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On a new kind of λ-Bernstein-Kantorovich operators for univariate and bivariate functions

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Abstract. This paper presents a novel class of λ-Bernstein operators, wherein the parameter $\lambda \in [-1, 1]$. An approximation theorem of the Korovkin type is explored, a local approximation theorem is established and an asymptotic formula of the Voronovskaja type is derived. In addition, the bivariate tensor product operators are built, some approximation properties are discussed, including an asymptotic theorem of the Voronovskaja type and the order of convergence in relation to Peetre's K-functional. Finally, for certain continuous functions, numerical examples and plots to demonstrate our newly defined operators' convergence behavior are provided and there are also provided in comparison with the classical Kantorovich operators in terms of the approximation error.

1. Introduction

Bernstein polynomials and various structures derived from them are used in fields such as computer graphics, numerical analysis and approximation theory. Considering approximation theory, the first knowledge that comes to mind is that a new function approximating a continuous function ζ defined on $[0, 1]$ can be obtained by using Bernstein polynomials, and one of the most important results of this obtaining process is that the reconstructed function uniformly converges to this function ζ. Due to the widespread use, simplicity, and useful properties of Bernstein polynomials, they have attracted a lot of attention from researchers, as a result, it has contributed to the emergence of many studies and continues to do so. Inspired by Bernstein polynomials, numerous operators have been introduced and their convergence properties have been examined, you may refer to studies [3, 4, 12, 13, 18, 21, 24, 26, 28, 31–33, 36, 37, 42, 44]. Besides,

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these numerous operators' generalizations also have been developed and studied, sometimes by using the parameter λ (see [2, 6, 8–10, 14, 16, 19, 30, 34, 45]) and other times by creating of their q-analogues (see [20, 25, 38, 40]). Additionally, the following papers that define and analyze operators on two-variable function spaces are also worth reviewing:[1, 5, 7, 17, 22, 35, 39, 43]. Let us give brief information about some of them in particular. In [5], the authors have introduced the univariate and bivariate Bernstein-Schurer-type operators. They have presented the degree of convergence, Korovkin-type approximation theorem, and Voronovskaja-type asymptotic theorem for the univariate Bernstein-Schurer-Type operators; then they have given the order of convergence and the Voronovskaja-type asymptotic theorem for the bivariate of it. In [1], the bivariate extension of the Bernstein-Chlodovsky operators has been introduced. Weighted approximation properties for continuous functions in the weighted space have been explored by the researchers.

Now, let us explain the definition of famous Bernstein operators [11]. It is commonly known that the definition of Bernstein operators for $\zeta \in C[0,1]$ and $n \in \mathbb{N}$ is as follows:

$$B_n(\zeta;y) = \sum_{l=0}^{n} b_{n,l}(y)\zeta\left(\frac{l}{n}\right),\ y \in [0,1], \tag{1}$$

where

$$b_{n,l}(y) = \binom{n}{l} y^l(1-y)^{n-l},\ l = 0,1,...,n. \tag{2}$$

A type of λ-Bernstein operators was proposed by Cai et al. [16] in 2018. It was constructed using the following λ-Bézier basis:

$$\begin{cases} \widetilde{b}^{\lambda}_{n,0}(y) = b_{n,0}(y) - \frac{\lambda}{n+1}b_{n+1,1}(y), \\ \widetilde{b}^{\lambda}_{n,l}(y) = b_{n,l}(y) + \lambda\left(\frac{n-2l+1}{n^2-1}b_{n+1,l}(y) - \frac{n-2l-1}{n^2-1}b_{n+1,l+1}(y)\right),\ (1 \le l \le n-1), \\ \widetilde{b}^{\lambda}_{n,n}(y) = b_{n,n}(y) - \frac{\lambda}{n+1}b_{n+1,n}(y), \end{cases} \tag{3}$$

where $\lambda \in [-1,1]$ and $b_{n,l}(y)$ are defined in (2).

Very recently, Zhou et al. [45] discovered a novel class of λ-Bernstein operators, which are as follows:

$$B^{\lambda}_n(\zeta;y) = \sum_{l=0}^{n} b^{\lambda}_{n,l}(y)\ \zeta\left(\frac{l}{n}\right),\ y \in [0,1], \tag{4}$$

where the new λ-Bézier basis functions $b^{\lambda}_{n,l}(y)$ are provided by

$$\begin{cases} b^{\lambda}_{n,0}(y) = b_{n,0}(y) - \frac{\lambda}{n+1}b_{n+1,1}(y), \\ b^{\lambda}_{n,l}(y) = b_{n,l}(y) + \frac{\lambda}{n+1}\left(b_{n+1,l}(y) - b_{n+1,l+1}(y)\right),\ (1 \le l \le n-1,\ \lambda \in [-1,1]), \\ b^{\lambda}_{n,n}(y) = b_{n,n}(y) + \frac{\lambda}{n+1}b_{n+1,n}(y). \end{cases} \tag{5}$$

Obviously, this new λ-Bézier basis function is formally simpler than the original λ-Bézier basis function. They obtained that for some values of λ, the convergence effect of new operators (4) is better than that of the original λ-Bernstein operators defined in [16], including the classical Bernstein operators (1).

Now, motivated by the studies mentioned above, for $\zeta \in C[0,1]$, $y \in [0,1]$, we define a new kind of λ-Bernstein-Kantorovich operators as follows,

$$K^{\lambda}_n(\zeta;y) = (n+1)\sum_{l=0}^{n} b^{\lambda}_{n,l}(y) \int_{\frac{l}{n+1}}^{\frac{l+1}{n+1}} \zeta(t)dt,\ \lambda \in [-1,1], \tag{6}$$

where $b^{\lambda}_{n,l}(y)$ is defined in (4). Apparently, when $\lambda = 0$, the operators given by (6) reduce to the classical form proposed by Kantorovich [27]. The structure of this document is as follows: In section 2, moments and central moments are determined for the operators in (6); in section 3, we create a Korovkin type theorem, a local approximation theorem, and a Voronovkaja type asymptotic formula by first providing some essential definitions; in section 4, we define bivariate tensor product operators, we present the moments and central moments, we examine convergence properties via the Volkov theorem, obtain the Voronovkaja type asymptotic theorem, and compute the rate of convergence by using Peetre's K-functional for these operators. To enhance the significance of the research for our paper, in section 5, we will compare and analyze the approximation error of some continuous functions by the newly defined operators (6) with the classical Kantorovich operators (the case when $\lambda = 0$ for (6)) and the original λ-Bernstein-Kantorovich operators defined in the following:

$$\widetilde{K}_n^{\lambda}(\zeta;y) = (n+1)\sum_{l=0}^{n}\widetilde{b}_{n,l}^{\lambda}(y)\int_{\frac{l}{n+1}}^{\frac{l+1}{n+1}}\zeta(t)dt,$$

where $\lambda \in [-1,1]$ and $\widetilde{b}_{n,l}^{\lambda}(y)$ are defined in (3).

