

$$\max\{2, m^+\} < p^- \leq p^+ < \min\left\{m^+\left(1+\frac{4}{n}\right), (m^-)^*\right\}, \tag{2}$$

with

$$m^- = ess\inf_{x\in\Omega} m(x), \quad m^+ = ess\sup_{x\in\Omega} m(x),$$

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$$p^{-} = ess\inf_{x\in\Omega} p(x), \quad p^{+} = ess\sup_{x\in\Omega} p(x),$$

and

$$m^{-^{*}}(x) = \begin{cases} \frac{nm^{-}}{n-m^{-}}, & \text{if } m^{-} < n, \\ +\infty & \text{if } m^{-} \geq n. \end{cases}$$

(A2) $\forall \varkappa, \xi \in \Omega$, $|\varkappa - \xi| < \delta$, $\gamma > 0$ and $0 < \delta < 1$,

$$|m(\varkappa) - m(\xi)| + |p(\varkappa) - p(\xi)| \leq -\frac{\gamma}{\ln|\varkappa - \xi|}. \tag{3}$$

1.1. *Literature overview:*

Wu et al. [29] examined the subsequently semilinear parabolic equation with variable exponent

$$z_t - \Delta z = z^{q(x)}.$$

They proved the blow up of solutions. Then, many authors studied the blow up of solutions the same problem under different conditions (see [4, 16, 28]).

Boudjeriou [8] studied following heat equation

$$z_t - \Delta_{p(x)} z = |z|^{q(x)-2} z \ln|z|.$$

He proved local existence, global existence and finite time blow-up of solutions.

Liu et al. [18] examined the following $m(x)$-Laplacian parabolic equation

$$z_t = div\left(a|\nabla z|^{m(x)-2} \nabla z\right) + z\ln|z|.$$

The authors shows the non-extinction and the extinction in finite time of solutions.

Zhu et al. [31] investigated the following problem

$$z_t - \Delta z_t - div\left(|\nabla z|^{m(x)-2} \nabla z\right) = |z|^{m(x)-2} z.$$

They acquired global existence and blow-up outcomes for weak solutions characterized by arbitrarily high initial energy.

Chuong et al. [10] reviewed the following a pseudo-parabolic equation problem

$$z_t - \Delta z_t - div\left(|\nabla z|^{p(x)-2} \nabla z\right) = |z|^{q(x)-2} z.$$

They derived decay and blow up also show the asymptotic behavior of global solution.

Liu et al. [17] examined the following fourth-order pseudo-parabolic problem with $p(x)$-Laplacian

$$z_t - \Delta z_t + \Delta^2 z - div\left(|\nabla z|^{p(x)-2} \nabla z\right) = |z|^{q(x)-1} z.$$

They showed the classification of initial energy on the existence of blow-up, global and extinction solutions.

Pan et al. [20] studied the following a pseudo-parabolic equation problem

$$z_t - \Delta z_t - div\left(|\nabla z|^{p(x)-2} \nabla z\right) = |z|^{q(x)-2} z\ln|z|.$$

They obtain the global existence and blow-up results of weak solutions. Also, some authors studied the partial differential equations with variable exponents (see [2, 3, 6, 7, 15, 22, 23, 27]).

Choung et al. [11] studied the following $m(x)$ −Laplacian equations with logarithmic source terms

$$z_t - \Delta_{m(x)} z = |z|^{p(x)-2} z\ln|z|.$$

Inspired by these works, we consider the problem (1) with the logarithmic nonlinearity $|z|^{p(x)-2} z \ln |z|$. The primary challenges in addressing the problem arise due to the disparity between the norm and the modulus. Moreover, the inclusion of the term $|z|^{p(x)-2} z \ln |z|$ presents certain challenges in the application of the potential well method. The problem (1) occurs in many mathematical models of applied science, such as electro-rheological fluids, heat transfer, chemical reactions, population dynamics, etc. The interested readers may refer to [1, 5, 12, 14, 19, 21] and the references therein.

The goal of this study is as follows:

(i) In Section 2, we present function spaces, notations and lemmas. Additionally, we define variable function spaces pertaining to both Lebesgue and Sobolev type.

(ii) In Sections 3, we define the weak solutions to the problem (1) and outline the main results that will be derived in the subsequent sections.

2. Function spaces and notations

In this part, we present certain notations, lemmas and fundamental properties of the generalized Lebesgue space and Sobolev space [12, 25]. Let $\Omega \subset \mathbb{R}^n$ be a domain with a smooth boundary, and let $\mathcal{K}(\Omega)$ represent the set of all measurable functions $q : \Omega \to [1, \infty)$. For $q \in \mathcal{K}(\Omega)$, the Lebesgue space with a variable exponent $q(\cdot)$ is defined as follows:

$$L^{q(x)}(\Omega) = \left\{ z : \Omega \to R,\ z \text{ is measurable and } \rho_{q(.)}(\lambda z) < \infty,\ \text{for some } \lambda > 0 \right\},$$

here

$$\rho_{q(.)}(z) = \int_{\Omega} |z|^{q(x)} dx.$$

Also endowed with the Luxemburg-type norm

$$\|z\|_{q(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{z}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

Lemma 2.1. *[12]. Assume that $q, s \in \mathcal{K}(\Omega)$, the following result holds:*

1) Given that $1 < q^- \leq q^+ < \infty$, later $L^{q(\cdot)}(\Omega)$ is a separeble and uniformly convex Banach space.

2) Given that $q^+ < \infty$ later the relationship between the modular $\rho_{q(.)}(z)$ and the norm $\|z\|_{q(x)}$ is given by:

$$\min \left\{ \|z\|_{q(\cdot)}^{q^-}, \|z\|_{q(\cdot)}^{q^+} \right\} \leq \rho_{q(.)}(z) \leq \max \left\{ \|z\|_{q(\cdot)}^{q^-}, \|z\|_{q(\cdot)}^{q^+} \right\},$$

for every $z \in L^{q(\cdot)}(\Omega)$.

