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Some special parallel like curves in Galilean 3-space

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Abstract. In this research, we consider the relations between parallel-like curves and some special curves in a 3-dimensional Galilean space. We prove that, although parallel-like curves are not an involute-evolute curve couple, they become a Bertrand and Mannheim curve couple under certain conditions. We also present this practically and visually.

1. Introduction

The most important basic building block of differential geometry is the concept of a curve. From the first moments when the curve began to be studied, many researchers first examined the Frenet apparatus $\kappa, \tau, \mathrm{T}, \mathrm{N}, \mathrm{B}$ defined on the curve and a lot of information about the curves was obtained by using these expressions. For example, a curve is a helix curve if and only if the ratio of τ and κ of the curve is constant. If both κ and τ are non-zero and constants in this ratio, the curve is called a circular helix [1]. In addition to the helix curve, another important curve is the Bertrand curves, discovered by J. Bertrand in 1850. If the principal normal vector N at each point of a curve becomes the principal normal vector of another curve, this curve pair is called the Bertrand curve pair [2]. Mannheim curves, which were first proposed by Mannheim in 1878, have a similar feature to Bertrand curves. Here, the principal normal N of the curve is linearly dependent on the binormal vector B of the other curve [3]. Another special curve pair is the involute-evolute curve pair named by Chr. Huyghens in 1665. In short, if the tangent of a curve is the normal of the other curve, this curve pair is called involute-evolute curve pair [4, 5].

There are also different calculations of Frenet elements depending on the studied space. The Galilean space can be given as an example. The foundations of Galilean geometry, studied in Galilean space, were laid by I. Yaglom in 1979 [6]. O. Röschel made a great contribution, especially on the geometry of ruled surfaces in this space [7]. The three-dimensional Galilean space $\mathbb{G}_3$, which we will consider in this paper, belongs to the category of Cayley-Klein geometries characterized by the projective signature (0,0,+,+) summarized in reference [8]. The absolute configuration of this Galilean geometry is represented by a structured set $\{w, f, l\}$, wherein w signifies the ideal (absolute) plane, f denotes the (absolute) line within

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w, and l denotes the fixed elliptic involution of points on f. For further elaboration of this space and other spaces, consult the references [6–15].

The concept of parallel-like curves was introduced by H. Vogler in 1963 [16]. He defines this notion as "curve at a constant distance from the edge of regression on a curve". Afterward, Hacısalihoğlu derived a broader application of Vogler's findings [17]. In [17], this curve is defined in three-dimensional Euclidean space $\mathbb{E}^3$ as follows.

Let us consider a curve $\alpha(t)$ with $\{T, N, B\}$ as Frenet frame at the point $P = \alpha(s)$ of $\alpha(t)$ in $\mathbb{E}^3$. d is described as a vector tightly fastened to Frenet trihedron $\{T, N, B\}$ such that $d = d_1T + d_2N + d_3B$, where d_1, d_2, d_3 are constant numbers and $d_1^2 + d_2^2 + d_3^2 = 1$. k is described as a line tightly fastened to Frenet trihedron $\{T, N, B\}$ in the direction of d and passing through point P [17]. Let P_v denote a point on the line k at a constant distance v from P. During the movement of the Frenet trihedron along the curve $\alpha(t)$, $C_v(t)$ is a geometric place of $P_v(s)$ which is defined as parallel-like curve of α [17]. Recently, this definition has been extended to the 3-dimensional Galilean space, establishing relationships between the Frenet apparatus and the curvatures of parallel-like curve pairs [18].

In this paper, taking into account the findings in reference [18], we will investigate the criteria for parallel-like curves to generate involute-evolute, Bertrand, and Mannheim curve pairs.

2. Preliminaries

In this section, let's recall some background information in the Galilean 3-space.