2. Auxiliary results

Let $e_i(t) = t^i$, $i \in \{0,1,...,4\}$ and $\Psi_i(t,y) = (t-y)^i$, $i \in \{0,1,2,4\}$. To validate our major findings, we require the following lemmas.

Lemma 2.1. *[45] Let $y \in [0,1]$, $\lambda \in [-1,1]$, $n \in \mathbb{N}$.Then, we have the following equalities:*

$$\begin{aligned}
B_n^{\lambda}(e_0;y) &= 1,\\
B_n^{\lambda}(e_1;y) &= y + \lambda\frac{1-y^{n+1}-(1-y)^{n+1}}{n(n+1)},\\
B_n^{\lambda}(e_2;y) &= y^2 + \frac{y(1-y)}{n} + \lambda\left[\frac{2y(1-y^n)}{n^2} - \frac{1-y^{n+1}-(1-y)^{n+1}}{n^2(n+1)}\right],\\
B_n^{\lambda}(e_3;y) &= y^3 + \frac{3y^2(1-y)}{n} + \frac{y-3y^2+2y^3}{n^2} + \lambda\left[\frac{3y^2\left(1-y^{n-1}\right)}{n^2} + \frac{1-y^{n+1}-(1-y)^{n+1}}{n^3(n+1)}\right],\\
B_n^{\lambda}(e_4;y) &= y^4 + \frac{6y^3(1-y)}{n} + \frac{7y^2-18y^3+11y^4}{n^2} + \frac{y-7y^2+12y^3-6y^4}{n^3}\\
&\quad +\lambda\left[\frac{4y^3\left(1-y^{n-2}\right)}{n^2} + \frac{6y^2-4y^3-2y^{n+1}}{n^3} + \frac{2y(1-y^n)}{n^4} - \frac{1-y^{n+1}-(1-y)^{n+1}}{n^4(n+1)}\right].
\end{aligned}$$

Lemma 2.2. *For $y \in [0,1]$, $\lambda \in [-1,1]$, $n \in \mathbb{N}$, Lemma 2.1 enables us to derive the following equalities:*

$$K_n^{\lambda}(e_0;y) = 1, \tag{7}$$

$$K_n^{\lambda}(e_1;y) = y + \frac{1-2y}{2(n+1)} + \lambda\frac{1-y^{n+1}-(1-y)^{n+1}}{(n+1)^2}, \tag{8}$$

$$K_n^{\lambda}(e_2;y) = y^2 + \frac{2y-3y^2}{n+1} + \frac{1-6y+6y^2}{3(n+1)^2} + \lambda\frac{2y(1-y^n)}{(n+1)^2}, \tag{9}$$

$$\begin{aligned}
K_n^{\lambda}(e_3;y) &= y^3 + \frac{9y^2-12y^3}{2(n+1)} + \frac{7y-27y^2+22y^3}{2(n+1)^2} + \frac{1-14y+36y^2-24y^3}{4(n+1)^3}\\
&\quad +\lambda\left[\frac{3y^2\left(1-y^{n-1}\right)}{(n+1)^2} + \frac{3y(1-y)}{(n+1)^3} + \frac{1-y^{n+1}-(1-y)^{n+1}}{2(n+1)^4}\right],\\
K_n^{\lambda}(e_4;y) &= y^4 + \frac{8y^3-10y^4}{n+1} + \frac{15y^2-48y^3+35y^4}{(n+1)^2} + \frac{6y-45y^2+88y^3-50y^4}{(n+1)^3}
\end{aligned}$$

$$+\frac{1-30y+150y^2-240y^3+120y^4}{5(n+1)^4}+\lambda\left[\frac{4\left(y^3-y^{n+1}\right)}{(n+1)^2}+\frac{12y^2(1-y)}{(n+1)^3}\right.$$
$$\left.+\frac{6y-12y^2+8y^3-2y^{n+1}}{(n+1)^4}\right].$$

Proof. By the definition of $K_n^\lambda(\zeta)$ in (6), we have

$$K_n^\lambda(e_0;y)=(n+1)\sum_{l=0}^{n}b_{n,l}^\lambda(y)\int_{\frac{l}{n+1}}^{\frac{l+1}{n+1}}dt=\sum_{l=0}^{n}b_{n,l}^\lambda(y)=B_n^\lambda(e_0;y)=e_0(y).$$

Similarly, we can obtain the following equalities,

$$\begin{aligned}
K_n^\lambda(e_1;y) &= \frac{n}{n+1}B_n^\lambda(e_1;y)+\frac{1}{2(n+1)},\\
K_n^\lambda(e_2;y) &= \frac{n^2}{(n+1)^2}B_n^\lambda(e_2;y)+\frac{n}{(n+1)^2}B_n^\lambda(e_1;y)+\frac{1}{3(n+1)^2},\\
K_n^\lambda(e_3;y) &= \frac{n^3}{(n+1)^3}B_n^\lambda(e_3;y)+\frac{3n^2}{2(n+1)^3}B_n^\lambda(e_2;y)+\frac{n}{(n+1)^3}B_n^\lambda(e_1;y)+\frac{1}{4(n+1)^3},\\
K_n^\lambda(e_4;y) &= \frac{n^4}{(n+1)^4}B_n^\lambda(e_4;y)+\frac{2n^3}{(n+1)^4}B_n^\lambda(e_3;y)+\frac{2n^2}{(n+1)^4}B_n^\lambda(e_2;y)\\
&\quad+\frac{n}{(n+1)^4}B_n^\lambda(e_1;y)+\frac{1}{5(n+1)^4}.
\end{aligned}$$