3) Hölder's inequality also applies to the variable exponent case:

$$\|zv\|_{(.)} \leq 2 \|z\|_{q(.)} \|v\|_{r(.)} \text{ for all } z \in L^{q(\cdot)}(\Omega),\ v \in L^{r(\cdot)}(\Omega),$$

$$\frac{1}{s(x)} = \frac{1}{q(x)} + \frac{1}{r(x)} \text{for a.e. } x \in \Omega.$$

Lemma 2.2. *[12]. Suppose that $q, s \in \mathcal{K}(\Omega)$. If $q(x) \leq s(x)$ for a.e. $x \in \Omega$, then the embedding $L^{s(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous.*

We next define variable exponent Sobolev spaces

$$W^{m,p(\cdot)}(\Omega) = \left\{ z \in L^{p(\cdot)}(\Omega) \text{ such that } D^{\alpha} z \in L^{p(\cdot)}(\Omega),\ |\alpha| \leq m \right\}.$$

This space is a Banach space with respect to the norm

$$\|z\|_{W^{2,q(.)}(\Omega)} = \left(\|z\|_{q(.)}^2 + \|\nabla z\|_{q(.)}^2 + \|\Delta z\|_{q(.)}^2\right)^{1/2}.$$

Furthermore, let $W_0^{2,q(.)}(\Omega)$ be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,q(.)}(\Omega)$.

Lemma 2.3. *[12, 13]. Assume that $q, s \in \mathcal{K}(\Omega)$.*

1) Given that $2 < q^- \leq q^+ < \infty$, later $W^{1,q(.)}(\Omega)$ and $W_0^{2,q(.)}(\Omega)$ are separable and uniformly convex Banach spaces.

2) Given that $|\Omega| < \infty$ and $q \in C\left(\overline{\Omega}\right)$ fulfills ess $\inf_{x\in\Omega}(q^*(x) - s(x)) > 0$. Here

$$q^*(x) = \begin{cases} \frac{nq(x)}{(n-q(x))}, & \text{if } q(x) < n, \\ +\infty, & \text{if } q(x) \geq n. \end{cases}$$

Later the embedding $W_0^{2,q(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{s(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.4. *(Poincaré's Inequality, [12]). Assume that $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $q(\cdot)$ fulfills (4), later we have $\|z\|_{q(\cdot)} \leq C\|\Delta z\|_{q(\cdot)}$ for every $z \in W_0^{2,q(\cdot)}(\Omega)$.*

For $z \in W_0^{2,m(\cdot)}(\Omega)$ we define the energy functional $E(z)$ and Nehari functional $I(z)$ as follows:

$$E(z) = \frac{1}{2}\int_\Omega |\Delta z|^2\,dx + \int_\Omega \frac{1}{m(x)}|\Delta z|^{m(x)}\,dx - \int_\Omega \frac{1}{p(x)}|z|^{p(x)}\ln|z|\,dx + \int_\Omega \frac{1}{p^2(x)}|z|^{p(x)}\,dx$$

and

$$I(z) = \int_\Omega |\Delta z|^2\,dx + \int_\Omega |\Delta z|^{m(x)}\,dx - \int_\Omega |z|^{p(x)}\ln|z|\,dx.$$

These functionals are of class C^2 over $W_0^{2,m(x)}(\Omega)$ because of the condition (2). We also define the Nehari manifold

$$\mathcal{N} = \left\{z \in W_0^{2,m(x)}(\Omega)\setminus\{0\} \mid z \neq 0 \text{ and } I(z) = 0\right\},$$

with the potential well depth

$$d = \inf_{z\in\mathcal{N}} E(z).$$

The lemma below demonstrates that $\mathcal{N}$ is a nonempty set, ensuring the well-definedness of d.

Lemma 2.5. *Suppose that (2)-(3) are satisfied. For each $z \in W_0^{2,m(x)}(\Omega)\setminus\{0\}$ there exists a $\gamma_z \in (0,\infty)$ that depends on z, such that $\gamma_z z \in \mathcal{N}$.*

Proof. We first note that the function $s \to |a|s\ln|a|$ is increasing, and therefore

$$\begin{aligned} I(z) &= \int_\Omega |\Delta z|^2\,dx + \int_\Omega |\Delta z|^{m(x)}\,dx - \int_\Omega |z|^{p(x)}\ln|z|\,dx \\ &\leq \int_\Omega |\Delta z|^2\,dx + \int_\Omega |\Delta z|^{m(x)}\,dx - \int_\Omega |z|^{p^-}\ln|z|\,dx. \end{aligned}$$

Next, by replacing z with γz in the previous inequality for any $\gamma > 0$, we derive

$$\begin{aligned} I(\gamma z) &\leq \gamma^2 \int_\Omega |\Delta z|^2 dx + \gamma^{m(x)} \int_\Omega |\Delta z|^{m(x)} dx - \gamma^{p^-} \int_\Omega |z|^{p^-} \ln|z| \, dx - \gamma^{p^-} \ln|\gamma| \, \|z\|_{p^-}^{p^-} \\ &\leq \gamma^2 \int_\Omega |\Delta z|^2 dx + \max\left\{\gamma^{m^-}, \gamma^{m^+}\right\} \int_\Omega |\Delta z|^{m(x)} dx - \gamma^{p^-} \int_\Omega |z|^{p^-} \ln|z| \, dx - \gamma^{p^-} \ln|\gamma| \, \|z\|_{p^-}^{p^-}. \end{aligned}$$

Observe that $p^- > m^+$ and $\|z\|_{p^-}^{p^-} > 0$ because $z = 0$. From this and the previous inequality, we get $\lim_{\gamma\to\infty} I(\gamma z) = -\infty$. Similarly, we find that $I(\gamma z) > 0$ for sufficiently small $\gamma > 0$ due to the following estimate

$$\begin{aligned} I(\gamma z) &\geq \gamma^2 \int_\Omega |\Delta z|^2 dx + \gamma^{m(x)} \int_\Omega |\Delta z|^{m(x)} dx - \gamma^{p^+} \int_\Omega |z|^{p^+} \ln|z| \, dx - \gamma^{p+} \ln|\gamma| \, \|z\|_{p^+}^{p^+} \\ &\geq \gamma^2 \int_\Omega |\Delta z|^2 dx + \min\left\{\gamma^{m^-}, \gamma^{m^+}\right\} \int_\Omega |\Delta z|^{m(x)} dx - \gamma^{p^+} \int_\Omega |z|^{p^+} \ln|z| \, dx - \gamma^{p+} \ln|\gamma| \, \|z\|_{p^+}^{p^+}. \end{aligned}$$

Thus, by the intermediate value theorem, there exists a $\gamma z \in (0, \infty)$, such that $I(\gamma_z z) = 0$, implying $\gamma_z z \in \mathcal{N}$. This completes the proof. □

The lemma below can be proven with straightforward calculations.