Let $\vec{V} = (v_1, v_2, v_3)$ and $\vec{W} = (w_1, w_2, w_3)$ be two vectors in the Galilean 3-space $\mathbb{G}_3$. The Galilean scalar product of two vectors is defined by

$$\left\langle \vec{V}, \vec{W} \right\rangle_G = \vec{V}.\vec{W} = \begin{cases} v_1 w_1, & if \;\; v_1 \neq 0 \; or \; w_1 \neq 0 \\ v_2 w_2 + v_3 w_3, & if \;\; v_1 = 0 \; and \; w_1 = 0 \end{cases}$$

If $\vec{V}.\vec{W} = 0$, these two vectors are orthogonal in the Galilean case [6]. The norm of the vector $\vec{V}$ is defined by [6]

$$\left\| \vec{V} \right\|_G = \begin{cases} |v_1|, & v_1 \neq 0 \\ \sqrt{v_2^2 + v_3^2}, & v_1 = 0 \end{cases}$$

The Galilean vector product of $\vec{V}$ and $\vec{W}$ in $\mathbb{G}_3$ is

$$\vec{V} \times_G \vec{W} = \begin{vmatrix} 0 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

if $v_1 = w_1 = 0$,

$$V \times_G W = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{vmatrix},$$

where $\times_G$ represents the vector product of two vectors [7, 20].

For $\vec{V} = (v_1, v_2, v_3) \in \mathbb{G}_3$, if $v_1 = 0$, then $\vec{V}$ is isotropic; if not, it is non-isotropic [20].

2.1. Curves in $\mathbb{G}_3$

Let γ be a curve given by$\gamma : I \to \mathbb{G}_3$, $\gamma(t) = (u(t), v(t), w(t))$, where $u(t), v(t), w(t)$ are continuously differentiable functions, and $t \in I$. If $u'(t) \neq 0$, $\gamma(t)$ is a regular curve.

Let $\gamma : I \to \mathbb{G}_3$ be a regular curve in $\mathbb{G}_3$. In this case, $ds = |u'(t)dt| = |du|$. Then, we have $s = u$. Hence, we can given by $\gamma(s) = (s, v(s), w(s))$[21].

Here, the functions $v, w : I \to \mathbb{R}$ are coordinate functions of the curve. By differentiating $\gamma(s) = (s, v(s), w(s))$ according to s, we have

$$\left\|\gamma'(x)\right\|_G = 1. \tag{1}$$

Then, it is claimed that the curve γ is a unit speed curve.

Let $\gamma : I \to \mathbb{G}_3,\ \gamma(s) = (s, v(s), w(s))$ be a regular unit speed curve in $\mathbb{G}_3$. Differentiating $\gamma(s)$, we obtain

$$\gamma'(s) = (1, v'(s), w'(s)). \tag{2}$$

This means that $\gamma'(s)$ is the unit tangent vector field of $\gamma(s)$. Then, we can write

$$T(s) = (1, v'(s), w'(s)). \tag{3}$$

Since $\gamma'(s).\gamma''(s) = 0$, the unit normal vector field is described as

$$\nu(s) = \frac{\gamma''(s)}{\left\|\gamma''(s)\right\|_G} = \frac{1}{\sqrt{v''^2(s) + w''^2(s)}}(0, v''(s), w''(s)). \tag{4}$$

Finally, the unit binormal vector field $\beta(s)$ of $\gamma(s)$ is

$$\beta(s) = \frac{1}{\sqrt{v''^2(s) + w''^2(s)}}(0, -w''(s), v''(x)), \tag{5}$$

and then in $\mathbb{G}_3$, the frame $\{T(s), \nu(s), \beta(s)\}$ selected in this manner is referred to as the Frenet-Serret frame for unit speed curves [22].

Proposition 2.1. *A unit speed curve $\gamma(s)$ in $\mathbb{G}_3$ has Frenet-Serret formulas that are provided by*

$$\begin{pmatrix} T'(s) \\ \nu'(s) \\ \beta'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ \nu(s) \\ \beta(s) \end{pmatrix}, \tag{6}$$

where

$$\kappa(s) = \sqrt{v''^2(s) + w''^2(s)} \tag{7}$$

is the curvature of γ and

$$\tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa^2(s)} \tag{8}$$

is the torsion of γ [23].