Then, by using Lemma 2.1 and some computations, we can get the desired results of Lemma 2.2. □

Lemma 2.3. *For $y\in[0,1]$, $\lambda\in[-1,1]$, $n\in\mathbb{N}$, we can get*

$$\begin{aligned}
K_n^\lambda(\Psi_1(t,y);y) &= \frac{1-2y}{2(n+1)}+\lambda\frac{1-y^{n+1}-(1-y)^{n+1}}{(n+1)^2}:=\alpha_n(y),\\
K_n^\lambda(\Psi_2(t,y);y) &= \frac{y(1-y)}{n+1}+\frac{1-6y+6y^2}{3(n+1)^2}-\lambda\frac{2y\left(y^n-y^{n+1}-(1-y)^{n+1}\right)}{(n+1)^2}\\
&:=\beta_n(y),\\
K_n^\lambda(\Psi_4(t,y);y) &= \frac{3y^2-6y^3+3y^4}{(n+1)^2}+\frac{5y-31y^2+52y^3-26y^4}{(n+1)^3}+\frac{1-30y+150y^2-240y^3+120y^4}{5(n+1)^4}\\
&\quad+\lambda\left\{\frac{4y^3(1-y)^{n+1}-4y^{n+1}(1-y)^3}{(n+1)^2}+\frac{4y-12y^2+8y^3-2y\left[y^n+y^{n+1}+(1-y)^{n+1}\right]}{(n+1)^4}\right\}.
\end{aligned}$$

Proof. We can readily set up this lemma by using Lemma 2.2 and the linearity of the operators provided by (6). □

Owing to Lemma 2.3, we can easily obtain the following conclusion. Our purpose in presenting this finding is to facilitate the proof of our theorems.

Corollary 2.4. *Let $y\in[0,1]$, $\lambda\in[-1,1]$, $n\in\mathbb{N}$, so the following limits are hold.*

$$\lim_{n\to\infty}nK_n^\lambda(\Psi_1(t,y);y)=\frac{1-2y}{2},$$
$$\lim_{n\to\infty}nK_n^\lambda(\Psi_2(t,y);y)=y(1-y),$$
$$\lim_{n\to\infty}n^2K_n^\lambda(\Psi_4(t,y);y)=3y^2-6y^3+3y^4.$$

3. Approximation of the newly defined operators for univariate functions

In our consideration, the space of all continuous functions on the closed interval $[0,1]$ is represented by the notation $C[0,1]$. Also, $C[0,1]$ is a normed space equipped with the norm

$$||\zeta||_{C[0,1]} = \sup\{|\zeta(y)| : y \in [0,1]\}.$$

Moreover, we use the symbol $C^2[0,1]$ to imply the space of functions $\zeta \in C[0,1]$ such that ζ' and ζ'' also belong to $C[0,1]$.

Now, we give other concepts related to our subject. For a function $\zeta \in C[0,1]$, the first modulus of continuity is defined by

$$\omega(\zeta,\delta) = \sup_{0<h\leq\delta,\ y\in[0,1]} |\zeta(y+h) - \zeta(y)|$$

and also the second order modulus of smoothness on the same interval is presented by

$$\omega_2(\zeta, \sqrt{\delta}) = \sup_{0<h\leq \sqrt{\delta}} \sup_{y\in[0,1]} |\zeta(y+2h) - 2\zeta(y+h) + \zeta(y)|.$$

Additionally, the Peetre's $K-$functional is given by

$$K_2(\zeta,\delta) = \inf\{||\zeta - g||_{C[0,1]} + \delta||g''||_{C[0,1]} \ : \ g, g', g'' \in C[0,1]\}.$$

Firstly, we present a Korovkin type approximation theorem for the modified λ-Bernstein-Kantorovich operators $K_n^\lambda(\zeta;y)$.

Theorem 3.1. *Suppose that $\zeta \in C[0,1]$ and $\lambda \in [-1,1]$. Then, the operators $K_n^\lambda(\zeta;y)$ converge uniformly to this function ζ.*

Proof. By advantage of Korovkin theorem [29], it is adequate to demonstrate that

$$\lim_{n\to\infty} ||K_n^\lambda(e_i;y) - y^i||_{C[0,1]} = 0, \ \text{for } i = 0,1,2.$$

Thanks to the equalities given in (7), (8), and (9) of Lemma 2.2, it is easy to conclude that these conditions are hold. Hence, the proof is finalized. □

Secondly, we present a direct result for the operators $K_n^\lambda(\zeta;y)$. To do that, at the outset, we define a new beneficial operator $\overline{K}_n^\lambda$ which facilitates our job. It is defined by

$$\overline{K}_n^\lambda(\zeta;y) = K_n^\lambda(\zeta;y) - \zeta(K_n^\lambda(t;y)) + \zeta(y). \tag{10}$$

We give the following lemma, whose proof is ignored due to its prevalence.

Lemma 3.2. *Based on the definition of $\overline{K}_n^\lambda$, the subsequent equalities are obtained:*

i) $\overline{K}_n^\lambda(e_0;y) = 1,$

ii) $\overline{K}_n^\lambda(e_1;y) = y,$

iii) $\overline{K}_n^\lambda(\Psi_1(t,y);y) = 0.$

Lemma 3.3. *Let $\zeta \in C^2[0,1]$. Then, it is obtained that*

$$\left|\overline{K}_n^\lambda(\zeta;y) - \zeta(y)\right| \leq \delta_n(y)||\zeta''||_{C[0,1]},$$

where $\delta_n(y) = \frac{1}{2}\left\{\beta_n(y) + \alpha_n^2(y)\right\}.$

Proof. Let us take advantage of Taylor expansion given as

$$\zeta(t) = \zeta(y) + \Psi_1(t,y)\zeta'(y) + \int_y^t \Psi_1(t,u)\zeta''(u)du.$$