Lemma 2.6. *[24]. The inequality below holds for all $a > 0$ with $s > 0$*

$$\ln s \leq \frac{s^a}{ea}.$$

The following lemma will be crucial in establishing our main results.

Lemma 2.7. *Assume (2)–(3) hold. Then*

$$E(z) - \frac{1}{p^-} I(z) \geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \int_\Omega |\Delta z|^2 dx + \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{(p^+)^2} \int_\Omega |z|^{p(x)} dx - K_0.$$

Here, K_0 is a non-negative constant defined by

$$K_0 = \frac{1}{e} \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) \frac{1}{p(x)} dx. \tag{4}$$

Proof. For $z \in W_0^{2,m(\cdot)}(\Omega)$. According to Lemma 2.6., we have

$$\begin{aligned} -\ln|z| &= \ln \frac{1}{|z|} \\ &\leq \frac{1}{ep(x)|z|^{p(x)}}, \end{aligned}$$

thus, this implies that

$$|z|^{p(x)} \ln|z| \geq -\frac{1}{ep(x)}.$$

Later, by the definition of I and E, we derive

$$\begin{aligned} E(z)-\frac{1}{p^-}I(z) &= \int_\Omega \left(\frac{1}{2}-\frac{1}{p^-}\right)|\Delta z|^2\,dx+\int_\Omega \left(\frac{1}{m(x)}-\frac{1}{p^-}\right)|\Delta z|^{m(x)}\,dx \\ &\quad +\frac{1}{p^2(x)}\int_\Omega |z|^{p(x)}\,dx+\int_\Omega \left(\frac{1}{p^-}-\frac{1}{p(x)}\right)|z|^{p(x)}\ln|z|\,dx \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^2\,dx+\left(\frac{1}{m^+}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)}\,dx \\ &\quad +\frac{1}{(p^+)^2}\int_\Omega |z|^{p(x)}\,dx-K_0. \end{aligned}$$

This concludes the proof. □

Next, we define

$$\mathcal{N}_-=\left\{W_0^{2,m(\cdot)}(\Omega)\setminus\{0\}:I(z)\leq 0\right\},$$

and demonstrate that 0 is not contained in the set $\mathcal{N}_-$.

Lemma 2.8. *Suppose that (2)–(3) are satisfied. Later*

$$dist(0,\mathcal{N}_-)=\inf_{z\in\mathcal{N}_-}\|\Delta z\|_{m(\cdot)}\geq\mu_0>0.$$

Where, μ_0 represents the constant as defined in (5).

Proof. Since (2), we observe that $W_0^{2,m(\cdot)}(\Omega)\hookrightarrow L^{p^++\epsilon}(\Omega)$ with $\epsilon>0$ fulfilling

$$0<\epsilon<(m^-)^*-p^+.$$

Consider any $z\in\mathcal{N}_-$. Utilizing Lemma 2.6. and noting that the function $s\to|a|s\ln|a|$ is increasing, we derive

$$\begin{aligned} \min\left\{\|\Delta z\|_{m(\cdot)}^{m^-},\|\Delta z\|_{m(\cdot)}^{m^+}\right\} &\leq \int_\Omega |\Delta z|^{m(x)}\,dx\leq\int_\Omega |z|^{p(x)}\ln|z|\,dx \\ &\leq \int_\Omega |z|^{p^+}\ln|z|\,dx \\ &\leq \frac{1}{e\epsilon}\|z\|_{m^++\epsilon}^{m^++\epsilon} \\ &\leq \frac{1}{e\epsilon}K_\epsilon^{m^++\epsilon}\|\Delta z\|_{m(\cdot)}^{p^++\epsilon}. \end{aligned}$$

Where, K_ϵ represents the optimal embedding constant of $W_0^{2,m(\cdot)}(\Omega)\hookrightarrow L^{p^++\epsilon}(\Omega)$. Consequently, we deduce that$\|\Delta z\|_{m(\cdot)}\geq\mu_0$ here

$$\begin{aligned} \mu_0 &= \min\left\{(e\epsilon)^{\frac{1}{p^++\epsilon-m^-}}K_\epsilon^{\frac{p^++\epsilon}{p^++\epsilon-m^-}},(e\epsilon)^{\frac{1}{p^++\epsilon-m^+}}K_\epsilon^{\frac{p^++\epsilon}{p^++\epsilon-m^+}}\right\} \\ &> 0. \end{aligned} \tag{5}$$

Thus

$$dist\,(0,\mathcal{N}_-) = \inf_{z\in\mathcal{N}_-} \|\Delta z\|_{m(\cdot)} \geq \mu_0 > 0.$$

The demonstration is concluded. □

The subsequent lemma provides a lower limit for the potential well depth, implying that d and thus $d > -\infty$.

Lemma 2.9. *Suppose that (2)–(3) are satisfied. Let*

$$\begin{aligned} d_0 &= \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^2\,dx + \left(\frac{1}{m^+}-\frac{1}{p^-}\right)\min\left\{\mu_0^{m^-},\mu_0^{m^+}\right\} - K_0 \\ &> -K_0. \end{aligned} \tag{6}$$

In this scenario, μ_0 and K_0 represent the constants provided in (5) and (4) correspondingly. As a result, we have $d \geq d_0$ and

$$E(z) - \frac{1}{p^-}I(z) \geq d_0,\ \ \forall z \in \mathcal{N}_-.$$

Proof. Utilizing Lemma 2.7. and Lemma 2.8. we derive that for any $z \in \mathcal{N}_-$

$$\begin{aligned} E(z) - \frac{1}{p^-}I(z) &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^2\,dx - K_0 + \left(\frac{1}{m^+}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)}\,dx \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^2\,dx - K_0 + \left(\frac{1}{m^+}-\frac{1}{p^-}\right)\min\left\{\|\Delta z\|_{m(\cdot)}^{m^-},\|\Delta z\|_{m(\cdot)}^{m^+}\right\} \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^2\,dx - K_0 + \left(\frac{1}{m^+}-\frac{1}{p^-}\right)\min\left\{\mu_0^{m^-},\mu_0^{m^+}\right\} \\ &= d_0. \end{aligned}$$

Later, because $\mathcal{N} \subset \mathcal{N}_-$, we get

$$E(z) = E(z) - \frac{1}{p^-}I(z) \geq d_0,\ \ \forall z \in \mathcal{N}.$$

As a result, we have $d = \inf_{z\in\mathcal{N}} E(z) \geq d_0$. The demonstration concludes here. □

3. Main Results

In this section, we offer findings concerning the global existence and blow-up of weak solutions within the subcritical case where $E(u_0) < d$. Initially, we outline the definition of weak solutions for the problem presented in (1).