2.2. Some Special Curves in $\mathbb{G}_3$

Definition 2.2. *Let γ be a curve given by $\gamma : I \to \mathbb{G}_3$ and the Frenet-Serret frame $\{T(s), \nu(s), \beta(s)\}$. If the curvatures κ and τ of γ are positive constants along γ, γ is called circular helix with respect to Frenet-Serret frame [24].*

Definition 2.3. *Let $\gamma : I \to \mathbb{G}_3$ and $\gamma^* : I \to \mathbb{G}_3$ be given by the curvatures and torsions*

$$\kappa_\gamma(s) \neq 0, \kappa_\gamma{}^*(s) \neq 0, \tau_\gamma(s) \neq 0, \tau_\gamma^*(s) \neq 0$$

and Frenet-Serret frames $\left\{T_\gamma, \nu_\gamma, \beta_\gamma\right\}$, $\left\{T_\gamma^*, \nu_\gamma^*, \beta_\gamma^*\right\}$ respectively. If ν_γ and ν_γ^* are linearly dependent, these curves are called Bertrand curves, for $\forall s \in$ I. Also (γ, γ^*) is defined Bertrand curve pair [9]. In this case, we can write [9]

$$\gamma^*(s) = \gamma(s) + u(s)\,\nu_\gamma(s).$$

Definition 2.4. *Let $\gamma : I \to \mathbb{G}_3$ and $\gamma^* : I \to \mathbb{G}_3$ be given by Frenet-Serret frames $\left\{ T_\gamma, \nu_\gamma, \beta_\gamma \right\}$, $\left\{ T^*_\gamma, \nu^*_\gamma, \beta^*_\gamma \right\}$ respectively. If the tangent vector of the curve γ at the point $\gamma(s)$ passes through the tangent vector of the curve γ^* at the point $\gamma^*(s)$ and $\langle T, T^* \rangle_G = 0$, the curve γ^* is defined involute of the curve γ. In addition, the curve γ is defined as the evolute of the curve γ^* [25]. In this case, we can write [25]*

$$\gamma^*(s) = \gamma(s) + \lambda\, T_\gamma(s).$$

Definition 2.5. *Let $\gamma : I \to \mathbb{G}_3$ and $\gamma^* : I \to \mathbb{G}_3$ be given by the curvatures and torsions*

$$\kappa_\gamma(s) \neq 0, \kappa_\gamma{}^*(s) \neq 0, \tau_\gamma(s) \neq 0, \tau^*_\gamma(s) \neq 0$$

and Frenet-Serret frames $\left\{ T_\gamma, \nu_\gamma, \beta_\gamma \right\}$, $\left\{ T^*_\gamma, \nu^*_\gamma, \beta^*_\gamma \right\}$ respectively. If ν_γ and β^*_γ are linearly dependent, these curves are called Mannheim curves for $\forall s \in$ I. Also, (γ, γ^*) is defined Mannheim curve pair [26].

Definition 2.6. *Let γ be a curve given by $\gamma : I \to \mathbb{G}_3$ and the Frenet-Serret frame $\{ T(s), \nu(s), \beta(s) \}$ at the point $P = \gamma(s)$ of γ. Let P_r be a point at a constant distance r from P. Throughout the movement of the Frenet trihedron on the curve γ, the position of points P_r is defined by $\gamma_r = \gamma + rd$, where*

$$d = \begin{cases} d_1 T, & \text{if } d \text{ is non-isotropic} \\ d_2\nu + d_3\beta, & \text{if } d \text{ is isotropic} \end{cases}$$

such that $d_1^2 + d_2^2 + d_3^2 = 1$, $d_1, d_2, d_3 \in \mathbb{R}$ and $d^2 = 1$, $|d| = 1$. In this case, γ_r is called parallel like curve of γ [18].

According to Definition 2.6, there are two cases. d is isotropic or non-isotropic [18].

Case 1: d is non-isotropic. In this case,

$$\gamma_r(s) = \gamma(s) + rT(s), \tag{9}$$

where $d_1^2 = 1$, $d_1 = 1$, $d_2 = d_3 = 0$. Then, we derive $d(s) = T(s)$ [18].