Since $\overline{K}_n^{\lambda}$ is linear, we can write

$$\overline{K}_n^{\lambda}(\zeta(t);y) - \zeta(y) = \overline{K}_n^{\lambda}\left(\int_y^t \Psi_1(t,u)\zeta''(u)du;\ y\right)$$

thanks to Lemma 3.2. By the Lemma 2.3 and the inequality

$$\left|\int_y^t \Psi_1(t,u)\zeta''(u)du\right| \leq \|\zeta''\|_{C[0,1]}\frac{\Psi_2(t,y)}{2},$$

we obtain

$$\begin{aligned}\left|\overline{K}_n^{\lambda}(\zeta;y) - \zeta(y)\right| &\leq \left|K_n^{\lambda}\left(\int_y^t \Psi_1(t,u)\zeta''(u)du\ ;\ y\right) - \int_y^{K_n^{\lambda}(e_1;y)}\left(K_n^{\lambda}(e_1;y) - u\right)\zeta''(u)du\right| \\ &\leq \frac{\|\zeta''\|_{C[0,1]}}{2}\left\{K_n^{\lambda}\left(\Psi_2(t,y);y\right) + \left[K_n^{\lambda}(\Psi_1(t,y);y)\right]^2\right\} \\ &= \frac{\|\zeta''\|_{C[0,1]}}{2}\left\{\beta_n(y) + \alpha_n^2(y)\right\}.\end{aligned}$$

Thus, we deduce the intended outcome. □

Theorem 3.4. *For $y \in [0,1]$ and $\zeta \in C[0,1]$, we have*

$$\left|K_n^{\lambda}(\zeta;y) - \zeta(y)\right| \leq 2C\omega_2(\zeta, \sqrt{\delta_n(y)}) + \omega\left(\zeta, \alpha_n(y)\right), \qquad (C \text{ is a constant}).$$

Proof. Using the definition of $\overline{K}_n^{\lambda}$ given in (10) , for any $g \in C^2[0,1]$, we acquire

$$\left|K_n^{\lambda}(\zeta;y) - \zeta(y)\right| \leq \left|\overline{K}_n^{\lambda}(\zeta - g;y) - (\zeta - g)(y) + \overline{K}_n^{\lambda}(g;y) - g(y)\right| + \left|\zeta(K_n^{\lambda}(e_1;y)) - \zeta(y)\right|.$$

By using Lemma 3.3 and the modulus of continuity, we get

$$\left|K_n^{\lambda}(\zeta;y) - \zeta(y)\right| \leq 2\|\zeta - g\|_{C[0,1]} + \delta_n(y)\|g''\|_{C[0,1]} + \omega\left(\zeta, (K_n^{\lambda}(e_1;y) - y)\right).$$

In light of Theorem 2.4 in [23], p.177, if we take the infimum over $g \in C[0,1]$, we obtain the result

$$\left|K_n^{\lambda}(\zeta;y) - \zeta(y)\right| \leq 2C\omega_2(\zeta;\ \sqrt{\delta_n(y)}) + \omega(\zeta;\alpha_n(y)).$$

Thus, we deduce the intended outcome. □

Now, we give the Voronovskaja-type asymptotic theorem.

Theorem 3.5. *For any $\zeta \in C^2[0,1]$, we have*

$$\lim_{n\to\infty} n\left(K_n^{\lambda}(\zeta;y) - \zeta(y)\right) = \zeta'(y)\frac{1-2y}{2} + \zeta''(y)\frac{y(1-y)}{2},$$

uniformly on $C[0,1]$.

Proof. Let $y \in [0,1]$ be fixed. According to Taylor formula, we have

$$\zeta(t) = \zeta(y) + \zeta'(y)\Psi_1(t,y) + \frac{1}{2}\zeta''(y)\Psi_2(t,y) + \epsilon(t,y)\Psi_2(t,y)$$

where $\epsilon(t,y)$ represents the Peano form of the remainder. Since $\epsilon(\cdot,y) \in C[0,1]$, it is guaranteed that $\lim_{t\to y}\epsilon(t,y) = 0$.

Considering that K_n^λ is linear, we have

$$K_n^\lambda(\zeta(t);y) = \zeta(y)K_n^\lambda(e_0;y) + \zeta'(y)K_n^\lambda(\Psi_1(t,y);y) + \frac{1}{2}\zeta''(y)K_n^\lambda(\Psi_2(t,y);y) + K_n^\lambda(\epsilon(t,y)\Psi_2(t,y);y).$$

Lemma 2.3 allows us to acquire

$$\begin{aligned} K_n^\lambda(\zeta(t);y) - \zeta(y) =& \zeta'(y)\left\{\frac{1-2y}{2(n+1)} + \lambda\frac{1-y^{n+1}-(1-y)^{n+1}}{(n+1)^2}\right\} \\ &+ \frac{1}{2}f''(y)\left(\frac{y(1-y)}{n+1} + \frac{1-6y+6y^2}{3(n+1)^2} - \lambda\frac{2y\left(y^n - y^{n+1} - (1-y)^{n+1}\right)}{(n+1)^2}\right) \\ &+ K_n^\lambda(\epsilon(t,y)\Psi_2(t,y);y). \end{aligned}$$

Following the application of Cauchy-Schwarz inequality to the final term of the right side, we obtain

$$\lim_{n\to\infty} n\left\{K_n^\lambda(\epsilon(t,y)\Psi_2(t,y);y)\right\} \le \sqrt{\lim_{n\to\infty} K_n^\lambda(\epsilon^2(t,y);y)}\sqrt{\lim_{n\to\infty} n^2 K_n^\lambda(\Psi_4(t,y);y)}.$$

Since $\lim_{n\to\infty} K_n^\lambda(\epsilon^2(t,y);y) = 0$ and $\lim_{n\to\infty} n^2 K_n^\lambda(\Psi_4(t,y);y)$ is finite because of Corollary 2.4, we get

$$\lim_{n\to\infty} n\left\{K_n^\lambda(\epsilon(t,y)\Psi_2(t,y);y)\right\} = 0.$$

Hence we conclude that

$$\lim_{n\to\infty} n(K_n^\lambda(\zeta;y) - \zeta(y)) = \zeta'(y)\frac{1-2y}{2} + \zeta''(y)\frac{y(1-y)}{2}$$

and the proof is completed. □

4. Approximation of the bivariate tensor product operators

Let $D = [0,1]\times[0,1]$, we use the symbol $C(D)$ to state the set of all real-valued continuous functions on D. For $\zeta \in C(D)$, the usual norm is defined by

$$\|\zeta\|_{C(D)} = \sup_{(y,z)\in D} |\zeta(y,z)|. \tag{11}$$

Let us assume that

$$C^2(D) = \left\{\zeta \in C(D) : \frac{\partial^i\zeta}{\partial y^i}, \frac{\partial^i\zeta}{\partial z^i} \in C(D), i = 1,2\right\}.$$

