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On cycle decompositions of complete 3-uniform hypergraphs

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Abstract. The complete 3-uniform hypergraph $K_n^{(3)}$ of order n has a set V of cardinality n as its vertex set and the set of all 3 element subsets of V as its edge set. For $n \geq 2$, let $\mathbb{Z}_n$ denote the set of integers modulo n. For $m > 3$, let $LC_m^{(3)}$ (respectively, $TC_m^{(3)}$) denote the 3-uniform hypergraph with vertex set $\mathbb{Z}_{2m}$ (respectively, $\mathbb{Z}_m$) and edge set $\{\{2i, 2i+1, 2i+2\} : i \in \{0, 1, 2, \dots, m-1\}\}$ (respectively, $\{\{i, i+1, i+2\} : i \in \mathbb{Z}_m\}$). Any hypergraph isomorphic to $LC_m^{(3)}$ (respectively, $TC_m^{(3)}$) is a 3-*uniform loose m-cycle* (respectively, 3-*uniform tight m-cycle*). A decomposition of $K_n^{(3)}$ is a partition of the edge set of $K_n^{(3)}$. We show that there exists a decomposition of $K_n^{(3)}$ into subhypergraphs isomorphic to $LC_7^{(3)}$ if and only if $n \geq 14$ and $n \equiv 0, 1$ or 2 (mod 7). Next, we show that, for $\ell \geq 1$ and $m \in \{8, 16, 20, 28, 32, 40, 44\}$, there exists a decomposition of $K_{2^\ell m}^{(3)}$ into subhypergraphs isomorphic to $TC_m^{(3)}$.

1. Introduction

A hypergraph F consists of a finite nonempty set V of *vertices* and a set E of nonempty subsets of V called *hyperedges* or simply *edges*.

A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \dots, H_b\}$ of subhypergraphs of K such that $E(H_1) \cup E(H_2) \cup \dots \cup E(H_b) = E(K)$ and $E(H_i) \cap E(H_j) = \emptyset$ for all i and j with $1 \leq i < j \leq b$. We denote this fact by $K = H_1 \oplus H_2 \oplus \cdots \oplus H_b$. It follows from the definition that

$$|E(H_1)| + |E(H_2)| + \cdots + |E(H_b)| = |E(K)|.$$

If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an H-*block*, and Δ is called an H-*decomposition* of K. In this case, we say that H *decomposes* K, and we write $H \,|\, K$. Also, in this case, we have

$$b|E(H)| = |E(K)|.$$

Hence, a necessary condition for the existence of an H-decomposition of K is that

$$|E(H)| \text{ divides } |E(K)|.$$

The *degree* of a vertex x in a hypergraph F is the number of edges of F containing x.

Another necessary condition for the existence of an H-decomposition of K is that

the g.c.d. of the degrees of vertices in H divides the g.c.d. of the degrees of vertices in K.

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If each vertex x in a hypergraph F has the same degree, then we say that the hypergraph F is regular, or F is *k-regular* if the degree of x is k.

If for each edge e in a hypergraph F, we have $|e| = t$, then F is said to be *t-uniform*. Thus simple graphs are 2-uniform hypergraphs.

A *cycle* of length m, in a hypergraph F is a sequence of the form $v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1$ where $v_1, v_2, \ldots, v_m$ are distinct vertices and $e_1, e_2, \ldots, e_m$ are distinct edges satisfying $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, \ldots, m-1\}$ and $v_m, v_1 \in e_m$. This cycle is known as a *Berge cycle* having been introduced by Berge in [5]. For $i \in \{1, 2, \ldots, m\}$, if $|e_i| = t$, then we denote this Berge cycle by $BC_m^{(t)}$.

For $n \geq 2$, let $\mathbb{Z}_n$ denote the set of integers modulo n.

For $m > t \geq 2$, let $LC_m^{(t)}$ denote the t-uniform hypergraph with vertex set $\mathbb{Z}_{(t-1)m}$ and edge set $\{\{it-i, it-i+1, it-i+2, \ldots, it-i+(t-1)\} : i \in \{0, 1, \ldots, m-1\}\}$. Any hypergraph isomorphic to $LC_m^{(t)}$ is a *t-uniform loose m-cycle*. In particular, for $t = 3$, a 3-uniform loose m-cycle $LC_m^{(3)}$ is a 3-uniform hypergraph with vertex set $\mathbb{Z}_{2m}$ and edge set $\{\{2i, 2i+1, 2i+2\} : i \in \{0, 1, \ldots, m-1\}\}$.

For $m > t \geq 2$, let $TC_m^{(t)}$ denote the t-uniform hypergraph with vertex set $\mathbb{Z}_m$ and edge set $\{\{i, i+1, i+2, \ldots, i+t-1\} : i \in \mathbb{Z}_m\}$. Any hypergraph isomorphic to $TC_m^{(t)}$ is a *t-uniform tight m-cycle*. In particular, for $t = 3$, a 3-uniform tight m-cycle $TC_m^{(3)}$ is a 3-uniform hypergraph with vertex set $\mathbb{Z}_m$ and edge set $\{\{i, i+1, i+2\} : i \in \mathbb{Z}_m\}$.

Let F be a t-uniform hypergraph. It follows from the definitions that every loose cycle of F is a Berge cycle of F and every tight cycle of F is a Berge cycle of F. Observe that, for $t = 2$, $BC_m^{(2)} \cong LC_m^{(2)} \cong TC_m^{(2)}$.

Let K be a t-uniform hypergraph, $t \geq 3$. The necessary conditions for the existence of:

$BC_m^{(t)}$-decomposition of K are $|V(K)| \geq m$ and m divides $|E(K)|$;

$LC_m^{(t)}$-decomposition of K are $|V(K)| \geq (t-1)m$ and m divides $|E(K)|$;

$TC_m^{(t)}$-decomposition of K are $|V(K)| \geq m$, m divides $|E(K)|$ and t divides the degree of each vertex of K.

As both loose cycle of K and tight cycle of K are Berge cycles of K, we have: every $LC_m^{(t)}$-decomposition of K is a $BC_m^{(t)}$-decomposition of K and every $TC_m^{(t)}$-decomposition of K is a $BC_m^{(t)}$-decomposition of K.

A t-uniform hypergraph $F = (V, E)$ is said to be *complete* if every t-element subset of V is in E. We denote such a hypergraph by $K_V^{(t)}$ or by $K_n^{(t)}$ if $|V| = n$. $K_n^{(t)}$ is $\binom{n-1}{t-1}$-regular and it has $\binom{n}{t}$ edges. An H-decomposition of $K_n^{(t)}$ is also known as an *H-design* of order n. Given a t-uniform hypergraph H, the problem of determining all values of n for which there exists an H-design of order n is known as the *spectrum problem* for H.

If $K = K_n^{(t)}$, then the above necessary conditions for the existence of:

$BC_m^{(t)}$-decomposition of $K_n^{(t)}$ are $n \geq m$ and $m \mid \binom{n}{t}$;

$LC_m^{(t)}$-decomposition of $K_n^{(t)}$ are $n \geq (t-1)m$ and $m \mid \binom{n}{t}$;

$TC_m^{(t)}$-decomposition of $K_n^{(t)}$ are $n \geq m$, $m \mid \binom{n}{t}$ and $t \mid \binom{n-1}{t-1}$.

Assume $3 \leq t < n$. A $BC_n^{(t)}$ of $K_n^{(t)}$ is called a *Hamilton cycle* of $K_n^{(t)}$ and a $BC_n^{(t)}$-decomposition of $K_n^{(t)}$ is called a *Hamilton cycle decomposition* of $K_n^{(t)}$. Since a $TC_n^{(t)}$ of $K_n^{(t)}$ is a $BC_n^{(t)}$ of $K_n^{(t)}$, a $TC_n^{(t)}$-decomposition of $K_n^{(t)}$ is also a $BC_n^{(t)}$-decomposition of $K_n^{(t)}$, and so it is a special type of Hamilton cycle decomposition of $K_n^{(t)}$.

The necessary condition for the existence of $BC_n^{(t)} \mid K_n^{(t)}$ is $n \mid \binom{n}{t}$. In [4], Bermond et al. conjectured that this necessary condition is sufficient and proved this conjecture for n a prime. In [17], Kühn and Osthus, proved that for $t \geq 4$ and $n \geq 30$, if $n \mid \binom{n}{t}$, then $BC_n^{(t)} \mid K_n^{(t)}$. For $t = 3$, the necessary condition $n \mid \binom{n}{3}$ is: $n \equiv 1$, 2, 4 or 5 (mod 6); in [3], Bermond proved that: if $n \equiv 2$, 4 or 5 (mod 6), then $BC_n^{(3)} \mid K_n^{(3)}$, and in [25], Verrall proved that: if $n \equiv 1$ (mod 6), then $BC_n^{(3)} \mid K_n^{(3)}$.