Theorem 2.7. *Given that $\gamma(s)$ represents a curve parameterized by its arc length s, it follows that the parameterization of the arc length for curve γ_r remains s [18].*

Theorem 2.8. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves with arc length s in $\mathbb{G}_3$. If the Frenet vectors of γ and γ_r are $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$, the curvatures are κ, τ and κ_r, τ_r respectively, then the subsequent relations are true [18]:*

$$T_r = T + r\kappa\nu, \tag{10}$$

$$\nu_r = \frac{1}{\kappa_r}\left[(\kappa + r\kappa')\nu + r\kappa\tau\beta\right], \tag{11}$$

$$\beta_r = \frac{1}{\kappa_r}\left[-r\kappa\tau\nu + (\kappa + r\kappa')\beta\right], \tag{12}$$

$$\kappa_r = \sqrt{\kappa^2 + 2r\kappa\kappa' + r^2(\kappa'^2 + \kappa^2\tau^2)} \tag{13}$$

and

$$\tau_r = \frac{\kappa_r^2\tau + r\kappa^2\tau' + r^2\left(\kappa'^2\tau + \kappa\kappa'\tau' - \kappa\kappa''\tau\right)}{\kappa_r^2}. \tag{14}$$

Case 2: d is isotropic. Hence, we have $d_2^2 + d_3^2 = 1$ and $d = d_2\nu + d_3\beta$. In this case,

$$\gamma_r = \gamma(s) + r_2\nu + r_3\beta, \tag{15}$$

where $r_2 = rd_2$ and $r_3 = rd_3$ [18].

Theorem 2.9. *If $\gamma(s)$ is a curve with the arc length parameter s, then the arc length parameter of the curve γ_r is also s [18].*

Theorem 2.10. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$. If the Frenet vectors of γ and γ_r are $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$, the curvatures are κ, τ and κ_r, τ_r respectively, then the subsequent relations are true [18]:*

$$T_r = T - r_3\tau\nu + r_2\tau\beta, \tag{16}$$

$$\nu_r = \frac{1}{\kappa_r}\left[\left(\kappa - r_2\tau^2 - r_3\tau'\right)\nu + \left(r_2\tau' - r_3\tau^2\right)\beta\right], \tag{17}$$

$$\beta_r = \frac{1}{\kappa_r}\left[\left(-r_2\tau' + r_3\tau^2\right)\nu + \left(\kappa - r_2\tau^2 - r_3\tau'\right)\beta\right], \tag{18}$$

$$\kappa_r = \sqrt{\kappa^2 + \left(r_2^2 + r_3^2\right)(\tau^4 + \tau'^2) - 2\kappa\left(r_2\tau^2 + r_3\tau'\right)}, \tag{19}$$

and

$$\tau_r = \frac{\kappa_r^2\tau + \left(r_2^2 + r_3^2\right)\left(2\tau\tau'^2 - \tau^2\tau''\right) + r_2\left(\kappa\tau'' - \kappa'\tau'\right) + r_3\left(\kappa'\tau^2 - 2\kappa\tau\tau'\right)}{\kappa_r^2}. \tag{20}$$

3. The Relationships Between Parallel Like Curves and Some Special Curves

In this chapter, some relations will be obtained between special curves and parallel like curves in Galilean 3- space by helping with Frenet-Serret frame apparatus. Since the parallel like curve varies according to the vector d (isotropic or non-isotropic), it will be examined under two headings.

3.1. The Relationships Between Some Special Curves and Parallel Like Curves generated by the Non-isotropic Vector-d

In this title, it will be investigated whether a regular curve and its parallel like curve depending on the case where the vector d is non-isotropic are a Bertrand, Mannheim and involute-evolute curves pair or not.