For $\zeta \in C^2(D)$, this space is equipped with the norm as follows:

$$\|\zeta\|_{C^2(D)} = \|\zeta\|_{C(D)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i\zeta}{\partial y^i}\right\|_{C(D)} + \left\|\frac{\partial^i\zeta}{\partial z^i}\right\|_{C(D)}\right).$$

The usual modulus of continuity of $\zeta \in C(D)$ are given as follows:

$$\overline{\omega}(\zeta,\xi) = \sup\left\{|\zeta(s_1,t_1)-\zeta(s_2,t_2)| : (s_1,t_1),(s_2,t_2)\in C(D),\ \sqrt{(s_1-s_2)^2+(t_1-t_2)^2}\le \xi\right\}.$$

In addition, the Peetre's K-functional of $\zeta \in C^2(D)$ is given as follows:

$$K(\zeta,\xi) = \inf\left\{\|\zeta-g\|_{C(D)} + \xi\|g\|_{C^2(D)} : g\in C^2(D)\right\},$$

where $\xi > 0$.

There exists a constant $M > 0$ such that

$$K(\zeta,\xi) \le M\left\{\overline{\omega_2}(\zeta,\sqrt{\xi}) + \min(1,\xi)\|\zeta\|_{C^2(D)}\right\}, \tag{12}$$

where $\overline{\omega_2}$ denotes the second order modulus of continuity of $\zeta \in C(D)$ (see [15]). The constant M is independent of ζ and ξ.

For $\zeta \in C(D)$, $(y,z)\in D$, $m,n\in\mathbb{N}$ and $\lambda_1,\lambda_2\in[-1,1]$, we construct the following bivariate tensor product operators,

$$K_{m,n}^{\lambda_1,\lambda_2}(\zeta;y,z) = (m+1)(n+1)\sum_{i=0}^{m}\sum_{j=0}^{n} b_{m,i}^{\lambda_1}(y)b_{n,j}^{\lambda_2}(z)\int_{\frac{i}{m+1}}^{\frac{i+1}{m+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}\zeta(u,v)dudv, \tag{13}$$

where $b_{m,i}^{\lambda_1}(y)$ and $b_{n,j}^{\lambda_2}(z)$ are defined in (5). We can say that the operators given in (13) are positive and linear.

Let $e_{ij}(s,t) = s^i t^j,\ i,j\in\{0,1,...,4\}$ and $\Psi_{ij}(s,t;y,z) = (s-y)^i(t-z)^j,\ i,j\in\{0,1,2,4\}$.

Lemma 4.1. *For $\zeta \in C(D)$, $(y,z)\in D$, $m,n\in\mathbb{N}$ and $\lambda_1,\lambda_2\in[-1,1]$, the following equations are valid:*

$$\begin{aligned}
K_{m,n}^{\lambda_1,\lambda_2}(e_{00};y,z) &= 1,\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{10};y,z) &= y + \frac{1-2y}{2(m+1)} + \lambda_1\frac{1-y^{m+1}-(1-y)^{m+1}}{(m+1)^2},\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{01};y,z) &= z + \frac{1-2z}{2(n+1)} + \lambda_2\frac{1-z^{n+1}-(1-z)^{n+1}}{(n+1)^2},\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{11};y,z) &= K_{m,n}^{\lambda_1,\lambda_2}(e_{10};y,z)K_{m,n}^{\lambda_1,\lambda_2}(e_{01};y,z),\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{20};y,z) &= y^2 + \frac{2y-3y^2}{(m+1)} + \frac{1-6y+6y^2}{3(m+1)^2} + \lambda_1\frac{2y(1-y^m)}{(m+1)^2},\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{02};y,z) &= z^2 + \frac{2z-3z^2}{(n+1)} + \frac{1-6z+6z^2}{3(n+1)^2} + \lambda_2\frac{2z(1-z^n)}{(n+1)^2},\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{22};y,z) &= K_{m,n}^{\lambda_1,\lambda_2}(e_{20};y,z)K_{m,n}^{\lambda_1,\lambda_2}(e_{02};y,z),\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{30};y,z) &= y^3 + \frac{9y^2-12y^3}{2(m+1)} + \frac{7y-27y^2+22y^3}{2(m+1)^2} + \frac{1-14y+36y^2-24y^3}{4(m+1)^3}\\
&\quad +\lambda_1\left[\frac{3y^2\left(1-y^{m-1}\right)}{(m+1)^2} + \frac{3y(1-y)}{(m+1)^3} + \frac{1-y^{m+1}-(1-y)^{m+1}}{2(m+1)^4}\right],\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{03};y,z) &= z^3 + \frac{9z^2-12z^3}{2(n+1)} + \frac{7z-27z^2+22z^3}{2(n+1)^2} + \frac{1-14z+36z^2-24z^3}{4(n+1)^3}\\
&\quad +\lambda_2\left[\frac{3z^2\left(1-z^{n-1}\right)}{(n+1)^2} + \frac{3z(1-z)}{(n+1)^3} + \frac{1-z^{n+1}-(1-z)^{n+1}}{2(n+1)^4}\right],\\
K_{m,n}^{\lambda_1,\lambda_2}(e_{40};y,z) &= y^4 + \frac{8y^3-10y^4}{m+1} + \frac{15y^2-48y^3+35y^4}{(m+1)^2} + \frac{6y-45y^2+88y^3-50y^4}{(m+1)^3}
\end{aligned}$$

$$+\frac{1-30y+150y^2-240y^3+120y^4}{5(m+1)^4}+\lambda_1\left[\frac{4\left(y^3-y^{m+1}\right)}{(m+1)^2}+\frac{12y^2(1-y)}{(m+1)^3}+\frac{6y-12y^2+8y^3-2y^{m+1}}{(m+1)^4}\right],$$

$$K_{m,n}^{\lambda_1,\lambda_2}(e_{04};y,z) = z^4+\frac{8z^3-10z^4}{n+1}+\frac{15z^2-48z^3+35z^4}{(n+1)^2}+\frac{6z-45z^2+88z^3-50z^4}{(n+1)^3}+\frac{1-30z+150z^2-240z^3+120z^4}{5(n+1)^4}+\lambda_2\left[\frac{4\left(z^3-z^{n+1}\right)}{(n+1)^2}+\frac{12z^2(1-z)}{(n+1)^3}+\frac{6z-12z^2+8z^3-2z^{n+1}}{(n+1)^4}\right].$$