Let $\mathscr{E}_n^{(t)}$ be the set of all t element subsets of $\mathbb{Z}_n$, where $1 < t < n$. If $E \in \mathscr{E}_n^{(t)}$ and $r \in \mathbb{Z}_n$, let $E + r$ be formed by replacing each element $x \in E$ with $x + r$; so $(r, E) \mapsto E + r$ maps $\mathbb{Z}_n \times \mathscr{E}_n^{(t)}$ into $\mathscr{E}_n^{(t)}$. It can be seen that the group $\mathbb{Z}_n$ acts on the set $\mathscr{E}_n^{(t)}$ partitioning it into $\mathbb{Z}_n$-orbits, where $E_1, E_2 \in \mathscr{E}_n^{(t)}$ are in the same orbit if and only if $E_1 + r = E_2$ for some $r \in \mathbb{Z}_n$. We define $[E]$ to be $\{E + r : r \in \mathbb{Z}_n\}$, which we refer to as the $\mathbb{Z}_n$*-orbit* of E. If $\mathscr{S} \subseteq \mathscr{E}_n^{(t)}$ and $r \in \mathbb{Z}_n$, let $\mathscr{S} + r = \{E + r : E \in \mathscr{S}\}$. By *clicking* $\mathscr{S}$, we shall mean replacing $\mathscr{S}$ with $\mathscr{S} + 1$.

Let H be a subhypergraph of $K_n^{(t)}$, where $V(K_n^{(t)}) = \mathbb{Z}_n$ and let Γ be a H-decomposition of $K_n^{(t)}$. Then Γ is said to be *cyclic* if Γ is closed under clicking. Thus if $H_i \in \Gamma$, then $H_i + 1 \in \Gamma$. If we partition $\mathscr{E}_n^{(t)}$ into k distinct $\mathbb{Z}_n$-orbits each of size n and if H is a subhypergraph of $K_n^{(t)}$ consisting of one edge from each k distinct $\mathbb{Z}_n$-orbits, then $\Gamma = \{H + i : i \in \mathbb{Z}_n\}$ is a *cyclic* H-decomposition of $K_n^{(t)}$.

Petecki [22], showed that $K_n^{(t)}$ admits a cyclic Hamilton cycle decomposition if and only if $g.c.d.(n, t) = 1$ and $\lambda = min\,\{d > 1 : d \,|\, n\} > \frac{n}{t}$.

The necessary condition for the existence of $TC_n^{(t)} \,|\, K_n^{(t)}$ is $n \,|\, \binom{n}{t}$ and $t \,|\, \binom{n-1}{t-1}$. The problem of determining the existence of a $TC_n^{(3)}$-decomposition of $K_n^{(3)}$ was first investigated by Bailey and Stevens in [2]; also proved that for $n \in \{7, 8, 9, 10, 11, 16\}$. Meszka and Rosa [21] obtained $TC_n^{(3)} \,|\, K_n^{(3)}$, for all admissible $n \leq 32$. Huo et al. [14], obtained $TC_n^{(3)} \,|\, K_n^{(3)}$, for all admissible $32 < n \leq 46$ and $n \neq 43$.

A 1-*factor* of a hypergraph F is a spanning subhypergraph I of F, in which each of the n vertices of F has degree 1 in I. We denote the complete t-uniform hypergraph on n vertices, less a 1-factor I, by $K_n^{(t)} - I$. $K_n^{(t)} - I$ is $\left(\binom{n-1}{t-1} - 1\right)$-regular and it has $\binom{n}{t} - \frac{n}{t}$ edges.

If $K = K_n^{(t)} - I$, then the necessary conditions for the existence of:

$BC_m^{(t)}$-decomposition of $K_n^{(t)} - I$ are $n \geq m$ and $m \,|\, \left(\binom{n}{t} - \frac{n}{t}\right)$;

$LC_m^{(t)}$-decomposition of $K_n^{(t)} - I$ are $n \geq (t-1)m$ and $m \,|\, \left(\binom{n}{t} - \frac{n}{t}\right)$;

$TC_m^{(t)}$-decomposition of $K_n^{(t)} - I$ are $n \geq m$, $m \,|\, \left(\binom{n}{t} - \frac{n}{t}\right)$ and $t \,|\, \left(\binom{n-1}{t-1} - 1\right)$.

Verrall [25] proved that the necessary condition for $BC_n^{(3)} \,|\, (K_n^{(3)} - I)$ is sufficient. (The necessary condition $n \,|\, \left(\binom{n}{3} - \frac{n}{3}\right)$ is $n \equiv 0$ or 3 (mod 6).)

Keszler et al. [16] showed that $TC_6^{(3)} \,|\, (K_n^{(3)} - I)$ if and only if $n \equiv 0$, 3 or 6 (mod 12); also proved that $TC_9^{(3)} \,|\, (K_n^{(3)} - I)$ if and only if n is a multiple of 3.

Jordon et al. [15] proved that the necessary conditions are sufficient for the existence of a $BC_4^{(3)}$-decomposition of $K_n^{(3)}$. In [18, 19], Lakshmi and Poovaragavan proved that the necessary conditions are sufficient for the existence of a $BC_6^{(3)}$-decomposition of $K_n^{(3)}$ and for the existence of a $BC_p^{(3)}$-decomposition of $K_n^{(3)}$, for $p \geq 5$ is prime.

In [6], Bryant et al. proved that there exists an $LC_3^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 0$, 1 or 2 (mod 9). Bunge et al. [10] shown that there exists an $LC_4^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 0$, 1, 2, 4 or 6 (mod 8) and $n \notin \{4, 6\}$. In [9], Bunge et al. shown that there exists a $LC_3^{(4)}$-decomposition of $K_n^{(4)}$ if and only if $n \equiv 1$, 2, 3 or 6 (mod 9) and $n \geq 9$.

Meszka and Rosa [21] introduced the idea of $TC_m^{(3)}$-decompositions of $K_n^{(3)}$ for $m \neq n$; also obtained a $TC_5^{(3)} \,|\, K_n^{(3)}$, for all admissible $n \leq 17$, and for all $n = 4^m + 1$, m a positive integer. It is noted in [21] that as a consequence of Hanani's classical result on the existence of Steiner quadruple systems [13], there exists a $TC_4^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 2$ or 4 (mod 6). In [1], Akin et al. shown that there exists a $TC_6^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 1$, 2, 10, 20, 28 or 29 (mod 36). Bunge et al. [8] proved that there exists a $TC_9^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 1$ or 2 (mod 27). For $t \in \{5, 7\}$, $TC_t^{(3)}$-decomposition of $K_n^{(3)}$ is studied in [12, 20]. For $t \in \{5, 7\}$, the problem of finding a $TC_t^{(3)}$-decomposition of $K_n^{(3)}$ is still open.

A hypergraph F is *simple* if no edge appears more than once in $E(F)$. If F is a simple hypergraph and if λ is a positive integer, then the λ-*fold* of F, denoted $\lambda\, F$, is the multi-hypergraph obtained from F by repeating each edge exactly λ times.

If L is a subhypergraph of M with edge set $E(L)$ and Δ is a H-decomposition of $M \setminus E(L)$, then Δ is called a H-*packing* of M with leave L. Such a H-packing is *maximum* if no other possible H-packing of M has a leave of a smaller size than that of L. Clearly, if $|E(L)| < |E(H)|$, then the H-packing is maximum. Moreover, a H-decomposition of M can be viewed as a maximum H-packing with an empty leave.

In [7], Bunge et al. studied maximum $LC_3^{(3)}$-packings of $\lambda K_n^{(3)}$ and showed that if λ and $n \geq 6$ are positive integers, then there exists a maximum $LC_3^{(3)}$-packing of $\lambda K_n^{(3)}$ where the leave has two or fewer edges. In [11], Bunge et al. studied $LC_5^{(3)}$ decompositions, pacings and coverings of $\lambda K_n^{(3)}$.

In this paper, we prove the following results:

Theorem 1.1. *Let $m \geq 3$ be an odd integer and $n \geq 2m$ be an integer with $n \equiv 0, 1$ or $2 \pmod{m}$. If $LC_m^{(3)}|K_{2m}^{(3)}$, $LC_m^{(3)}|K_{2m+1}^{(3)}$, $LC_m^{(3)}|K_{2m+2}^{(3)}$, $LC_m^{(3)}|K_{3m}^{(3)}$, $LC_m^{(3)}|K_{3m+1}^{(3)}$ and $LC_m^{(3)}|K_{3m+2}^{(3)}$, then $LC_m^{(3)}|K_n^{(3)}$.*

Theorem 1.2. *$LC_7^{(3)}|K_n^{(3)}$ if and only if $n \geq 14$ and $n \equiv 0, 1$ or $2 \pmod{7}$.*

Theorem 1.3. *If $\ell \geq 1$, and $m \in \{8, 16, 20, 28, 32, 40, 44\}$, then $TC_m^{(3)} \,|\, K_{2^\ell m}^{(3)}$.*

2. Preliminaries

In what follows, $\mathbb{N}$ denote the set of positive integers.

2.1. Hypergraphs

For disjoint sets X and Y, the hypergraph with vertex set $X \cup Y$ and edge set consisting of all 3-sets having at most 2 vertices in each of X and Y is denoted either by $K_{X,Y}^{(3)}$ or by $K_{|X|,|Y|}^{(3)}$. We partition the edge set of $K_{X,Y}^{(3)}$ into two sets one consisting of all 3-sets having exactly 2 vertices in X and the other consisting of all 3-sets having exactly 2 vertices in Y. We denote the subhypergraph induced by the former edge set by $K_{X,\overline{Y}}^{(3)}$ or by $K_{|X|,\overline{|Y|}}^{(3)}$ and the latter by $K_{\overline{X},Y}^{(3)}$ or $K_{\overline{|X|},|Y|}^{(3)}$. Clearly, $K_{X,Y}^{(3)} = K_{X,\overline{Y}}^{(3)} \oplus K_{\overline{X},Y}^{(3)}$.