Theorem 3.1. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$ and $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ be Serret-Frenet vector fields, respectively, where $\gamma_r(s) = \gamma(s) + rT(s)$. Then, the pair (γ, γ_r) is not Bertrand curve pair.*

Proof. From Definition 2.3, it is known that if $(\gamma(s), \gamma_r(s))$ is Bertrand curve pair, ν and ν_r are linearly dependent. Conversely, in light of Theorem 2.8 and equation (11),

$$\nu \times_G \nu_r \neq 0.$$

If we actually replace ν_r and calculate, we have

$$\nu \times_G \nu_r = \frac{r\kappa\tau}{\kappa_r}T.$$

This also shows that (γ, γ_r) is not Bertrand curve pair. Hence, the proof is completed. □

Corollary 3.2. $(\gamma(s), \gamma_r(s))$ *is Bertrand curves pair if and only if $\tau = 0$, where τ is torsion of $\gamma(s)$).*

Theorem 3.3. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$ and $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ be Serret-Frenet vector fields, respectively, where $\gamma_r(s) = \gamma(s) + rT(s)$. Then, the pair (γ, γ_r) is not involut-evolut curve pair.*

Proof. From Definition 2.4, we know that if $(\gamma(s), \gamma_r(s))$ is involut-evolut curve pair, T and T_r are perpendicular to each other in the Galilean viewpoint. Then, in light of Theorem 2.8 and equation (10), we get

$$\langle T, T_r\rangle_G = \langle T, T + r\kappa\nu\rangle_G \neq 0.$$

In that case, $\langle T, T_r\rangle \neq 0$ and $T \not\perp T_r$. This means that (γ, γ_r) involut-evolut curve pair cannot exist under any circumstances.

This completes the proof. □

Theorem 3.4. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$ and $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ be Serret-Frenet vector fields, respectively, where $\gamma_r(s) = \gamma(s) + rT(s)$. Then, the pair (γ, γ_r) is the Mannheim curve pair if and only if*

$$\kappa(s) = c_1 e^{-\frac{s}{r}},$$

where $\kappa(s)$ is the curvature of $\gamma(s)$ and $c_1 \in \mathbb{R}$.

Proof. Let $(\gamma(s), \gamma_r(s))$ be the Mannheim curve pair. From Definition 2.5, it is known that if $(\gamma(s), \gamma_r(s))$ is the Mannheim curve pair, ν and β_r are linearly dependent. We considering Theorem 2.8 and equation (12), we get

$$\nu \times_G \beta_r = (\kappa + r\kappa')T. \tag{21}$$

In that case, $\kappa + r\kappa' = 0$. If we solve the differential equation in equation (21), we obtain

$$\kappa(s) = c_1 e^{-\frac{s}{r}}.$$

Conversely, let us $\kappa(s) = c_1 e^{-\frac{s}{r}}$. In that case, it is obvious that $\kappa + r\kappa' = 0$ and vectors ν and β_r are linearly dependent. Therefore, $(\gamma(s), \gamma_r(s))$ is the Mannheim curve pair.

This concludes the proof.

□

For $r = 1$ and $c_1 = 1$, $\kappa(s) = e^{-s}$. Furthermore, setting $\tau(s) = s$, the Mannheim curve, as delineated using Mathematica software, is depicted in Figure 1.

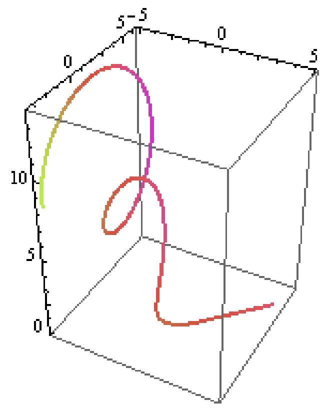

Figure 1: Mannheim curve with $\kappa(s) = e^{-s}$ and $\tau(s) = s$ in $\mathbb{G}_3$

3.2. The Relationships Between Some Special Curves and Parallel Like Curves generated by the Isotropic Vector-d

In this title, we examine whether a regular curve and its corresponding parallel-like curve, under the condition that the vector d is isotropic, constitute pairs of Bertrand, involute-evolute, and Mannheim curves.