Proof. Using Lemma 2.2 and the description of the bivariate operators in (13), İt is clear that the previously mentioned equalities hold. □

Corollary 4.2. *Thanks to Lemma 4.1, the central moments of the operators $K_{m,n}^{\lambda_1,\lambda_2}$ are given as follows:*

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{10}(s,t;y,z);y,z) =\frac{1-2y}{2(m+1)}+\lambda_1\frac{1-y^{m+1}-(1-y)^{m+1}}{(m+1)^2},$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{11}(s,t;y,z);y,z) =K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{10}(s,t;y,z);y,z)K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{01}(s,t;y,z);y,z),$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{01}(s,t;y,z);y,z) =\frac{1-2z}{2(n+1)}+\lambda_2\frac{1-z^{n+1}-(1-z)^{n+1}}{(n+1)^2},$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{20}(s,t;y,z);y,z) =\frac{y-y^2}{(m+1)}+\frac{1-6y+6y^2}{3(m+1)^2}+\lambda_1\left[\frac{2y^{m+2}-2y^{m+1}+2y(1-y)^{m+1}}{(m+1)^2}\right],$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{02}(s,t;y,z);y,z) =\frac{z-z^2}{(n+1)}+\frac{1-6z+6z^2}{3(n+1)^2}+\lambda_2\left[\frac{2z^{n+2}-2z^{n+1}+2z(1-z)^{n+1}}{(n+1)^2}\right],$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{40}(s,t;y,z);y,z) =\frac{3y^2-6y^3+3y^4}{(m+1)^2}+\frac{5y-31y^2+52y^3-26y^4}{(m+1)^3}+\frac{1-30y+150y^2-240y^3+120y^4}{5(m+1)^4}+\lambda_1\left[\frac{4y^3(1-y)^{m+1}-4y^{m+1}(1-3y+3y^2-y^3)}{(m+1)^2}+\frac{4y-12y^2+8y^3-2y(y^m+y^{m+1}+(1-y)^{m+1})}{(m+1)^4}\right],$$

$$K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{04}(s,t;y,z);y,z) =\frac{3z^2-6z^3+3z^4}{(n+1)^2}+\frac{5z-31z^2+52z^3-26z^4}{(n+1)^3}+\frac{1-30z+150z^2-240z^3+120z^4}{5(n+1)^4}+\lambda_2\left[\frac{4z^3(1-z)^{n+1}-4z^{n+1}(1-3z+3z^2-z^3)}{(n+1)^2}+\frac{4z-12z^2+8z^3-2z(z^n+z^{n+1}+(1-z)^{n+1})}{(n+1)^4}\right].$$

Theorem 4.3. *For any $\zeta\in C(D)$, we have*

$$\lim_{m,n\to\infty}||K_{m,n}^{\lambda_1,\lambda_2}(\zeta;y,z)-\zeta(y,z)||_{C(D)}=0.$$

Proof. Using Lemma 4.1 and the norm given in (11), for $m, n \to \infty$, we get

$$\|K_{m,n}^{\lambda_1,\lambda_2}(e_{00}) - e_{00}\|_{C(D)} \to 0;$$
$$\|K_{m,n}^{\lambda_1,\lambda_2}(e_{10}) - e_{10}\|_{C(D)} \to 0;$$
$$\|K_{m,n}^{\lambda_1,\lambda_2}(e_{01}) - e_{01}\|_{C(D)} \to 0;$$
$$\|K_{m,n}^{\lambda_1,\lambda_2}(e_{20} + e_{02}) - (e_{20} + e_{02})\|_{C(D)} \to 0.$$

From the Volkov theorem [41], proof of the theorem is completed. □

Now, we will show following Voronovskaja-type theorem for these operators' asymptotic approximation.

Theorem 4.4. *If $\zeta \in C^2(D)$, then we get the following limit*

$$\lim_{m\to\infty} m\left[K_{m,m}^{\lambda_1,\lambda_2}(\zeta(s,t);y,z) - \zeta(y,z)\right] = \zeta_y(y,z)(\frac{1}{2} - y) + \zeta_z(y,z)(\frac{1}{2} - z) + \frac{1}{2}\left\{\zeta_{yy}(y,z)(y - y^2) + \zeta_{zz}(y,z)(z - z^2)\right\}.$$

Proof. Using the Taylor formula for any $(y,z) \in D$, we obtain

$$\begin{aligned}\zeta(s,t) &= \zeta(y,z) + \zeta_y(y,z)\Psi_{10}(s,t;y,z) + \zeta_z(y,z)\Psi_{01}(s,t;y,z)\\ &+ \frac{1}{2}\left[\zeta_{yy}(y,z)\Psi_{20}(s,t;y,z) + 2\zeta_{yz}(y,z)\Psi_{11}(s,t;y,z) + \zeta_{zz}(y,z)\Psi_{02}(s,t;y,z)\right]\\ &+ \phi(s,t;y,z)\sqrt{\Psi_{40}(s,t;y,z) + \Psi_{04}(s,t;y,z)}\end{aligned} \tag{14}$$

for $(s,t) \in D$, where $\phi(.,.;y,z) \in C(D)$ and $\phi(s,t;y,z) \to 0$ when $(s,t) \to (y,z)$.
If we apply the $K_{m,m}^{\lambda_1,\lambda_2}$ on (14), we yield

$$\begin{aligned}&K_{m,m}^{\lambda_1,\lambda_2}(\zeta(s,t);y,z)\\ &= \zeta(y,z) + \zeta_y(y,z)K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{10}(s,t;y,z):y,z) + \zeta_z(y,z)K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{01}(s,t;y,z);y,z)\\ &+ \frac{1}{2}\left[\zeta_{yy}(y,z)K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{20}(s,t;y,z);y,z) + 2\zeta_{yz}(y,z)K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{11}(s,t;y,z);y,z)\right.\\ &\left.+\zeta_{zz}(y,z)K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{02}(s,t;y,z);y,z)\right]\\ &+ K_{m,m}^{\lambda_1,\lambda_2}(\phi(s,t;y,z)\sqrt{(\Psi_{40}(s,t;y,z) + \Psi_{04}(s,t;y,z)};y,z).\end{aligned}$$