Lemma 3.1. *Let $(.,.)$ denote the inner product in $L^2(\Omega)$ and suppose $T \in (0,\infty)$. A function $z \in L^\infty\left(0,T;W_0^{2,m(\cdot)}(\Omega)\right)$ is termed a weak solution to problem (1) with $z_t \in L^2\left(0,T;L^2(\Omega)\right)$ if it meets the initial condition $z(\cdot,0) = z_0$ and*

$$(z_t,w) + (\Delta z,\Delta w) + \left(|\Delta z|^{m(x)-2}\Delta z,\Delta w\right) = \left(|z|^{p(x)-2}z\ln|z|,w\right), \tag{7}$$

for almost every $t \in (0,T)$ and for any test-function $w \in W_0^{2,m(\cdot)}(\Omega)$. Additionally, z also fulfills the subsequent inequality, for almost every $t \in (0,T)$

$$\int_0^t \|z'(s)\|_2^2 \, ds + E(z) \leq E(z_0).$$

Weak solutions local existence can be acquired through the Galerkin method, as demonstrated in references such as [9] or [30]. Subsequently, we introduce the definition for the maximal duration of weak solutions existence.

Lemma 3.2. *The maximal existence time $T_{\max}$ of the weak solution $z(t)$ of (1) is specified as described:*
(i) Given that $z(t)$ is specified on $[0,\infty)$, later $T_{\max} = \infty$.
(ii) Given that $z(t)$ is specified on $[0,T_0)$, but it cannot be extended to T_0, later $T_{\max} = T_0$.

The unstable set U and stable set (potential well) W are defined similarly to Sattinger [26].

$$\begin{aligned} U &= \left\{ z \in W_0^{2,m(\cdot)}(\Omega) : E(z) < d \text{ with } I(z) < 0 \right\}, \\ W &= \left\{ z \in W_0^{2,m(\cdot)}(\Omega) : E(z) > d \text{ with } I(z) \geq 0 \right\}. \end{aligned}$$

Now, we present our main findings as follows. Initially, we examine the scenario where the initial energy is negative $E(z_0) < -K_0$. Here, the constant $K_0 \geq 0$ is specified in (4).

Theorem 3.3. *Suppose that (2)–(3) are satisfied. Given that $E(z_0) < -K_0$, later*

$$T_{\max} \leq C \max\left\{ \|z_0\|_2^{2-p^-}, \|z_0\|_2^{2-p+} \right\},$$

here

$$C = \frac{(p^+)^2 \max\left\{ K_1^{p^-}, K_1^{p^+} \right\}}{p^-(p^- - 2)} > 0, \tag{8}$$

where K_1 represents the optimal embedding constant from $L^{p(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$, defined as

$$K_1 = \sup_{z \in L^{p(\cdot)}(\Omega)\setminus\{0\}} \frac{\|z\|_2}{\|z\|_{p(\cdot)}}. \tag{9}$$

Our subsequent demonstration establishes the instability of the solution to the problem (1) assuming the initial data $z_0 \in U$.

Proof. Consider the function $h : [0, T_{\max}) \to \mathbb{R}$ defined as

$$h(t) = \|z\|_2^2.$$

By employing Lemma 2.7., we acquire

$$E(z) - \frac{1}{p^-} I(z) \geq -K_0 + \frac{1}{(p^+)^2} \int_\Omega |z|^{p(x)} \, dx.$$

Conversely, we have $-K_0 > E(z_0) \geq E(z)$. Hence

$$0 \leq \int_\Omega |z|^{p(x)} \, dx \leq -\frac{(p^+)^2}{p^-} I(z) = \frac{(p^+)^2}{2p^-} h'(t), \tag{10}$$

this implies that h is monotonically increasing on $[0, T_{\max})$, thus

$$h(t) \geq h(0) = \|z_0\|_2^2 > 0, \quad \forall t \in [0, T_{\max}).$$

Later

$$\begin{aligned}\int_\Omega |z|^{p(x)}\,dx &\geq \min\left\{\|z\|_{p(\cdot)}^{p^-}, \|z_0\|_{p(\cdot)}^{p^+}\right\} \\ &\geq \min\left\{K_1^{-p^-}\|z\|_2^{p^-}, K_1^{-p^+}\|z_0\|_2^{p^+}\right\} \\ &\geq \min\left\{K_1^{-p^-}, K_1^{-p^+}\right\}\min\left\{1, h^{(p^+-p^-)/2}(t)\right\} h^{p^-/2}(t) \\ &\geq \min\left\{K_1^{-p^-}, K_1^{-p^+}\right\}\min\left\{1, \|z_0\|_2^{p^+-p^-}\right\} h^{p^-/2}(t),\end{aligned}$$

where K_1 is specified in (9). This, along with (10) suggests that

$$h'(t)\,h^{p^-/2}(t) \geq C_0, \tag{11}$$

here

$$C_0 = \frac{2p^-}{(p^+)^2}\min\left\{K_1^{-p^-}, K_1^{-p^+}\right\}\min\left\{1, \|z_0\|_2^{p^+-p^-}\right\} > 0.$$

Integrating (11) over $[0, t]$, we derive

$$0 < h^{1-p^-/2}(t) \leq h^{1-p^-/2}(0) + C_0\left(1 - \frac{p^-}{2}\right)t, \quad \forall t \in [0, T_{\max}),$$

thus indicating that for $\forall t \in [0, T_{\max})$

$$t < \frac{2}{C_0(p^- - 2)}\|z_0\|_2^{2-p^-} = C\max\left\{\|z\|_{p(\cdot)}^{2-p^-}, \|z_0\|_{p(\cdot)}^{2-p^+}\right\},$$

where C is defined in (8). Letting $t \to T_{\max}^-$, we attain the necessary outcome. Hence, the proof is concluded. □

Theorem 3.4. *Let (2)–(3) are satisfied. Given that $z_0 \in U$, later $T_{\max} < \infty$. Furthermore, in the case where $E(z_0) < d_0$ and $I(z_0) < 0$, we obtain the subsequent upper limit for $T_{\max}$:*

$$T_{\max} \leq \frac{4(p^- - 1)\|z_0\|_2^2}{p^-(p^- - 2)^2(d_0 - E(z_0))}.$$

In this context, $d_0 \leq d$ represents the constant specified in (6).