Theorem 3.5. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$ and $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ be Serret-Frenet vector fields, respectively, where $\gamma_r(s) = \gamma(s) + r_2\nu(s) + r_3\beta(s)$. Then, the pair (γ, γ_r) is the Bertrand curve pair if and only if*

$$\tau(s) = \frac{r_2}{c_1 r_2 - r_3 s}$$

where $c_1 \in \mathbb{R}$, $s \neq \frac{c_1 r_2}{r_3}$ and $\tau(s)$ is torsion of $\gamma(s)$.

Proof. Let $(\gamma(s), \gamma_r(s))$ be the Bertrand curve pair. It is obvious from the Definition 2.3 that ν and ν_r are linearly dependent if $(\gamma(s), \gamma_r(s))$ is a pair of Bertrand curves. We considering Theorem 2.10 and equation (17), we get

$$\nu \times_G \nu_r = \frac{(r_2\tau' - r_3\tau^2)}{\kappa_r} T \tag{22}$$

If equation (22) equals to zero, then ν and ν_r are linearly dependent. In that case, $r_2\tau' - r_3\tau^2 = 0$. If differential equation in this last equation is solved, for $c_1 \in \mathbb{R}$

$$\tau(s) = \frac{r_2}{c_1 r_2 - r_3 s},$$

where $s \neq \frac{c_1 r_2}{r_3}$.

Conversely, let us $\tau(s) = \frac{r_2}{c_1 r_2 - r_3 s}$, for $c_1 \in \mathbb{R}$ and $s \neq \frac{c_1 r_2}{r_3}$. In that case, it is obvious that ν and ν_r are linearly dependent. This means that $(\gamma(s), \gamma_r(s))$ is Bertrand curve pair.

Hence, the proof is completed. □

For $r_2 = \frac{1}{2}$, $r_3 = \frac{\sqrt{3}}{2}$ and $c_1 = 1$, $\tau(s) = \frac{1}{1-\sqrt{3}s}$, $s \neq \frac{1}{\sqrt{3}}$. Additionally, by specifying $\kappa(s) = 1$, the Bertrand curve, as rendered with the Mathematica software, is illustrated in Figure 2.

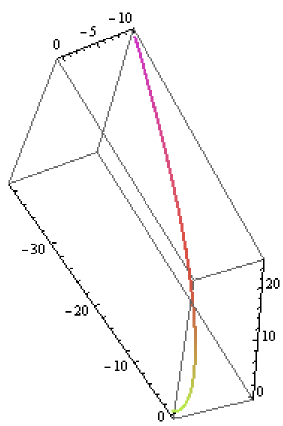

Figure 2: Bertrand curve with $\kappa(s) = 1$ and $\tau(s) = \frac{1}{1-\sqrt{3}s}$ in $\mathbb{G}_3$

Theorem 3.6. *Let $(\gamma(s), \gamma_r(s))$ represent the pair of curves parameterized by arc length s in $\mathbb{G}_3$, with $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ denoting the respective Serret-Frenet vector fields. Consequently, the pair (γ, γ_r) does not constitute an involute-evolute curve pair such that $\gamma_r(s) = \gamma(s) + r_2\nu(s) + r_3\beta(s)$.*

Proof. From Definition 2.4, we know that if $(\gamma(s), \gamma_r(s))$ is involut-evolut curve pair, T and T_r are perpendicular to each other in the Galilean viewpoint. Then, in light of Theorem 2.10 and equation (16), we get

$$\langle T,\, T - r_3\tau\nu + r_2\tau\beta\rangle_G \neq 0$$

In that case, $\langle T, T_r\rangle \neq 0$ and $T \not\perp T_r$. Therefore, it can be conclusively determined that the pair of involut-evolut curves (γ, γ_r) is not feasible under any conceivable conditions.

This completes the proof. □

Theorem 3.7. *Let $(\gamma(s), \gamma_r(s))$ be the pair of curves given with the arc length s in $\mathbb{G}_3$ and $\{T, \nu, \beta\}$ and $\{T_r, \nu_r, \beta_r\}$ be Serret-Frenet vector fields, respectively, where $\gamma_r(s) = \gamma(s) + r_2\nu(s) + r_3\beta(s)$. Then, the pair (γ, γ_r) is the Mannheim curve pair if and only if*

$$\kappa = r_2\tau^2 + r_3\tau',$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of $\gamma(s)$ respectively.