For pairwise disjoint sets X, Y and Z, the hypergraph with vertex set $X \cup Y \cup Z$ and edge set consisting of all 3-sets having exactly one vertex in each of X, Y and Z is denoted by $K_{X,Y,Z}^{(3)}$ or $K_{|X|,|Y|,|Z|}^{(3)}$.

2.2. Graphs

Graphs K_n, C_n, P_n and $K_{m,n}$, respectively, denote the complete graph with n vertices, the cycle with n ($n \geq 3$) vertices, the path with n vertices and the complete bipartite graph with partite sizes m and n.

We need the following:

Theorem 2.1. ([24]). *Let $m \geq n$ and let one of the following conditions hold:*
(1) m is even, n is odd and k divides $2n$,
(2) m is odd, n is even and k divides n,
(3) m is odd, n is even, $k < 2n$ and k divides m,
(4) $m = n$ or $m = n + 1$ and k divides m,
(5) m and n are odd, $m \geq (3n + 1)/2$ and k divides n.
Then $K_{m,n}$ has a decomposition into paths of length k.

3. Loose odd cycle decompositions

3.1. Decompositions of $K_{m,\overline{n}}^{(3)}$

Theorem 3.1. *Let $q, s, m \in \mathbb{N}$ and $m \geq 3$ be odd. If $C_q \,|\, K_m$ and $s \geq q$, then $LC_q^{(3)} \,|\, K_{m,\overline{s}}^{(3)}$.*

Proof. Let $K_{m,\overline{s}}^{(3)} = K_{X,\overline{Y}}^{(3)}$, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_0, y_1, \ldots, y_{s-1}\}$. Here, the subscript of y is expressed modulo s. By hypothesis, $C_q \,|\, K_m$. Let $\mathscr{C}$ be a collection of q-cycles in a C_q-decomposition of K_m. For each q-cycle C in $\mathscr{C}$, we produce s loose q-cycles in $K_{X,\overline{Y}}^{(3)}$ as follows: Suppose $C := x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_q} x_{i_1}$, then the loose q-cycles are:

$$((x_{i_1}, y_j, x_{i_2}), (x_{i_2}, y_{j+1}, x_{i_3}), (x_{i_3}, y_{j+2}, x_{i_4}), \ldots, (x_{i_{q-1}}, y_{j+q-2}, x_{i_q}), (x_{i_q}, y_{j+q-1}, x_{i_1})),$$

where $j \in \mathbb{Z}_s$. The collection of the resulting $\frac{m(m-1)}{2q} s$ loose q-cycles yield the required decomposition. □

Corollary 3.2. *Let $s, m \in \mathbb{N}$ and $m \geq 3$ be odd. If $s \geq m$, then $LC_m^{(3)} \,|\, K_{m,\bar{s}}^{(3)}$.*

Proof. Follows from Theorem 3.1 (take $q = m$) since $C_m \,|\, K_m$. □

Theorem 3.3. *Let $q, s, m \in \mathbb{N}$ and $m \geq 3$. If $P_{q+1} \,|\, K_{m+1}$, $q \geq 3$ and $s \geq q - 1$, then $LC_q^{(3)} \,|\, K_{m+1,\bar{s}}^{(3)}$.*

Proof. Let $K_{m+1,\bar{s}}^{(3)} = K_{X,\overline{Y}}^{(3)}$, where $X = \{x_1, x_2, \dots, x_{m+1}\}$ and $Y = \{y_0, y_1, \dots, y_{s-1}\}$. Here, the subscript of y is expressed modulo s. By hypothesis, $P_{q+1} \,|\, K_{m+1}$. Let $\mathscr{P}$ be a collection of paths of order $q + 1$ in a P_{q+1}-decomposition of K_{m+1}. For each path P in $\mathscr{P}$, we produce s loose q-cycles in $K_{X,\overline{Y}}^{(3)}$ as follows: Suppose $P := x_{i_1} x_{i_2} x_{i_3} \dots x_{i_{q+1}}$, then the loose q-cycles are:

$$((y_j, x_{i_1}, x_{i_2}), (x_{i_2}, y_{j+1}, x_{i_3}), (x_{i_3}, y_{j+2}, x_{i_4}), \dots, (x_{i_{q-1}}, y_{j+q-2}, x_{i_q}), (x_{i_q}, x_{i_{q+1}}, y_j)),$$

where $j \in \mathbb{Z}_s$. The collection of the resulting $\frac{(m+1)m}{2q}\, s$ loose q-cycles yield the required decomposition. □

Corollary 3.4. *Let $s, m \in \mathbb{N}$ and $m \geq 3$ be odd. If $s \geq m - 1$, then $LC_m^{(3)} \,|\, K_{m+1,\bar{s}}^{(3)}$.*

Proof. Follows from Theorem 3.3 (take $q = m$) since $P_{m+1} \,|\, K_{m+1}$. □

3.2. *Decompositions of $K_{m,n}^{(3)}$*

Replacing s by m in Corollary 3.2, we have: if $m \geq 3$ is an odd integer, then $LC_m^{(3)} \,|\, K_{m,\overline{m}}^{(3)}$. This together with $K_{m,m}^{(3)} = K_{m,\overline{m}}^{(3)} \oplus K_{\overline{m},m}^{(3)}$ and $K_{m,\overline{m}}^{(3)} \cong K_{\overline{m},m}^{(3)}$ imply the following:

Corollary 3.5. *Let $m \geq 3$ be an odd integer. We have $LC_m^{(3)} | K_{m,m}^{(3)}$.*

Replacing s by $m + 1$ in Corollary 3.2 and s by m in Corollary 3.4, we have, respectively: if $m \geq 3$ is an odd integer, then $LC_m^{(3)} \,|\, K_{m,\overline{m+1}}^{(3)}$ and $LC_m^{(3)} \,|\, K_{m+1,\overline{m}}^{(3)}$. This together with $K_{m,m+1}^{(3)} = K_{m,\overline{m+1}}^{(3)} \oplus K_{\overline{m},m+1}^{(3)}$ and $K_{m+1,\overline{m}}^{(3)} \cong K_{\overline{m},m+1}^{(3)}$ imply the following:

Corollary 3.6. *Let $m \geq 3$ be an odd integer. We have $LC_m^{(3)} \,|\, K_{m,m+1}^{(3)}$.*

3.3. *Decompositions of $K_{m,n,s}^{(3)}$*

Theorem 3.7. *Let $m, n, r, s \in \mathbb{N}$. If $P_{m+1} \,|\, K_{r,s}$, $m \geq 3$ and $n \geq m - 1$, then $LC_m^{(3)} \,|\, K_{r,s,n}^{(3)}$.*

Proof. Let $K_{r,s,n}^{(3)} = K_{X,Y,Z}^{(3)}$, where $X = \{x_1, x_2, \dots, x_r\}$, $Y = \{y_1, y_2, \dots, y_s\}$ and $Z = \{z_0, z_1, \dots, z_{n-1}\}$. Here, the subscript of z is expressed modulo n. By hypothesis, $P_{m+1} \,|\, K_{r,s}$. Let $\mathscr{P}$ be a collection of paths of order $m + 1$ in a P_{m+1}-decomposition of $K_{r,s}$. For each path P in $\mathscr{P}$, we produce n loose m-cycles in $K_{X,Y,Z}^{(3)}$ as follows:
Case 1. m is odd.

Then, $P := x_{i_1} y_{i_2} x_{i_3} y_{i_4} \dots x_{i_m} y_{i_{m+1}}$. The loose m-cycles are:

$$((z_j, x_{i_1}, y_{i_2}), (y_{i_2}, z_{j+1}, x_{i_3}), (x_{i_3}, z_{j+2}, y_{i_4}), \dots, (y_{i_{m-1}}, z_{j+m-2}, x_{i_m}), (x_{i_m}, y_{i_{m+1}}, z_j)),$$

where $j \in \mathbb{Z}_n$.
Case 2. m is even.

If $P := x_{i_1} y_{i_2} x_{i_3} y_{i_4} \dots y_{i_m} x_{i_{m+1}}$, then the loose m-cycles are:

$$((z_j, x_{i_1}, y_{i_2}), (y_{i_2}, z_{j+1}, x_{i_3}), (x_{i_3}, z_{j+2}, y_{i_4}), \dots, (x_{i_{m-1}}, z_{j+m-2}, y_{i_m}), (y_{i_m}, x_{i_{m+1}}, z_j)),$$

where $j \in \mathbb{Z}_n$.