Proof. Let's suppose $z_0 \in U$. We aim to establish $T_{\max} < \infty$. Assuming the contrary, let's assume $T_{\max} = \infty$. According to Theorem 3.3., we deduce $E(z) \geq -K_0$ for all $t \geq 0$, and thus

$$\int_0^t \|z'(\delta)\|_2^2\,d\delta \leq E(z_0) - E(z) \leq E(z_0) + K_0 < \infty.$$

As $t \to \infty$, we derive $\int_0^\infty \|z'(\delta)\|_2^2\,d\delta$. Consequently, there exists a sequence $t_n \nearrow \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty}\|z'(t_n)\|_2 = 0. \tag{12}$$

For sufficiently large n, we have

$$\begin{aligned} I(z(t_n)) &= |(z'(t_n), z(t_n))| \\ &\leq \|z'(t_n)\|_2 \|z(t_n)\|_2 \\ &\leq \|z'(t_n)\|_2 K_2 \|\Delta z(t_n)\|_{m(\cdot)} \qquad (13) \\ &\leq \|\Delta z(t_n)\|_{m(\cdot)}, \qquad (14) \end{aligned}$$

where K_2 denotes the optimal embedding constant of $W_0^{2,m(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$, defined as

$$K_2 = \sup_{z \in W_0^{2,m(\cdot)}(\Omega)\setminus\{0\}} \frac{\|z\|_2}{\|\Delta z\|_{m(\cdot)}}. \qquad (15)$$

Utilizing Lemma 2.7. and (14), one has

$$\begin{aligned} E(z_0) &\geq E(z(t_n)) \geq \frac{1}{p^-} I(z(t_n)) - K_0 + \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \int_\Omega |\Delta z(t_n)|^{m(x)} dx \\ &\geq -\frac{1}{p^-} \|\Delta z(t_n)\|_{m(\cdot)} - K_0 + \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \min\left\{\|\Delta z(t_n)\|_{m(\cdot)}^{m^-}, \|\Delta z(t_n)\|_{m(\cdot)}^{m^+}\right\}. \end{aligned}$$

The inequality above indicates that the set $\{z(t_n)\}$ is bounded in $W_0^{2,m(\cdot)}(\Omega)$ given that $m^- > 1$. Later, since $W_0^{2,m(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)+\epsilon}(\Omega)$ with $\epsilon > 0$ is small enough, there is a $\varphi \in W_0^{2,m(\cdot)}(\Omega)$ and a subsequence of $\{z(t_n)\}$, which still denoted by itself, so that

$$\begin{aligned} z(t_n) &\rightharpoonup \varphi \text{ weakly in } W_0^{2,m(\cdot)}(\Omega), \\ z(t_n) &\to \varphi \text{ strongly in } L^{p(\cdot)+\epsilon}(\Omega), \\ z(t_n) &\to \varphi \text{ a.e. in } \Omega. \qquad (16) \end{aligned}$$

Replacing z with $z(t_n)$ in (7), we find for every $w \in W_0^{2,m(\cdot)}(\Omega)$.

$$\begin{aligned} &\left| \begin{array}{c} (\Delta z(t_n), \Delta w) + \left(|\Delta z(t_n)|^{m(x)-2} \Delta z(t_n), \Delta w\right) \\ -\left(|z(t_n)|^{p(x)-2} z(t_n) \ln |z(t_n)|, w\right) \end{array} \right| \\ &= |(z'(t_n), w)| \\ &\leq \|z'(t_n)\|_2 \|w\|_2. \end{aligned}$$

As $n \to \infty$ and observing (12) and (16), we derive

$$(\Delta\varphi, \Delta w) + \left(|\Delta\varphi|^{m(x)-2} \Delta\varphi, \Delta w\right) - \left(|\varphi|^{p(x)-2} \varphi \ln|\varphi|, w\right) = 0.$$

By setting $w = \varphi$ in the aforementioned equation, we deduce $I(\varphi) = 0$. Conversely, employing the weak lower semi-continuity of E, we infer from (16) that

$$E(\varphi) \leq \lim_{n\to\infty} \inf E(z(t_n)) \leq E(z_0) < d.$$

From this with $I(\varphi) = 0$, we get

$$\varphi = 0. \qquad (17)$$

Utilizing (12), (13) and noting that$\{z(t_n)\}$ is bounded in $W_0^{2,m(\cdot)}(\Omega)$, we deduce $\lim_{n\to\infty} I(z(t_n)) = 0$. This, coupled with (16) and (17), suggests

$$z(t_n) \to 0 \text{ strongly in } W_0^{2,m(\cdot)}(\Omega) \text{ as } n \to \infty. \qquad (18)$$

To proceed, we establish $I(z(t)) < 0$ for all $t \geq 0$. If this were not the case, then there would exist a $t_* > 0$, such that $I(z(t)) < 0$ for $0 \leq t < t_*$ and $I(z(t_*)) = 0$. Considering this scenario and recognizing that $z(t_*) \notin \mathcal{N}$ due to $d > E(z_0) \geq E(z(t_*))$, we derive $z(t_*) = 0$. Furthermore, Lemma 2.8. implies that $\|\Delta z(t)\|_{m(\cdot)} \geq \mu_0$, for $0 \leq t < t_*$. By letting $t \nearrow t_*$, we have $\|\Delta z(t_*)\|_{m(\cdot)} \geq \mu_0 > 0$ for all n. This contradicts (18). Thus, $T_{\max} = \infty$.