Upon applying the Cauchy-Schwarz inequality to

$$K_{m,m}^{\lambda_1,\lambda_2}(\phi(s,t;y,z)\sqrt{(\Psi_{40}(s,t;y,z) + \Psi_{04}(s,t;y,z)};y,z),$$

we have

$$\begin{aligned}&m\left|K_{m,m}^{\lambda_1,\lambda_2}(\phi(s,t;y,z)\sqrt{(\Psi_{40}(s,t;y,z) + \Psi_{04}(s,t;y,z)};y,z)\right|\\ &\leq \sqrt{K_{m,m}^{\lambda_1,\lambda_2}(\phi^2(s,t;y,z);y,z)}\sqrt{m^2\left\{K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{40}(s,t;y,z);y,z) + K_{m,m}^{\lambda_1,\lambda_2}(\phi(\Psi_{04}(s,t;y,z);y,z)\right\}}.\end{aligned}$$

Taking into account Theorem 4.3 and $\phi(\cdot,\cdot;y,z) \in C(D)$, we have $\phi(s,t;y,z) \to 0$ when $(s,t) \to (y,z)$. Then, we yield

$$\lim_{m\to\infty} K_{m,m}^{\lambda_1,\lambda_2}(\phi^2(s,t;y,z);y,z) = 0,$$

uniformly on D.

Intercalarily, from Corollary 4.2, we obtain that

$$\begin{aligned}\lim_{m\to\infty} m^2 K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{40}(s,t;y,z);y,z) &= 3y^2(1-y)^2,\\ \lim_{m\to\infty} m^2 K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{04}(s,t;y,z);y,z) &= 3z^2(1-z)^2.\end{aligned}$$

Hence,

$$\lim_{m\to\infty} m\left\{K_{m,m}^{\lambda_1,\lambda_2}\left(\phi(s,t;y,z)\sqrt{\Psi_{40}(s,t;y,z)+\Psi_{04}(s,t;y,z)};y,z\right)\right\}=0. \tag{15}$$

By Corollary 4.2, we obtain

$$\lim_{m\to\infty} m\left(K_{m,m}^{\lambda_1,\lambda_2}(\Psi_{11}(s,t;y,z);y,z)\right)=0. \tag{16}$$

Thanks to (15), (16), and Corollary 4.2, the proof of theorem is made as follows:

$$\lim_{m\to\infty} m\left\{K_{m,m}^{\lambda_1,\lambda_2}\left(\zeta(s,t);y,z\right)-\zeta(y,z)\right\}=\zeta_y(y,z)(\frac{1}{2}-y)+\zeta_z(y,z)(\frac{1}{2}-z)+\frac{1}{2}\left\{\zeta_{yy}(y,z)(y-y^2)+\zeta_{zz}(y,z)(z-z^2)\right\}.$$

Theorem 4.4 is proved. □

Theorem 4.5. *Let $\zeta\in C^2(D)$, then we obtain the following inequality:*

$$\left|K_{m,n}^{\lambda_1,\lambda_2}\left(\zeta(s,t);y,z\right)-\zeta(y,z)\right|\le\overline{M}\left\{\overline{\omega_2}\left(\zeta;\frac{\sqrt{S_{m,n}(y,z)}}{2}\right)+\min\left(1,\frac{S_{m,n}(y,z)}{4}\right)\|\zeta\|_{C^2(D)}\right\}+\overline{\omega}(\zeta;\xi_{m,n}(y,z)),$$

where $\overline{M}>0$ is a constant, and it is independent of ζ and $\xi_{m,n}$.

Proof. We take advantage of the auxiliary operators defined by

$$\overline{K_{m,n}^{\lambda_1,\lambda_2}}\left(\zeta(s,t);y,z\right) = K_{m,n}^{\lambda_1,\lambda_2}\left(\zeta(s,t);y,z\right)+\zeta(y,z)-\zeta\left(K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z),K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)\right).$$

From Lemma 4.1, we yield

$$\begin{aligned}&\overline{K_{m,n}^{\lambda_1,\lambda_2}}(e_{00};y,z)=1,\\ &\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\Psi_{10}(s,t;y,z);y,z)=0,\\ &\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\Psi_{01}(s,t;y,z);y,z)=0.\end{aligned}$$

Now, let $\zeta\in C^2(D)$ and $(s,t)\in D$. By the Taylor formula, we can obtain

$$\begin{aligned}\zeta(s,t)-\zeta(y,z)=\frac{\partial\zeta(y,z)}{\partial y}\Psi_{10}(s,t;y,z)+\int_y^s\Psi_{10}(s,t;u,z)\frac{\partial^2\zeta(u,z)}{\partial u^2}du\\ +\frac{\partial\zeta(y,z)}{\partial z}\Psi_{01}(s,t;y,z)+\int_z^t\Psi_{01}(s,t;y,v)\frac{\partial^2\zeta(y,v)}{\partial v^2}dv.\end{aligned} \tag{17}$$

If we operate $\overline{K_{m,n}^{\lambda_1,\lambda_2}}$ on the Taylor formula given in (17), we obtain

$$\begin{aligned}
&\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\zeta(s,t);y,z)-\zeta(y,z)\overline{K_{m,n}^{\lambda_1,\lambda_2}}(e_{00}(s,t);y,z)\\
&=\frac{\partial\zeta(y,z)}{\partial y}\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\Psi_{10}(s,t;y,z);y,z)+\overline{K_{m,n}^{\lambda_1,\lambda_2}}\left(\int_y^s\Psi_{10}(s,t;u,z)\frac{\partial^2\zeta(u,z)}{\partial u^2}du\;;y,z\right)\\
&\quad+\frac{\partial\zeta(y,z)}{\partial z}\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\Psi_{01}(s,t;y,z);y,z)+\overline{K_{m,n}^{\lambda_1,\lambda_2}}\left(\int_z^t\Psi_{01}(s,t;y,v)\frac{\partial^2\zeta(y,v)}{\partial v^2}dv\;;y,z\right)\\
&=K_{m,n}^{\lambda_1,\lambda_2}\left(\int_y^s\Psi_{10}(s,t;u,z)\frac{\partial^2\zeta(u,z)}{\partial u^2}du\;;y,z\right)-\int_y^{K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)}\left(K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)-u\right)\frac{\partial^2\zeta(u,z)}{\partial u^2}du\\
&\quad+K_{m,n}^{\lambda_1,\lambda_2}\left(\int_z^t\Psi_{01}(s,t;y,v)\frac{\partial^2\zeta(y,v)}{\partial v^2}dv\;;y,z\right)-\int_z^{K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)}\left(K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)-v\right)\frac{\partial^2\zeta(y,v)}{\partial v^2}dv,
\end{aligned}$$