Otherwise, $P := y_{i_1} x_{i_2} y_{i_3} x_{i_4} \dots x_{i_m} y_{i_{m+1}}$, then the loose m-cycles are:

$$((z_j, y_{i_1}, x_{i_2}), (x_{i_2}, z_{j+1}, y_{i_3}), (y_{i_3}, z_{j+2}, x_{i_4}), \dots, (y_{i_{m-1}}, z_{j+m-2}, x_{i_m}), (x_{i_m}, y_{i_{m+1}}, z_j)),$$

where $j \in \mathbb{Z}_n$.

The collection of the resulting $\frac{rs}{m}\, n$ loose m-cycles yield the required decomposition. □

Corollary 3.8. *Let $p, m, n \in \mathbb{N}$. If $m \geq 3$ and $n \geq m - 1$, then $LC_m^{(3)} \,|\, K_{pm,pm,n}^{(3)}$.*

Proof. Take $r = s = pm$ in Theorem 3.7 and apply Theorem 2.1 (4) for its hypothesis. □

Taking $p = 1$ in Corollary 3.8, we have:

Corollary 3.9. *Let $m, n \in \mathbb{N}$. If $m \geq 3$ and $n \geq m-1$, then $LC_m^{(3)} \mid K_{m,m,n}^{(3)}$, and so $LC_m^{(3)}$ divides both $K_{m,m,m+1}^{(3)}$ and $K_{m,m,m}^{(3)}$.*

Corollary 3.10. *Let $m, n \in \mathbb{N}$ and $m \geq 3$ be odd. If $n \geq m-1$, then $LC_m^{(3)} | K_{m+1,m,n}^{(3)}$.*

Proof. Take $r - 1 = s = m$ in Theorem 3.7 and apply Theorem 2.1 (1) for its hypothesis. □

3.4. More on decompositions of $K_{r,s}^{(3)}$

Corollary 3.11. *Let $m \geq 3$ be an odd integer, $r \equiv 0 \pmod m$ and $s \equiv x \pmod m$, where $x \in \{0, 1, 2\}$, and $s \geq 2m + x$. We have $LC_m^{(3)} \mid K_{r,s}^{(3)}$.*

Proof. Then $r = mp$ and $s = mq + x$ for some integers $p \geq 1$ and $q \geq 2$, and therefore $K_{r,s}^{(3)} = K_{mp,mq+x}^{(3)} = K_{X,Y}^{(3)}$, where $X = X_1 \cup X_2 \cup \ldots \cup X_p$ and $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_q$ be pairwise disjoint union of sets $X_1, X_2, \ldots, X_p$ and $Y_1, Y_2, \ldots, Y_q$, respectively, with $|X_1| = |X_2| = \cdots = |X_p| = m = |Y_3| = |Y_4| = \cdots = |Y_q|$; and $|Y_1| = |Y_2| = m$, if $x = 0$; $|Y_1| = m + 1$ and $|Y_2| = m$, if $x = 1$; $|Y_1| = |Y_2| = m + 1$, if $x = 2$. We consider three cases.

Case 1. $x = 0$.

Write $K_{X,Y}^{(3)}$ as an edge-disjoint union of $K_{X_i,Y_j}^{(3)} \cong K_{m,m}^{(3)}$, $K_{X_{i_1},X_{i_2},Y_j}^{(3)} \cong K_{m,m,m}^{(3)}$ and $K_{X_i,Y_{j_1},Y_{j_2}}^{(3)} \cong K_{m,m,m}^{(3)}$, where i, i_1, $i_2 \in \{1, 2, \ldots, p\}$, $i_1 \neq i_2$ and j, j_1, $j_2 \in \{1, 2, \ldots, q\}$, $j_1 \neq j_2$.

Case 2. $x = 1$.

Write $K_{X,Y}^{(3)}$ as an edge-disjoint union of $K_{X_i,Y_j}^{(3)} \cong K_{m,m}^{(3)}$, $K_{X_i,Y_1}^{(3)} \cong K_{m,m+1}^{(3)}$, $K_{X_{i_1},X_{i_2},Y_j}^{(3)} \cong K_{m,m,m}^{(3)}$, $K_{X_{i_1},X_{i_2},Y_1}^{(3)} \cong K_{m,m,m+1}^{(3)}$, $K_{X_i,Y_{j_1},Y_{j_2}}^{(3)} \cong K_{m,m,m}^{(3)}$, $K_{X_i,Y_j,Y_1}^{(3)} \cong K_{m,m,m+1}^{(3)}$, where i, i_1, $i_2 \in \{1, 2, \ldots, p\}$, $i_1 \neq i_2$ and j, j_1, $j_2 \in \{2, 3, \ldots, q\}$, $j_1 \neq j_2$.

Case 3. $x = 2$.

Write $K_{X,Y}^{(3)}$ as an edge-disjoint union of $K_{X_i,Y_j}^{(3)} \cong K_{m,m}^{(3)}$, $K_{X_i,Y_\ell}^{(3)} \cong K_{m,m+1}^{(3)}$, $K_{X_{i_1},X_{i_2},Y_j}^{(3)} \cong K_{m,m,m}^{(3)}$, $K_{X_{i_1},X_{i_2},Y_\ell}^{(3)} \cong K_{m,m,m+1}^{(3)}$, $K_{X_i,Y_{j_1},Y_{j_2}}^{(3)} \cong K_{m,m,m}^{(3)}$, $K_{X_i,Y_j,Y_\ell}^{(3)} \cong K_{m,m,m+1}^{(3)}$, $K_{X_i,Y_1,Y_2}^{(3)} \cong K_{m,m+1,m+1}^{(3)}$, where i, i_1, $i_2 \in \{1, 2, \ldots, p\}$, $i_1 \neq i_2$ and $\ell \in \{1, 2\}$, j, j_1, $j_2 \in \{3, 4, \ldots, q\}$, $j_1 \neq j_2$.

By Corollaries 3.5, 3.6 and 3.9, $LC_m^{(3)} \mid K_{m,m}^{(3)}$, $LC_m^{(3)} \mid K_{m,m+1}^{(3)}$ and $LC_m^{(3)}$ divides both $K_{m,m,m}^{(3)}$ and $K_{m,m,m+1}^{(3)}$. By Corollary 3.10, $LC_m^{(3)} \mid K_{m+1,m,m+1}^{(3)}$, and so $LC_m^{(3)} \mid K_{m,m+1,m+1}^{(3)}$. Hence, in all the three cases, $LC_m^{(3)} \mid K_{r,s}^{(3)}$. □

3.5. Decompositions of $K_n^{(3)}$ - Proof of Theorem 1.1

Proof. Assume that $m \geq 3$ is an odd integer, $n \geq 2m$ is an integer, $n \equiv 0, 1, 2, m, m+1$ or $m + 2 \pmod{2m}$, and for $t \in \{2m, 2m+1, 2m+2, 3m, 3m+1, 3m+2\}$, $LC_m^{(3)} | K_t^{(3)}$. Then, $n = 2mk + x$ for some integer $k \geq 1$, where $x \in \{0, 1, 2, m, m+1, m+2\}$. Therefore $K_n^{(3)} = K_{2mk+x}^{(3)} = K_X^{(3)}$, where $X = X_1 \cup X_2 \cup \cdots \cup X_k$ be pairwise disjoint union of sets $X_1, X_2, \ldots, X_k$ with $|X_1| = 2m + x$ and $|X_2| = |X_3| = \cdots = |X_k| = 2m$. Write $K_X^{(3)}$ as an edge-disjoint union of $K_{X_1}^{(3)} \cong K_{2m+x}^{(3)}$, $K_{X_i}^{(3)} \cong K_{2m}^{(3)}$, $K_{X_i,X_1}^{(3)} \cong K_{2m,2m+x}^{(3)}$, $K_{X_{i_1},X_{i_2}}^{(3)} \cong K_{2m,2m}^{(3)}$, $K_{X_{i_1},X_{i_2},X_1}^{(3)} \cong K_{2m,2m,2m+x}^{(3)}$, $K_{X_{i_1},X_{i_2},X_{i_3}}^{(3)} \cong K_{2m,2m,2m}^{(3)}$, where i, i_1, i_2, $i_3 \in \{2, 3, \ldots, k\}$, $i_1 \neq i_2$, $i_1 \neq i_3$ and $i_2 \neq i_3$. By hypothesis, $LC_m^{(3)} \mid K_{2m+x}^{(3)}$; by Corollary 3.11, $LC_m^{(3)} \mid K_{2m,2m+x}^{(3)}$; and by Corollary 3.8, $LC_m^{(3)} \mid K_{2m,2m,2m+x}^{(3)}$. Hence, $LC_m^{(3)} \mid K_n^{(3)}$. □

4. Difference technique

Following 'difference technique' method was introduced by Gionfriddo et al. [12]. Assume that the vertices of $K_n^{(3)}$ are $0, 1, \ldots, n-1$ and that they are arranged in a cyclic order. The distance between vertices i and j is defined to be

$$\|i-j\| = min\{|i-j|, n-|i-j|\}.$$

Using this, define a difference triplet

$$t_{i,j,k} = (\|i-j\|, \|j-k\|, \|k-i\|)$$

to any three vertices i, j, k with $0 \le i < j < k \le n-1$.