In the particular case here $E(z_0) < d_0$ and $I(z_0) < 0$, we will provide an upper bound estimate for $T_{\max}$. Let's examine the function R defined as

$$R(t) = \int_0^t \|z(\delta)\|_2^2 \, d\delta + (T_{\max} - t)\|z_0\|_2^2 + \psi(t), \text{ for } t \in [0, T_{\max}).$$

Where, $\psi(t) \in C^2[0, T_{\max})$ is a positive function given later. We get

$$\begin{aligned} R'(t) &= \|z(t)\|_2^2 - \|z_0\|_2^2 + \psi'(t), \\ R'' &= -2I(z(t)) + \psi''(t). \end{aligned}$$

Utilizing Cauchy–Schwarz inequality, we get for every $\epsilon_1 > 0$

$$\begin{aligned} &\left(\int_0^t \|z(\delta)\|_2^2 \, d\delta + \psi(t)\right)\left(\int_0^t \|z'(\delta)\|_2^2 \, d\delta + \epsilon_1\right) \\ \geq &\left(\int_0^t (z(\delta), z'(\delta)) \, d\delta + \sqrt{\epsilon_1 \psi(t)}\right)^2 \\ = &\frac{1}{4}\left(\|z(t)\|_2^2 - \|z_0\|_2^2 + 2\sqrt{\epsilon_1 \psi(t)}\right)^2. \end{aligned}$$

We select $\psi(t)$, such that $\psi'(t) = 2\sqrt{\epsilon_1 \psi(t)}$, imply $\psi(t) = \epsilon_1 (t + \epsilon_2)^2$ and $\epsilon_2 > 0$. Later

$$\begin{aligned} (R'(t))^2 &= \left(\|z(t)\|_2^2 - \|z_0\|_2^2 + 2\sqrt{\epsilon_1 \psi(t)}\right)^2 \\ &\leq 4\left(\int_0^t \|z(\delta)\|_2^2 \, d\delta + \psi(t)\right)\left(\int_0^t \|z'(\delta)\|_2^2 \, d\delta + \epsilon_1\right) \\ &\leq 4R(t)\left(\int_0^t \|z'(\delta)\|_2^2 \, d\delta + \epsilon_1\right). \end{aligned}$$

From this, it follows that

$$\begin{aligned} R''(t)R(t) - \frac{p^-}{2}(R'(t))^2 &\geq R(t)\left[R''(t) - 2p^-\left(\int_0^t \|z'(\delta)\|_2^2 \, d\delta + \epsilon_1\right)\right] \\ &\geq R(t)\left[R''(t) - 2p^-(E(z_0) - E(z(t)) + \epsilon_1)\right] \\ &= R(t)\left[\begin{array}{c} -2I(z(t)) + 2p^-E(z(t)) \\ -2p^-E(z_0) - 2\epsilon_1(p^- - 1) \end{array}\right]. \end{aligned} \tag{19}$$

However, since $z(t) \in \mathcal{N}_-$ applying Lemma 2.9. yields

$$E(z(t)) \geq \frac{1}{p^-} I(z(t)) + d_0.$$

This, combined with (19) implies

$$R''(t)R(t)-\frac{p^-}{2}(R'(t))^2\geq 2R(t)\left[p^-(d_0-E(z_0))-\epsilon_1(p^--1)\right].$$

Selecting $\epsilon_1=\frac{p^-(d_0-E(z_0))}{(p^--1)}>0$, we get

$$R''(t)R(t)-\frac{p^-}{2}(R'(t))^2\geq 0.$$

Observe that $R(0)=T_{\max}\|z_0\|_2^2+\epsilon_1\epsilon_2^2>0$ and $R'(0)=2\epsilon_1\epsilon_2>0$. From the aforementioned inequality, it follows that

$$T_{\max}\leq\frac{R(0)}{\left(\frac{p^-}{2}-1\right)R'(0)}=\frac{T_{\max}\|z_0\|_2^2+\epsilon_1\epsilon_2^2}{(p^--2)\epsilon_1\epsilon_2}.$$

Utilizing selecting $\epsilon_2>\frac{\|z_0\|_2^2}{(p^--2)\epsilon_1}$, we derive

$$T_{\max}\leq\frac{\epsilon_1\epsilon_2^2}{(p^--2)\epsilon_1\epsilon_2-\|z_0\|_2^2}=\varphi(\epsilon_2).$$

Thus

$$\begin{aligned}T_{\max}&\leq\min_{\epsilon_2>\frac{\|z_0\|_2^2}{(p^--2)\epsilon_1}}\varphi(\epsilon_2)=\varphi\left(\frac{2\|z_0\|_2^2}{(p^--2)\epsilon_1}\right)\\&=\frac{4(p^--1)\|z_0\|_2^2}{p^-(p^--2)^2(d_0-E(z_0))},\end{aligned}$$

and thus concludes the proof. □

Theorem 3.5. *Suppose that (2)–(3) hold. Given that $z_0\in W$, later $T_{\max}=\infty$ and the global weak solution z of the problem (1) tends to 0 strongly in $L^2(\Omega)$ as $t\to\infty$. Additionally, there exists a constant $C>0$ and a sufficiently large time t_0 large enough, such that the following decay estimates hold, for all $t\geq t_0$:*

i) Given that $m^+\leq 2$, later

$$\|z(t)\|_2^2\leq\|z(t_0)\|_2^2e^{-C(t-t_0)}.$$

ii) Given that $m^+>2$, later

$$\|z(t)\|_2^2\leq\left[\frac{m^+-2}{2}C(t-t_0)+\|z(t_0)\|^{2-m^+}\right]^{-\frac{2}{m^+-2}}.$$

Proof. To begin, we establish $z(t)\in W$ for all $t\in[0,T_{\max})$. Suppose this is not the case; then, there exists a $t^*\in(0,T_{\max})$, such that $z(t^*)\in\partial W$, implying either $E(z(t))=d$ or $I(z(t^*))=0$. The former case is impossible since, because $E(z(t^*))\leq E(z_0)<d$, so $I(z(t^*))=0$. This along with $E(z(t^*))<d$ imply that $z(t^*)=0$. On the other hand, we deduce from Lemma 2.8. that $B(0,r)\subset W$ for r sufficiently small, and thus 0 is an interior point of W. However, this contradicts $0=z(t^*)\in\partial W$. Hence, $z(t)\in W$ for all $t\in[0,T_{\max})$.