Proof. Let $(\gamma(s), \gamma_r(s))$ be the Mannheim curve pair. From Definition 2.5, it is known that if $(\gamma(s), \gamma_r(s))$ is Mannheim curve pair, ν and β_r are linearly dependent. We considering Theorem 2.10 and equation (18), we get

$$\nu \times_G \beta_r = (\kappa - r_2\tau^2 - r_3\tau')T. \tag{23}$$

Since ν and β_r are linearly dependent, equation (23) equals to zero. In that case, $\kappa - r_2\tau^2 - r_3\tau' = 0$. Then, we have $\kappa = r_2\tau^2 + r_3\tau'$.

Conversely, let us $\kappa = r_2\tau^2 + r_3\tau'$. In that case, it is obvious that the vectors ν and β_r are linearly dependent. That is, $(\gamma(s), \gamma_r(s))$ is a Mannheim curve pair.

Hence, the proof is finished. □

For $r_2 = \frac{1}{2}$, $r_3 = \frac{\sqrt{3}}{2}$ and $\tau(s) = s$, $\kappa(s) = \frac{\sqrt{3}}{2} + \frac{1}{2}s^2$. In view of these values, the Mannheim curve can be drawn as in Figure 3.

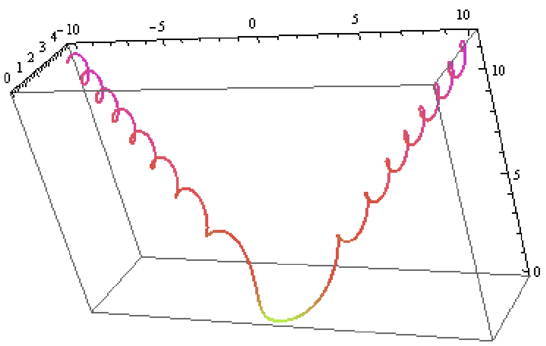

Figure 3: Mannheim curve with $\kappa(s) = \frac{\sqrt{3}}{2} + \frac{1}{2}s^2$ and $\tau(s) = s$ in $\mathbb{G}_3$

4. Conclusion

In this manuscript, the focus was on parallel-like curve pairs in 3-dimensional Galilean space. By definition, parallel-like curves produced in isotropic and non-isotropic vector directions were examined under two headings. Within these classifications, the conditions under which parallel-like curve pairs exhibit Bertrand, involute-evolute, and Mannheim relationships were rigorously examined. The results of this investigation are enumerated as follows.

1. Parallel-like curve pairs generated in both isotropic and non-isotropic vector directions do not form involute-evolute pairs under any conditions.

2. Parallel-like curve pairs produced by non-isotropic vector formed Bertrand curve pairs with the condition $\tau = 0$. On the other hand, parallel-like curve pairs produced by isotropic vectors formed Bertrand curve pairs under the condition $\tau(s) = \frac{r_2}{c_1 r_2 - r_3 s}$.

3. Parallel-like curve pairs produced by non-isotropic vectors became Mannheim curve pairs with condition $\kappa(s) = c_1 e^{-\frac{s}{r}}$, while parallel-like curve pairs produced by isotropic vectors formed Mannheim curve pairs under condition $\kappa = r_2\tau^2 + r_3\tau'$.