Note that the ordering condition $i < j < k$ is important in the definition. By taking $t_{j,k,i} = (\|j-k\|, \|k-i\|, \|i-j\|)$ and $t_{k,i,j} = (\|k-i\|, \|i-j\|, \|j-k\|)$, we assume that $t_{i,j,k} = t_{j,k,i} = t_{k,i,j}$ for all choices of $\{i, j, k\}$. Moreover, difference triplets are rotation-invariant, i.e. $t_{i,j,k} = t_{i+1,j+1,k+1}$ holds for all $\{i, j, k\}$.

From [12], we have: if n is not a multiple of 3, then there can occur two kinds of difference triplets:
• *symmetric triplets*: of the form (a, a, b), where $2a = b$ or $2a + b = n$, and
• *reflected triplets*: of the form (a, b, c) or (a, c, b), where $a + b = c$ or $a + b + c = n$, and $a \neq b \neq c \neq a$.
(If n is a multiple of 3, then we have an additional triplet $(\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$.)

In what follows, the decompositions in Lemmas 5.1 to 5.6 are obtained by using the method of difference triplets; in particular, in Lemmas 5.3, 5.5 and 5.6, the decompositions are cyclic.

5. A loose 7-cycle decomposition of $K_n^{(3)}$

Lemma 5.1. $LC_7^{(3)} | K_{14}^{(3)}$.

Proof. Let $V(K_{14}^{(3)}) = \mathbb{Z}_{14}$. Symmetric triplets are: $(1,1,2)$, $(2,2,4)$, $(3,3,6)$, $(4,4,6)$, $(4,5,5)$ and $(2,6,6)$, and reflected triplets are: $(1,2,3), (1,3,2), (1,3,4), (1,4,3), (1,4,5), (1,5,4), (1,5,6), (1,6,5), (1,6,7), (1,7,6), (2,3,5)$, $(2,5,3)$, $(2,4,6)$, $(2,6,4)$, $(2,5,7)$, $(2,7,5)$, $(3,4,7)$, $(3,7,4)$, $(3,5,6)$ and $(3,6,5)$. Following $LC_7^{(3)}$'s decompose $K_{14}^{(3)}$:
For each $i \in \mathbb{Z}_{14}$, consider
$i + [(0,1,4), (4,9,5), (5,11,6), (6,13,7), (7,8,2), (2,3,10), (10,12,0)]$
(edges having difference triplets $(1,3,4), (1,4,5), (1,5,6), (1,6,7), (1,6,5), (1,7,6), (2,2,4)$, respectively),
$i + [(12,7,9), (9,11,3), (3,10,8), (8,6,13), (13,1,5), (5,0,4), (4,2,12)]$
(edges having difference triplets $(2,3,5), (2,6,6), (2,7,5), (2,5,7), (2,4,6), (1,5,4), (2,6,4)$, respectively) and
$i + [(6,3,0), (0,10,7), (7,12,4), (4,1,5), (5,9,13), (13,8,2), (2,11,6)]$
(edges having difference triplets $(3,3,6), (3,4,7), (3,5,6), (1,4,3), (4,4,6), (3,6,5), (4,5,5)$, respectively).
In addition, for each $j \in \{0,1\}$, consider
$j + [(0,1,2), (2,3,4), (4,5,6), (6,7,8), (8,9,10), (10,11,12), (12,13,0)]$
(each edge has difference triplet $(1,1,2)$),
$j + [(1,0,3), (3,2,5), (5,4,7), (7,6,9), (9,8,11), (11,10,13), (13,12,1)]$
(each edge has difference triplet $(1,2,3)$),
$j + [(0,1,12), (12,13,10), (10,11,8), (8,9,6), (6,7,4), (4,5,2), (2,3,0)]$
(each edge has difference triplet $(1,3,2)$),
$j + [(0,11,2), (2,13,4), (4,1,6), (6,3,8), (8,5,10), (10,7,12), (12,9,0)]$
(each edge has difference triplet $(2,5,3)$),
$j + [(0,3,10), (10,13,6), (6,9,2), (2,5,12), (12,1,8), (8,11,4), (4,7,0)]$
(each edge has difference triplet $(3,7,4)$). □

Lemma 5.2. $LC_7^{(3)} | K_{15}^{(3)}$.

Proof. Let $V(K_{15}^{(3)}) = \mathbb{Z}_{15}$. Symmetric triplets are: $(1,1,2), (2,2,4), (3,3,6), (4,4,7), (5,5,5), (3,6,6)$ and $(1,7,7)$, and reflected triplets are: $(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(1,4,3)$, $(1,4,5)$, $(1,5,4)$, $(1,5,6)$, $(1,6,5)$, $(1,6,7)$, $(1,7,6)$, $(2,3,5)$, $(2,5,3)$, $(2,4,6)$, $(2,6,4)$, $(2,5,7)$, $(2,7,5)$, $(2,6,7)$, $(2,7,6)$, $(3,4,7)$, $(3,7,4)$, $(3,5,7)$, $(3,7,5)$, $(4,5,6)$ and $(4,6,5)$. Observe that there are exactly 5 edges, namely, $(0,5,10), (1,6,11), (2,7,12), (3,8,13), (4,9,14)$ having difference triplet $(5,5,5)$.
Following $LC_7^{(3)}$'s decompose $K_{15}^{(3)}$:
For each $i \in \mathbb{Z}_{15}$, consider

$i + [(2,1,5),(5,6,10),(10,9,0),(0,8,7),(7,14,13),(13,4,12),(12,3,2)]$
(edges having difference triplets $(1,3,4),(1,4,5),(1,5,6),(1,7,7),(1,7,6),(1,6,7),(1,6,5)$, respectively),
$i + [(9,0,5),(5,3,12),(12,14,7),(7,11,13),(13,10,1),(1,4,6),(6,2,9)]$
(edges having difference triplets $(4,6,5),(2,7,6),(2,7,5),(2,6,4),(3,3,6),(2,5,3),(3,7,4)$, respectively), and
$i + [(14,11,4),(4,10,1),(1,6,9),(9,2,13),(13,0,3),(3,7,12),(12,5,14)]$
(edges having difference triplets $(3,5,7),(3,6,6),(3,7,5),(4,4,7),(2,3,5),(4,5,6),(2,6,7)$, respectively).

The set of edges having remaining difference triplets $(1,1,2)$, $(1,2,3)$, $(2,2,4)$, $(1,3,2)$, $(1,4,3)$, $(1,5,4)$, $(2,4,6)$, $(2,5,7)$, $(3,4,7)$ and $(5,5,5)$ can be decomposed into $LC_7^{(3)}$ as follows:
For each $j \in \{0,1,\ldots,6\}$, consider
$j + [(14,0,10),(10,7,11),(11,12,9),(9,5,3),(3,1,8),(8,4,6),(6,2,14)]$,
for each $k \in \{0,1,2\}$, consider
$k + [(10,11,6),(6,3,7),(7,8,5),(5,1,14),(14,12,4),(4,0,2),(2,13,10)]$,
for each $\ell \in \{0,1\}$, consider
$\ell + [(0,1,3),(3,2,5),(5,4,7),(7,6,9),(9,8,11),(11,10,13),(13,14,0)]$,
$\ell + [(7,8,3),(3,0,4),(4,5,2),(2,13,11),(11,9,1),(1,12,14),(14,10,7)]$,
and
$[(0,14,2),(2,3,4),(4,5,6),(6,7,8),(8,9,10),(10,11,12),(12,13,0)]$,
$[(6,7,2),(2,14,3),(3,4,1),(1,12,10),(10,5,0),(0,11,13),(13,9,6)]$,
$[(9,10,5),(5,6,2),(2,7,0),(0,4,13),(13,11,3),(3,14,1),(1,12,9)]$,
$[(13,14,9),(9,6,10),(10,11,8),(8,4,2),(2,12,7),(7,3,5),(5,1,13)]$,
$[(0,10,8),(8,3,13),(13,12,14),(14,9,4),(4,7,6),(6,11,1),(1,2,0)]$,
$[(1,2,3),(3,4,5),(5,6,7),(7,8,9),(9,10,11),(11,12,13),(13,14,1)]$. □

Lemma 5.3. $LC_7^{(3)} | K_{16}^{(3)}$.