Now, we proceed to demonstrate that $T_{\max}=\infty$. Note that $I(z(t))\geq 0$, because $z(t)\in W$. Later, utilizing Lemma 2.7., we get

$$E(z_0)+K_0\quad\geq\quad E(z(t))+K_0$$

$$\geq \left(\frac{1}{m^+}-\frac{1}{p^-}\right)\min\left\{\|\Delta z(t)\|_{m(\cdot)}^{m^-},\|\Delta z(t)\|_{m(\cdot)}^{m^+}\right\},$$

this implies

$$\begin{aligned}\|\Delta z(t)\|_{m(\cdot)} &\leq \max\left\{\begin{array}{c}\left(\frac{m^+p^-(E(z_0)+K_0)}{p^--m^+}\right)^{\frac{1}{m^-}},\\ \left(\frac{m^+p^-(E(z_0)+K_0)}{p^--m^+}\right)^{\frac{1}{m^+}}\end{array}\right\}\\ &= C_1.\end{aligned}$$

The uniform estimate presented above indicates that the local solutions of (1) can be extended globally. Consequently, $T_{\max}=\infty$.

Next, we establish that the global weak solution z of the problem (1) tends to 0 strongly in $L^2(\Omega)$ as $t\to\infty$. Employing similar arguments to those in the proof of Theorem 3.4., we find a sequence $t_n \nearrow \infty$ as $n\to\infty$, such that

$$z(t_n)\to 0 \text{ strongly in } W_0^{2,m(\cdot)}(\Omega) \text{ as } n\to\infty.$$

This, together with the embedding $W_0^{2,m(\cdot)}(\Omega)\hookrightarrow L^2(\Omega)$ imply that

$$z(t_n)\to 0 \text{ strongly in } L^2(\Omega) \text{ as } n\to\infty. \tag{20}$$

Moreover, the function $t\mapsto \|z(t)\|_2$ is non-increasing, as

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|_2^2=-I(z(t))\leq 0.$$

This along with (20) imply that

$$z(t)\to 0 \text{ strongly in } L^2(\Omega) \text{ as } n\to\infty. \tag{21}$$

Lastly, we establish the decay property of $\|z(t)\|_2$. Given (2), we can select ϵ small enough, such that

$$\epsilon<\min\left\{m^+\left(1+\frac{4}{N}\right)-p^+,(m^-)^*-p^+\right\}.$$

Utilizing the Gagliardo–Nirenberg and Young inequality and noticing that the function $s\mapsto |a|s\ln|a|$ is increasing, we derive

$$\begin{aligned}I(z(t)) &\geq \|\Delta z(t)\|_2^2+\min\left\{\|\Delta z(t)\|_{m(\cdot)}^{m^-},\|\Delta z(t)\|_{m(\cdot)}^{m^+}\right\}-\frac{1}{e\epsilon}\int_\Omega |z(t)|^{p^++\epsilon}dx\\ &= \|\Delta z(t)\|_2^2+\min\left\{\|\Delta z(t)\|_{m(\cdot)}^{m^--m^+},1\right\}\|\Delta z(t)\|_{m(\cdot)}^{m^+}-\frac{1}{e\epsilon}\|z(t)\|_{p^++\epsilon}^{p^++\epsilon}\\ &\geq \min\{C_1,1\}\|\Delta z(t)\|_{m(\cdot)}^{m^+}-C_2\|\Delta z(t)\|_{m^-}^{\phi(p^++\epsilon)}\|z(t)\|_2^{(1-\phi)(p^++\epsilon)}\\ &\geq \min\{C_1,1\}\|\Delta z(t)\|_{m(\cdot)}^{m^+}-C_3\|\Delta z(t)\|_{m(\cdot)}^{\phi(p^++\epsilon)}\|z(t)\|_2^{(1-\phi)(p^++\epsilon)}\\ &\geq \min\{C_1,1\}\|\Delta z(t)\|_{m(\cdot)}^{m^+}-C_3\left(\mu\|\Delta z(t)\|_{m(\cdot)}^{m^+}+C(\mu)\|z(t)\|_2^{\lambda}\right)\\ &= (\min\{C_1,1\}-\mu C_3)\|\Delta z(t)\|_{m(\cdot)}^{m^+}-C_4\|z(t)\|_2^{\lambda}.\\ &\geq C\|z(t)\|_2^{m^+}-C_4\|z(t)\|_2^{\lambda}.\end{aligned} \tag{22}$$

$$\phi=\left(\frac{1}{2}-\frac{1}{p^++\epsilon}\right)\left(\frac{1}{n}-\frac{1}{m^-}+\frac{1}{2}\right)^{-1}\in(0,1),$$

$$\lambda = \frac{m^+ (p^+ + \epsilon)\left(1 - \phi\right)}{m^+ - \phi (p^+ + \epsilon)} = g\left(\phi\right),$$

$$C = (\min \{C_1, 1\} - \mu C_3) K_2^{-m^+} > 0,$$

where $\mu > 0$ is chosen small enough and K_2 is the constant specified in (15). It's worth noting that $\phi (p^+ + \epsilon) < m^+$, given that $\epsilon < m^+ \left(1 + \frac{4}{n}\right) - p^+$. This condition is necessary to employ Young inequality in the above estimate. Since

$$g'\left(\phi\right) = \frac{m^+ (p^+ + \epsilon)(p^+ - m^+ + \epsilon)}{\left(m^+ - \phi (p^+ + \epsilon)\right)^2} > 0,$$

later

$$\lambda = g\left(\phi\right) > g(0) = p^+ + \epsilon > m^+. \tag{23}$$

Given (21), there exists a time t_0, such that for every $t \geq t_0$

$$\|z(t)\|_2 \leq \min \left\{1, \left(\frac{C}{2C_4}\right)^{\frac{1}{\lambda - m^+}}\right\}. \tag{24}$$