References

[1] R. S. Millman, G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.
[2] A. Çakmak, S. Kızıltuğ, *Spherical indicatrices of a Bertrand curve in three-dimensional Lie groups*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68**, 2019, 1930–1938.
[3] L. Hiulu, W. Fan, *Mannheim Partner Curves in 3-space*, Journal of Geometry **88**, 2008, 120–126.
[4] E. Kreyszig, *Differential Geometry*, Dover Publications, New York, 1991.
[5] Ş. Kılıçoğlu, S. Şenyurt, *The Casual Characteristics of Offset Curves of a Non-Null Curve and Fibonacci Sequence*, Turkish Journal of Science, **8(3)**, 2023, 136-146.
[6] I. Yaglom, *A Simple Non-Euclidean Geometry and Physical Basis*, Springer, 1979.
[7] O. M. Röschel, *Die Geometrie des Galileischen raumes (Ph.D. Thesis)*, Habilitationsschrift, Leoben, 1984.
[8] B. Divjak, Z. M. Sipus, *Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces*, Acta Mathematica Hungarica, **98**, 2003, 203–215.
[9] A. O. Öğrenmiş, H. Öztekin, M. Ergüt, *Bertrand curves in Galilean space and their characterizations*, Kragujevac Journal of Mathematics, **32**, 2009, 139–147.
[10] Z. M. Sipus, B. Divjak, *Translation surfaces in the Galilean space*, Glasnik Matematicki, **46**, 2011, 455–469.
[11] M. Dede, C. Ekici, A. C. Çöken, *On the parallel surfaces in Galilean space*, Hacettepe Journal of Mathematics and Statistics, **42**, 2013, 605–615.
[12] S. Büyükkütük, G. Öztürk, *Non-null Translation-Homothetical surfaces in four-dimensional Minkowski space*, Turkish Journal of Science, **8(1)**, 2023, 1-10.
[13] A. Çakmak, M. K. Karacan, S. Kızıltuğ, D. W. Yoon, *Translation Surfaces in the 3-Dimensional Galilean Space Satisfying* $\Delta^{II} x_i = \lambda_i x_i$, Bulletin of the Korean Mathematical Society **54(4)**, 2017, 1241-1254
[14] A. Çakmak, *New type direction curves in 3-dimensional compact Lie group*, Symmetry, **11(3)**, 2019, 387.
[15] S. Kızıltuğ, A. Çakmak, *Developable ruled surfaces with Darboux frame in Minkowski 3-space*, Life Science Journal, **10(4)**, 2013, 1906-1914
[16] H. Vogler, *Über die pseudorektifizierenden torsen von raumkurven*, Springer in Komm., **171**, 1963, 173–178.
[17] H. H. Hacısalihoğlu, *On the motion of the Frenet thrihedron of a space curve*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **17**, 1968, 33–55.
[18] A. Çakmak, S. Kızıltuğ, G. Mumcu, *Notes on Curves at a Constant Distance from the Edge of Regression on a Curve in Galilean 3-Space*, CMES-Computer Modeling in Engineering & Sciences, **135**, 2023, 2731–2742.
[19] W. Rudin, *Real and complex analysis*, (3rd edition), McGraw-Hill, New York, 1986.
[20] Z. M. Sipus, *Ruled Weingarten surfaces in Galilean space*, Periodica Mathematica Hungarica, **56(2)**, 2008, 213–225.
[21] B. Divjak, *Curves in pseudo-Galilean geometry*, Annales Univ. Sci. Budapest, **41**, 1998, 117–128.
[22] B. J. Pavkovic, I. Kamenarovic, *The equiform differential geometry of curves in the Galilean space* G_3, Glasnik Matematicki, **22(42)**, 1987, 449–457.
[23] I. Kamenarovic, *Existence theorems for ruled surfaces in the Galilean space* G_3, Rad Hrvatske Akad. Znan. Umj. Math., **456(10)**, 1991, 183–196.
[24] A. O. Öğrenmis, M. Ergüt, M. Bektaş, *On The Helices in the Galilean Space* G_3, Iranian Journal of Science and Technology, Transaction A, **31(2)**, 2007, 177-181.
[25] M. Akyigit, A. Z. Azak, S. Ersoy, *Involute-evolute curves in Galilean space* G_3, Scientia Magna, **6(4)**, 2010, 75–80.
[26] A. O. Öğrenmiş, H. Öztekin, M. Ergüt, *Some Properties of Mannheim Curves in Galilean and Pseudo-Galilean Spaces*, Siauliai Mathematical Seminar, **7(15)**, 2012, 115-125.