Proof. Let $V(K_{16}^{(3)}) = \mathbb{Z}_{16}$. Symmetric triplets are: $(1,1,2)$, $(2,2,4)$, $(3,3,6)$, $(4,4,8)$, $(5,5,6)$, $(4,6,6)$ and $(2,7,7)$, and reflected triplets are: $(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(1,4,3)$, $(1,4,5)$, $(1,5,4)$, $(1,5,6)$, $(1,6,5)$, $(1,6,7)$, $(1,7,6)$, $(1,7,8)$, $(1,8,7)$, $(2,3,5)$, $(2,5,3)$, $(2,4,6)$, $(2,6,4)$, $(2,5,7)$, $(2,7,5)$, $(2,6,8)$, $(2,8,6)$, $(3,4,7)$, $(3,7,4)$, $(3,5,8)$, $(3,8,5)$, $(3,6,7)$, $(3,7,6)$, $(4,5,7)$ and $(4,7,5)$. Following $LC_7^{(3)}$'s decompose $K_{16}^{(3)}$:
For each $i \in \mathbb{Z}_{16}$, consider
$i + [(0,1,2),(2,3,5),(5,6,9),(9,10,14),(14,4,13),(13,8,7),(7,15,0)]$
(edges having difference triplets $(1,1,2),(1,2,3),(1,3,4),(1,4,5),(1,6,7),(1,5,6),(1,7,8)$, respectively),
$i + [(1,0,9),(9,2,8),(8,13,14),(14,15,10),(10,11,7),(7,4,6),(6,3,1)]$
(edges having difference triplets $(1,8,7),(1,7,6),(1,6,5),(1,5,4),(1,4,3),(1,3,2),(2,3,5)$, respectively),
$i + [(2,0,4),(4,6,10),(10,3,5),(5,13,7),(7,9,1),(1,15,8),(8,11,2)]$
(edges having difference triplets $(2,2,4),(2,4,6),(2,5,7),(2,6,8),(2,8,6),(2,7,7),(3,7,6)$, respectively),
$i + [(0,11,2),(2,14,4),(4,1,6),(6,9,12),(12,3,15),(15,7,10),(10,5,0)]$
(edges having difference triplets $(2,7,5),(2,6,4),(2,5,3),(3,3,6),(3,4,7),(3,5,8),(5,5,6)$, respectively), and
$i + [(0,9,3),(3,6,14),(14,2,5),(5,13,1),(1,12,8),(8,15,4),(4,10,0)]$
(edges having difference triplets $(3,6,7),(3,8,5),(3,7,4),(4,4,8),(4,5,7),(4,7,5),(4,6,6)$, respectively). □

Lemma 5.4. $LC_7^{(3)} | K_{21}^{(3)}$.

Proof. Let $V(K_{21}^{(3)}) = \mathbb{Z}_{21}$. Symmetric triplets are: $(1,1,2)$, $(2,2,4)$, $(3,3,6)$, $(4,4,8)$, $(5,5,10)$, $(6,6,9)$, $(7,7,7)$, $(5,8,8)$, $(3,9,9)$ and $(1,10,10)$, and reflected triplets are: $(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(1,4,3)$, $(1,4,5)$, $(1,5,4)$, $(1,5,6)$, $(1,6,5)$, $(1,6,7)$, $(1,7,6)$, $(1,7,8)$, $(1,8,7)$, $(1,8,9)$, $(1,9,8)$, $(1,9,10)$, $(1,10,9)$, $(2,3,5)$, $(2,5,3)$, $(2,4,6)$, $(2,6,4)$, $(2,5,7)$, $(2,7,5)$, $(2,6,8)$, $(2,8,6)$, $(2,7,9)$, $(2,9,7)$, $(2,8,10)$, $(2,10,8)$, $(2,9,10)$, $(2,10,9)$, $(3,4,7)$, $(3,7,4)$, $(3,5,8)$, $(3,8,5)$, $(3,6,9)$, $(3,9,6)$, $(3,7,10)$, $(3,10,7)$, $(3,8,10)$, $(3,10,8)$, $(4,5,9)$, $(4,9,5)$, $(4,6,10)$, $(4,10,6)$, $(4,7,10)$, $(4,10,7)$, $(4,8,9)$, $(4,9,8)$, $(5,6,10)$, $(5,10,6)$, $(5,7,9)$, $(5,9,7)$, $(6,7,8)$ and $(6,8,7)$. Observe that there are exactly 7 edges, namely, $(0,7,14)$, $(1,8,15)$, $(2,9,16)$, $(3,10,17)$, $(4,11,18)$, $(5,12,19)$, $(6,13,20)$ having difference triplet $(7,7,7)$.

Following $LC_7^{(3)}$'s decompose $K_{21}^{(3)}$:
For each $i \in \mathbb{Z}_{21}$, consider
$i + [(0, 9, 1), (1, 11, 2), (2, 13, 3), (3, 15, 4), (4, 17, 5), (5, 19, 6), (6, 7, 0)]$
(edges having difference triplets $(1, 8, 9), (1, 9, 10), (1, 10, 10), (1, 10, 9), (1, 9, 8), (1, 8, 7), (1, 7, 6)$, respectively),
$i + [(0, 1, 16), (16, 15, 11), (11, 7, 10), (10, 13, 12), (12, 14, 17), (17, 19, 2), (2, 4, 0)]$
(edges having difference triplets $(1, 6, 5), (1, 5, 4), (1, 4, 3), (1, 3, 2), (2, 3, 5), (2, 4, 6), (2, 2, 4)$, respectively),
$i + [(0, 2, 7), (7, 5, 13), (13, 4, 6), (6, 16, 8), (8, 10, 19), (19, 17, 9), (9, 11, 0)]$
(edges having difference triplets $(2, 5, 7), (2, 6, 8), (2, 7, 9), (2, 8, 10), (2, 9, 10), (2, 10, 8), (2, 10, 9)$, respectively),
$i + [(2, 0, 14), (14, 12, 6), (6, 11, 13), (13, 15, 9), (9, 7, 4), (4, 1, 8), (8, 5, 2)]$
(edges having difference triplets $(2, 9, 7), (2, 8, 6), (2, 7, 5), (2, 6, 4), (2, 5, 3), (3, 4, 7), (3, 3, 6)$, respectively),
$i + [(16, 11, 8), (8, 5, 14), (14, 7, 4), (4, 12, 1), (1, 19, 10), (10, 2, 13), (13, 6, 16)]$
(edges having difference triplets $(3, 5, 8), (3, 6, 9), (3, 7, 10), (3, 8, 10), (3, 9, 9), (3, 10, 8), (3, 10, 7)$, respectively),
$i + [(0, 12, 4), (4, 8, 17), (17, 7, 3), (3, 9, 13), (13, 18, 1), (1, 11, 6), (6, 16, 0)]$
(edges having difference triplets $(4, 8, 9), (4, 9, 8), (4, 10, 7), (4, 10, 6), (4, 9, 5), (5, 5, 10), (5, 6, 10)$, respectively),
$i + [(0, 5, 12), (12, 7, 20), (20, 6, 11), (11, 17, 1), (1, 10, 16), (16, 2, 8), (8, 14, 0)]$
(edges having difference triplets $(5, 7, 9), (5, 8, 8), (5, 9, 7), (5, 10, 6), (6, 6, 9), (6, 8, 7), (6, 7, 8)$, respectively), and
for each $j \in \{0, 1, \ldots, 19\}$, consider
$j + [(6, 3, 18), (18, 15, 10), (10, 14, 17), (17, 13, 0), (0, 4, 9), (9, 19, 2), (2, 12, 6)]$
(edges having difference triplets $(3, 9, 6), (3, 8, 5), (3, 7, 4), (4, 4, 8), (4, 5, 9), (4, 7, 10), (4, 6, 10)$, respectively).

The set of edges having remaining difference triplets $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 5, 6)$, $(1, 6, 7)$, $(1, 7, 8)$, and $(7, 7, 7)$ together with the set of remaining edges $\{(5, 2, 17), (17, 14, 9), (9, 13, 16), (16, 12, 20), (20, 3, 8), (8, 18, 1), (1, 11, 5)\}$ can be decomposed into $LC_7^{(3)}$ as follows:
For each $k \in \{0, 1, \ldots, 11\}$, consider
$k + [(13, 14, 15), (15, 18, 16), (16, 17, 20), (20, 19, 3), (3, 9, 4), (4, 11, 5), (5, 6, 13)]$,
for each $\ell \in \{0, 1, 2, 3\}$, consider
$\ell + [(5, 6, 7), (7, 10, 8), (8, 9, 12), (12, 11, 16), (16, 1, 17), (17, 3, 18), (18, 19, 5)]$,
for each $m \in \{0, 1\}$, consider
$m + [(9, 10, 11), (11, 14, 12), (12, 16, 13), (13, 6, 20), (20, 5, 0), (0, 7, 1), (1, 2, 9)]$,
and
$[(4, 5, 6), (6, 9, 7), (7, 11, 8), (8, 1, 15), (15, 0, 16), (16, 2, 17), (17, 18, 4)]$,
$[(12, 13, 14), (14, 17, 15), (15, 19, 16), (16, 9, 2), (2, 8, 3), (3, 10, 4), (4, 5, 12)]$,
$[(5, 2, 17), (17, 14, 9), (9, 13, 16), (16, 15, 20), (20, 3, 8), (8, 18, 1), (1, 11, 5)]$,
$[(11, 12, 13), (13, 16, 14), (14, 15, 18), (18, 17, 1), (1, 7, 2), (2, 9, 3), (3, 4, 11)]$,
$[(10, 3, 17), (17, 0, 16), (16, 20, 12), (12, 5, 19), (19, 2, 18), (18, 4, 11), (11, 15, 10)]$. □

Lemma 5.5. $LC_7^{(3)} | K_{22}^{(3)}$.