From this (22) and (23), we deduce

$$\frac{d}{dt} \|z(t)\|_2^2 = -2I(z(t)) \leq -C \|z(t)\|_2^{m^+}. \tag{25}$$

Given that $m^+ > 2$, later it follows from (25) that for every $t \geq t_0$

$$\|z(t)\|_2^2 \leq \left[\frac{m^+ - 2}{2} C (t - t_0) + \|z(t_0)\|^{2-m^+}\right]^{-\frac{2}{m^+ - 2}}.$$

If $m^+ \leq 2$, then it follows from (24) and (25), we obtain

$$\frac{d}{dt} \|z(t)\|_2^2 \leq -C \|z(t)\|_2^2,$$

this implies that for every $t \geq t_0$

$$\|z(t)\|_2^2 \leq \|z(t_0)\|_2^2 e^{-C(t - t_0)}.$$

This concludes the proof. □

References

[1] E. Acerbi, G. Mingione, Regularity results for stationary eletrorheological fluids, Archive for Rational Mechanics and Analysis, 164, (2002) 213-259.

[2] S. N. Antontsev, J. Ferreira, E. Pişkin, Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities, Electronic Journal of Differential Equations, (2021) 1-18.

[3] S. N. Antontsev, J. Ferreira, E. Pişkin, S. M. S. Cordeiro, Existence and non-existence of solutions for Timoshenko-type equations with variable exponents, Nonlinear Analysis: Real World Applications, 61, (2021) 1-13.

[4] K. Baghaei, M. B. Ghaemi, M. Hesaaraki, Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source, Appl. Math. Lett., 27, (2014) 49-52.

[5] I. Ben Omrane, M. Ben Slimane, S. Gala, M.A. Ragusa, A new regularity criterion for the 3D nematic liquid crystal flows, Anal. Appl.,(2024).

[6] G. Butakın, E. Pişkin, Existence and Blow up of Solutions of a Viscoelastic $m(x)$- Biharmonic Equation with Logarithmic Source Term, Miskolc Math. Notes, 25(2) (2024), 629–643.
[7] G. Butakın, E. Pişkin, Existence and Blow up of Solutions for $m(x)$- Biharmonic equation with Variable Exponent Sources, Filomat, 38 (22) (2024) 7871–7893.
[8] T. Boudjeriou, On the diffusion p(x)-Laplacian with logarithmic nonlinearity, J. Elliptic Parabol. Equ. 6, (2020) 773-794 .
[9] H. Chen, P. Luo, G. Liu, Global solution and blow up of a semilinear heat equation with logarithmic nonlinearity, J. Math. Anal. Appl., 422(1) (2015) 84-98.
[10] Q. V. Chuong, L. C. Nhan, L. X. Truong, Blow-up and Decay for a Pseudo-parabolic Equation with Nonstandard Growth Conditions, Taiwan. J. Math., (2024) 1-21.
[11] Q. V. Chuong, L. C. Nhan, L. X. Truong, On Global Solution for a Class of $p(x)$-Laplacian Equations with Logarithmic Nonlinearity, Mediterr. J. Math., (2024) 21-64.
[12] L. Diening, P. Harjulehto, P. Hasto, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents: Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 2011.
[13] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263, (2001) 424–446.
[14] A. Fidan, E. Pişkin, E. Çelik, Existence, decay, and blowup of solutions for a weighted m-biharmonic equation with nonlinear damping and source terms, J. Funct. Spaces, 2024, (2024).
[15] J. Ferreira, W. S. Panni, S. A. Messaoudi, E. Pişkin, M. Shahrouzi, Existence and Asymptotic Behavior of Beam-Equation Solutions with Strong Damping and $p(x)$- Biharmonic Operator, J. Math. Phys. Anal. Geom., 18(4), (2022) 488-513.
[16] A. Khelghati, K. Baghaei, Blow-up in a semilinear parabolic problem with variable source under positive initial energy, Appl. Anal., 94(9), (2015) 1888-1896.
[17] B. Liu, Y. Li, Note on a higher order pseudo-parabolic equation with variable exponents,Math. Methods Appl. Sci., 46, (2023) 16840-16856.
[18] B. Liu, ,M. Zhang, F. Li, Singular properties of solutions for a parabolic equation with variable exponents and logarithmic source, Nonlinear Anal. Real World Appl. 64, (2022) 103449.
[19] Q.T. Ou, On the partial boundary value condition basing on the diffusion coefficient, Filomat, 37 (18) , 5979-5992, (2023).
[20] R. Pan, Y. Gao, Q. Meng, Properties of Weak Solutions for a Pseudoparabolic Equation with Logarithmic Nonlinearity of Variable Exponents, Hindawi J. Math., (2023) 7441168.
[21] J. P. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, Nonlinear Anal. Real World Appl., 71, (2009) 1049-1058.
[22] E. Pişkin, Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon equation with variable exponents, Honam Math. J., 40(4), (2018) 771-783.
[23] E. Pişkin, G. Butakın, Blow-up phenomena for a $p(x)$-biharmonic heat equation with variable exponent, Math. Morav. 27(2), (2023) 25-32.
[24] E. Pişkin, T. Cömert, Qualitative analysis of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity, ODAM, 4(2), (2021) 1-10.
[25] E. Pişkin, B. Okutmuştur, An Introduction to Sobolev Spaces, Bentham Science, 2021.
[26] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Ration. Mech. Anal., 30(2), (1968) 148-172.
[27] M. Shahrouzi, J. Ferreira, E. Pişkin, K. Zennir, On the behavior of solutions for a class of nonlinear viscoelastic fourth-order $p(x)$-Laplacian equation, Mediterr. J. Math., 20, (2023) 1-28.
[28] H. Wang, Y. He, On blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy, Appl. Math. Lett., 26, (2013) 1008-1012.
[29] X. Wu, B. Guo, W. Gao, Blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy, Appl. Math. Lett., 26(5), (2013) 539-543.
[30] F. Zeng, Q. Deng, D. Wang, Global existence and blow-up for the pseudoparabolic $p(x)$-Laplacian equation with logarithmic nonlinearity, JNMP, 29 (2022) 41-57.
[31] X. Zhu, B. Guo, M. Liao, Global existence and blow-up of weak solutions for a pseudo-parabolic equation with high initial energy, Appl. Math. Lett., 104, (2020) 106270.