Proof. Let $V(K_{22}^{(3)}) = \mathbb{Z}_{22}$. Symmetric triplets are: $(1, 1, 2)$, $(2, 2, 4)$, $(3, 3, 6)$, $(4, 4, 8)$, $(5, 5, 10)$, $(6, 6, 10)$, $(7, 7, 8)$, $(6, 8, 8)$, $(4, 9, 9)$ and $(2, 10, 10)$, and reflected triplets are: $(1, 2, 3)$, $(1, 3, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(1, 4, 5)$, $(1, 5, 4)$, $(1, 5, 6)$, $(1, 6, 5)$, $(1, 6, 7)$, $(1, 7, 6)$, $(1, 7, 8)$, $(1, 8, 7)$, $(1, 8, 9)$, $(1, 9, 8)$, $(1, 9, 10)$, $(1, 10, 9)$, $(1, 10, 11)$, $(1, 11, 10)$, $(2, 3, 5)$, $(2, 5, 3)$, $(2, 4, 6)$, $(2, 6, 4)$, $(2, 5, 7)$, $(2, 7, 5)$, $(2, 6, 8)$, $(2, 8, 6)$, $(2, 7, 9)$, $(2, 9, 7)$, $(2, 8, 10)$, $(2, 10, 8)$, $(2, 9, 11)$, $(2, 11, 9)$, $(3, 4, 7)$, $(3, 7, 4)$, $(3, 5, 8)$, $(3, 8, 5)$, $(3, 6, 9)$, $(3, 9, 6)$, $(3, 7, 10)$, $(3, 10, 7)$, $(3, 8, 11)$, $(3, 11, 8)$, $(3, 9, 10)$, $(3, 10, 9)$, $(4, 5, 9)$, $(4, 9, 5)$, $(4, 6, 10)$, $(4, 10, 6)$, $(4, 7, 11)$, $(4, 11, 7)$, $(4, 8, 10)$, $(4, 10, 8)$, $(5, 6, 11)$, $(5, 11, 6)$, $(5, 7, 10)$, $(5, 10, 7)$, $(5, 8, 9)$, $(5, 9, 8)$, $(6, 7, 9)$ and $(6, 9, 7)$. Following $LC_7^{(3)}$'s decompose $K_{22}^{(3)}$:
For each $i \in \mathbb{Z}_{22}$, consider
$i + [(0, 1, 2), (2, 3, 5), (5, 4, 8), (8, 7, 12), (12, 18, 13), (13, 14, 21), (21, 6, 0)]$
(edges having difference triplets $(1, 1, 2), (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (1, 7, 8), (1, 6, 7)$, respectively),
$i + [(0, 1, 9), (9, 18, 8), (8, 20, 19), (19, 6, 7), (7, 16, 15), (15, 4, 14), (14, 21, 0)]$
(edges having difference triplets $(1, 8, 9), (1, 9, 10), (1, 10, 11), (1, 10, 9), (1, 9, 8), (1, 11, 10), (1, 8, 7)$, respectively),
$i + [(16, 0, 1), (1, 18, 2), (2, 20, 3), (3, 5, 6), (6, 9, 10), (10, 12, 14), (14, 19, 16)]$
(edges having difference triplets $(1, 7, 6), (1, 6, 5), (1, 5, 4), (1, 3, 2), (1, 4, 3), (2, 2, 4), (2, 3, 5)$, respectively),

$i + [(9,13,7),(7,5,12),(12,4,6),(6,15,8),(8,18,10),(10,0,20),(20,11,9)]$
(edges having difference triplets $(2,4,6),(2,5,7),(2,6,8),(2,7,9),(2,8,10),(2,10,10),(2,9,11)$, respectively),
$i + [(2,0,13),(13,11,3),(3,5,18),(18,16,10),(10,17,15),(15,19,21),(21,4,2)]$
(edges having difference triplets $(2,11,9),(2,10,8),(2,9,7),(2,8,6),(2,7,5),(2,6,4),(2,5,3)$, respectively),
$i + [(0,3,6),(6,9,13),(13,8,5),(5,2,11),(11,14,1),(1,4,12),(12,15,0)]$
(edges having difference triplets $(3,3,6),(3,4,7),(3,5,8),(3,6,9),(3,9,10),(3,8,11),(3,7,10)$, respectively),
$i + [(0,3,13),(13,10,2),(2,5,17),(17,4,1),(1,20,15),(15,11,18),(18,14,0)]$
(edges having difference triplets $(3,10,9),(3,11,8),(3,10,7),(3,9,6),(3,8,5),(3,7,4),(4,4,8)$, respectively),
$i + [(0,4,9),(9,15,5),(5,1,12),(12,20,8),(8,21,17),(17,3,7),(7,11,0)]$
(edges having difference triplets $(4,5,9),(4,6,10),(4,7,11),(4,8,10),(4,9,9),(4,10,8),(4,11,7)$, respectively),
$i + [(14,10,4),(4,21,8),(8,13,18),(18,7,12),(12,17,2),(2,11,19),(19,5,14)]$
(edges having difference triplets $(4,10,6),(4,9,5),(5,5,10),(5,6,11),(5,7,10),(5,9,8),(5,8,9)$, respectively), and
$i + [(0,5,15),(15,21,4),(4,10,16),(16,3,9),(9,17,1),(1,8,14),(14,7,0)]$
(edges having difference triplets $(5,10,7),(5,11,6),(6,6,10),(6,7,9),(6,8,8),(6,9,7),(7,7,8)$, respectively). □

Lemma 5.6. $LC_7^{(3)} | K_{23}^{(3)}$.

Proof. Let $V(K_{23}^{(3)}) = \mathbb{Z}_{23}$. Symmetric triplets are: $(1,1,2)$, $(2,2,4)$, $(3,3,6)$, $(4,4,8)$, $(5,5,10)$, $(6,6,11)$, $(7,7,9)$, $(7,8,8)$, $(5,9,9)$, $(3,10,10)$ and $(1,11,11)$, and reflected triplets are: $(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(1,4,3)$, $(1,4,5)$, $(1,5,4)$, $(1,5,6)$, $(1,6,5)$, $(1,6,7)$, $(1,7,6)$, $(1,7,8)$, $(1,8,7)$, $(1,8,9)$, $(1,9,8)$, $(1,9,10)$, $(1,10,9)$, $(1,10,11)$, $(1,11,10)$, $(2,3,5)$, $(2,5,3)$, $(2,4,6)$, $(2,6,4)$, $(2,5,7)$, $(2,7,5)$, $(2,6,8)$, $(2,8,6)$, $(2,7,9)$, $(2,9,7)$, $(2,8,10)$, $(2,10,8)$, $(2,9,11)$, $(2,11,9)$, $(2,10,11)$, $(2,11,10)$, $(3,4,7)$, $(3,7,4)$, $(3,5,8)$, $(3,8,5)$, $(3,6,9)$, $(3,9,6)$, $(3,7,10)$, $(3,10,7)$, $(3,8,11)$, $(3,11,8)$, $(3,9,11)$, $(3,11,9)$, $(4,5,9)$, $(4,9,5)$, $(4,6,10)$, $(4,10,6)$, $(4,7,11)$, $(4,11,7)$, $(4,8,11)$, $(4,11,8)$, $(4,9,10)$, $(4,10,9)$, $(5,6,11)$, $(5,11,6)$, $(5,7,11)$, $(5,11,7)$, $(5,8,10)$, $(5,10,8)$, $(6,7,10)$, $(6,10,7)$, $(6,8,9)$ and $(6,9,8)$.

Following $LC_7^{(3)}$'s decompose $K_{23}^{(3)}$:
For each $i \in \mathbb{Z}_{23}$, consider
$i + [(0,1,2),(2,3,5),(5,6,9),(9,10,14),(14,20,15),(15,16,22),(22,7,0)]$
(edges having difference triplets $(1,1,2),(1,2,3),(1,3,4),(1,4,5),(1,5,6),(1,6,7),(1,7,8)$, respectively),
$i + [(1,0,9),(9,10,19),(19,20,7),(7,6,18),(18,4,5),(5,15,16),(16,2,1)]$
(edges having difference triplets $(1,8,9),(1,9,10),(1,10,11),(1,11,11),(1,10,9),(1,11,10),(1,9,8)$, respectively),
$i + [(0,1,16),(16,15,9),(9,5,7),(7,8,2),(2,3,21),(21,17,20),(20,22,0)]$
(edges having difference triplets $(1,8,7),(1,7,6),(2,2,4),(1,6,5),(1,5,4),(1,4,3),(1,3,2)$, respectively),
$i + [(0,2,5),(5,3,9),(9,7,14),(14,6,8),(8,1,22),(22,11,13),(13,15,0)]$
(edges having difference triplets $(2,3,5),(2,4,6),(2,5,7),(2,6,8),(2,7,9),(2,9,11),(2,8,10)$, respectively),
$i + [(0,2,12),(12,1,22),(22,8,10),(10,20,18),(18,16,11),(11,3,9),(9,7,0)]$
(edges having difference triplets $(2,10,11),(2,11,10),(2,11,9),(2,10,8),(2,7,5),(2,8,6),(2,9,7)$, respectively),
$i + [(2,0,19),(19,17,14),(14,8,11),(11,4,7),(7,10,15),(15,6,9),(9,22,2)]$
(edges having difference triplets $(2,6,4),(2,5,3),(3,3,6),(3,4,7),(3,5,8),(3,6,9),(3,7,10)$, respectively),
$i + [(0,3,11),(11,8,20),(20,7,10),(10,13,1),(1,4,16),(16,19,9),(9,6,0)]$
(edges having difference triplets $(3,8,11),(3,9,11),(3,10,10),(3,11,9),(3,11,8),(3,10,7),(3,9,6)$, respectively),
$i + [(0,18,3),(3,22,6),(6,10,14),(14,5,9),(9,13,19),(19,8,12),(12,4,0)]$
(edges having difference triplets $(3,8,5),(3,7,4),(4,4,8),(4,5,9),(4,6,10),(4,7,11),(4,8,11)$, respectively),
$i + [(4,0,13),(13,22,3),(3,18,7),(7,11,1),(1,17,5),(5,10,14),(14,9,4)]$
(edges having difference triplets $(4,9,10),(4,10,9),(4,11,8),(4,10,6),(4,11,7),(4,9,5),(5,5,10)$, respectively),
$i + [(0,11,5),(5,17,10),(10,20,2),(2,7,16),(16,1,6),(6,13,18),(18,12,0)]$

(edges having difference triplets $(5,6,11),(5,7,11),(5,8,10),(5,9,9),(5,10,8),(5,11,7),(5,11,6)$, respectively), and
$i+[(6,0,12),(12,18,2),(2,8,16),(16,1,7),(7,14,20),(20,4,13),(13,21,6)]$
(edges having difference triplets $(6,6,11),(6,7,10),(6,8,9),(6,9,8),(6,10,7),(7,7,9),(7,8,8)$, respectively). □

Proof of Theorem 1.2.
The proof of necessity is obvious, and we prove the sufficiency. By Theorem 1.1, it is enough to find a loose 7-cycle decomposition of $K_r^{(3)}$, $r \in \{14,15,16,21,22,23\}$; this follows from Lemmas 5.1 to 5.6. □

6. Tight cycle decompositions

We use $[0,1,2,\dots,m-1]$ to denote any hypergraph isomorphic to $TC_m^{(3)}$ with vertex set $\mathbb{Z}_m$ and edge set $\{\{i,i+1,i+2\} : i \in \mathbb{Z}_m\}$.

Lemma 6.1. *If $r \geq 4$ and $s \geq 4$ are even integers and if $C_r \,|\, K_{s,s}$, then $TC_{2r}^{(3)} \,|\, K_{2s,2s}^{(3)}$.*

Proof. Consider $K_{X,Y}^{(3)} \cong K_{2s,2s}^{(3)}$ with $X = \{x_1,x_2,\dots,x_{2s}\}$ and $Y = \{y_1,y_2,\dots,y_{2s}\}$. We have to find $2\,\frac{s^2}{r}\,(2s-1)$ edge-disjoint $TC_{2r}^{(3)}$'s in $K_{2s,2s}^{(3)}$. Let $\{L_1,L_2,\dots,L_{2s-1}\}$ and $\{M_1,M_2,\dots,M_{2s-1}\}$ be 1-factorizations of the complete graphs $K_X \cong K_{2s}$ and $K_Y \cong K_{2s}$, respectively. For each $i \in \{1,2,\dots,2s-1\}$, consider the pair (L_i,M_i). The number of such pairs is $2s-1$. For convenience, let $L_i = \{x_1x_2, x_3x_4, x_5x_6, \dots, x_{2s-1}x_{2s}\}$ and $M_i = \{y_1y_2, y_3y_4, y_5y_6, \dots, y_{2s-1}y_{2s}\}$. Denote the edges $x_{2q-1}x_{2q}$ and $y_{2q-1}y_{2q}$ by new vertices u_q and v_q, respectively, where $q \in \{1,2,\dots,s\}$. Consider the complete bipartite graph $K_{\{u_1,u_2,u_3,\dots,u_s\},\{v_1,v_2,v_3,\dots,v_s\}} \cong K_{s,s}$. By hypothesis, $C_r \,|\, K_{s,s}$. Let $\mathscr{C}_i = \{C_{i1}, C_{i2}, \dots, C_{i\frac{s^2}{r}}\}$ be the collection of r-cycles in the decomposition of $K_{s,s}$. Now, corresponding to each C_{ij} in $\mathscr{C}_i$, we construct two edge-disjoint $TC_{2r}^{(3)}$'s, say C'_{ij} and C''_{ij}, of $K_{2s,2s}^{(3)}$ as follows, where $j \in \{1,2,\dots,\frac{s^2}{r}\}$: without loss of generality, let

$$C_{ij} = u_1v_1u_2v_2u_3v_3\dots u_{\frac{r}{2}}v_{\frac{r}{2}}u_1.$$

Then C'_{ij} is

$$[x_1,x_2,y_1,y_2,x_3,x_4,y_3,y_4,x_5,x_6,y_5,y_6,\dots,x_{r-3},x_{r-2},y_{r-3},y_{r-2},x_{r-1},x_r,y_{r-1},y_r]$$

and C''_{ij} is

$$[x_2,x_1,y_2,y_1,x_4,x_3,y_4,y_3,x_6,x_5,y_6,y_5,\dots,x_{r-2},x_{r-3},y_{r-2},y_{r-3},x_r,x_{r-1},y_r,y_{r-1}].$$

To complete the proof consider the collection $\{C'_{ij}\} \cup \{C''_{ij}\}$. □

Next, we use the following characterization of isomorphic cycle decompositions of complete bipartite graphs.

Theorem 6.2. *([23]) The complete bipartite graph $K_{a,b}$ can be decomposed into $2k$-cycles if and only if a and b are even, $a \geq k$, $b \geq k$, and $2k$ divides ab. In particular, $C_{2k}|K_{a,a}$ if and only if a is even, $a \geq k$, and $2k$ divides a^2.*

Lemma 6.3. *If $r \geq 4$ and $s \geq 4$ are even integers, $s \geq \frac{r}{2}$, and r divides s^2, then $TC_{2r}^{(3)} \,|\, K_{2s,2s}^{(3)}$.*

Proof. Follows from Lemma 6.1 and Theorem 6.2. □

Since,

$$K_{2(2p)}^{(3)} = K_{2p}^{(3)} \oplus K_{2p,2p}^{(3)} \oplus K_{2p}^{(3)},$$
$$K_{2(2^2p)}^{(3)} = K_{2^2p}^{(3)} \oplus K_{2^2p,2^2p}^{(3)} \oplus K_{2^2p}^{(3)},$$
$$K_{2(2^3p)}^{(3)} = K_{2^3p}^{(3)} \oplus K_{2^3p,2^3p}^{(3)} \oplus K_{2^3p}^{(3)},$$

$\vdots$

$K^{(3)}_{2(2^{\ell-1}p)} = K^{(3)}_{2^{\ell-1}p} \oplus K^{(3)}_{2^{\ell-1}p,2^{\ell-1}p} \oplus K^{(3)}_{2^{\ell-1}p}$,
we can write, for $\ell \geq 2$,

$$K^{(3)}_{2^{\ell}p} = (K^{(3)}_{2^{\ell-1}p,2^{\ell-1}p} \oplus 2K^{(3)}_{2^{\ell-2}p,2^{\ell-2}p} \oplus 2^2K^{(3)}_{2^{\ell-3}p,2^{\ell-3}p} \oplus \ldots \oplus 2^{\ell-4}K^{(3)}_{2^3p,2^3p} \oplus 2^{\ell-3}K^{(3)}_{2^2p,2^2p} \oplus 2^{\ell-2}K^{(3)}_{2p,2p}) \oplus (\underbrace{K^{(3)}_{2p} \oplus K^{(3)}_{2p} \oplus \cdots \oplus K^{(3)}_{2p}}_{2^{\ell-1} \text{ times}}).$$

By Lemma 6.3, if $p \geq 4$ is an even integer, then we have the following decompositions: $TC^{(3)}_{2p} \,|\, K^{(3)}_{2p,2p}$, $TC^{(3)}_{2p} \,|\, K^{(3)}_{2^2p,2^2p}$, $TC^{(3)}_{2p} \,|\, K^{(3)}_{2^3p,2^3p}, \ldots, TC^{(3)}_{2p} \,|\, K^{(3)}_{2^{\ell-3}p,2^{\ell-3}p}$, $TC^{(3)}_{2p} \,|\, K^{(3)}_{2^{\ell-2}p,2^{\ell-2}p}$, $TC^{(3)}_{2p} \,|\, K^{(3)}_{2^{\ell-1}p,2^{\ell-1}p}$. Consequently, if $TC^{(3)}_{2p} \,|\, K^{(3)}_{2p}$ for some even integer $p \geq 4$, then $TC^{(3)}_{2p} \,|\, K^{(3)}_{2^{\ell}p}$, for each $\ell \geq 1$. Thus, we collect known results on such decompositions.

Consider the decomposition $TC^{(3)}_{m} \,|\, K^{(3)}_{m}$. If $8 \leq m \leq 48$, then admissible m's for the existence of such decomposition with $m \equiv 0 \pmod 4$ are 8, 16, 20, 28, 32, 40 and 44. For each such m, $TC^{(3)}_{m} \,|\, K^{(3)}_{m}$ (see [2, 14, 21]). Hence, we have:

Lemma 6.4. *([2, 14, 21]). For* $m \in \{8, 16, 20, 28, 32, 40, 44\}$, $TC^{(3)}_{m} \,|\, K^{(3)}_{m}$.

Proof of Theorem 1.3.
Take $m = 2p$ to complete the proof. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

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