so we get

$$\begin{aligned}
&\left|\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\zeta(s,t);y,z)-\zeta(y,z)\right|\\
&\le K_{m,n}^{\lambda_1,\lambda_2}\left(\int_y^s|\Psi_{10}(s,t;u,z)|\left|\frac{\partial^2\zeta(u,z)}{\partial u^2}\right|du\;;y,z\right)+\int_y^{K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)}\left|K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)-u\right|\left|\frac{\partial^2\zeta(u,z)}{\partial u^2}\right|du\\
&\quad+K_{m,n}^{\lambda_1,\lambda_2}\left(\int_z^t|\Psi_{01}(s,t;y,v)|\left|\frac{\partial^2\zeta(y,v)}{\partial v^2}\right|dv\;;y,z\right)+\int_z^{K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)}\left|K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)-v\right|\left|\frac{\partial^2\zeta(y,v)}{\partial v^2}\right|dv\\
&\le\frac{\|\zeta\|_{C^2(D)}}{2}\left\{K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{20}(s,t;y,z);y,z)+\left[K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)-y\right]^2\right.\\
&\quad\left.+K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{02}(s,t;y,z);y,z)+\left[K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)-z\right]^2\right\}.
\end{aligned}$$

Let $\xi_{m,n}(y,z)=\sqrt{\left[K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z)-y\right]^2+\left[K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)-z\right]^2}$, $\gamma_m(y)=\sqrt{K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{20}(s,t;y,z);y,z)}$, and $\gamma_n(z)=\sqrt{K_{m,n}^{\lambda_1,\lambda_2}(\Psi_{02}(s,t;y,z);y,z)}$. If we choose

$$S_{m,n}(y,z)=\gamma_m^2(y)+\gamma_n^2(z)+\xi_{m,n}^2(y,z),$$

then we get

$$\left|\overline{K_{m,n}^{\lambda_1,\lambda_2}}(\zeta(s,t);y,z)-\zeta(y,z)\right|\le\frac{\|\zeta\|_{C^2(D)}}{2}S_{m,n}(y,z). \tag{18}$$

By the inequality (18), we reach the following result

$$\begin{aligned}
\left|K_{m,n}^{\lambda_1,\lambda_2}(\zeta(s,t);y,z)-\zeta(y,z)\right|&=\left|\overline{K_{m,n}^{\lambda_1,\lambda_2}}((\zeta-g)(s,t);y,z)-(\zeta-g)(y,z)\right|+\left|\overline{K_{m,n}^{\lambda_1,\lambda_2}}(g(s,t);y,z)-g(y,z)\right|\\
&\quad+\left|\zeta\left(K_{m,n}^{\lambda_1,\lambda_2}(e_{10}(s,t);y,z),K_{m,n}^{\lambda_1,\lambda_2}(e_{01}(s,t);y,z)\right)-\zeta(y,z)\right|\\
&\le 2\|\zeta-g\|_{C(D)}+\frac{\|\zeta\|_{C^2(D)}}{2}S_{m,n}(y,z)+\overline{\omega}(\zeta;\xi_{m,n}(y,z)).
\end{aligned}$$

Firstly, if we take over infimum on $g\in C^2(D)$ and secondly use the inequality given in (12), we obtain

$$\begin{aligned}
\left|K_{m,n}^{\lambda_1,\lambda_2}(\zeta(s,t);y,z)-\zeta(y,z)\right|&\le 2K\left(\zeta;\frac{S_{m,n}(y,z)}{4}\right)+\overline{\omega}(\zeta;\xi_{m,n}(y,z))\\
&\le\overline{M}\left\{\overline{\omega_2}\left(\zeta;\frac{\sqrt{S_{m,n}(y,z)}}{2}\right)+\min\left(1,\frac{S_{m,n}(y,z)}{4}\right)\|\zeta\|_{C^2(D)}\right\}+\overline{\omega}(\zeta;\xi_{m,n}(y,z)),
\end{aligned}$$

which provides the evidence. □

5. Graphical analysis

In this section, we present some numerical examples to illustrate the convergence properties of the modified λ-Bernstein-Kantorovich operators $K_n^{\lambda}(\zeta;y)$, we also compare the convergence effect with that of λ-Bernstein-Kantorovich operators $\widetilde{K}_n^{\lambda}(\zeta;y)$ and the classical Bernstein-Kantorovich operators $K_n(\zeta;y)$. In accordance with this purpose, we choose some functions and test its convergence behavior for different parameters. All experimental algorithms are coded using MATLAB R2019b.

Example 5.1. *We take the test function* $\zeta(y) = 1 - \cos(4e^y)$*. The graphs of* $K_n^{\lambda}(\zeta;y)$ *with* $n = 10$ *and* $\lambda = -1, 0, 1$ *are shown in Figure 1. In Figure 2, we fix* $\lambda = -1$*, operators* $K_n^{-1}(\zeta;y)$ *with* $n = 10, 50, 100$ *and* $\zeta(y)$ *are shown. It can be seen from Figure 2 that with the increase of n, the convergence effect of the operator on* $\zeta(y)$ *is getting better and better. For comparison with* λ*-Bernstein-Kantorovich operators and the classical form, Figure 3 shows the absolute error of* $K_n^{\lambda}(\zeta;y)$*,* $\widetilde{K}_n^{\lambda}(\zeta;y)$ *and* $K_n(\zeta;y)$ *with* $n = 10$*,* $\lambda = 1$ *on* $\zeta(y)$*.*

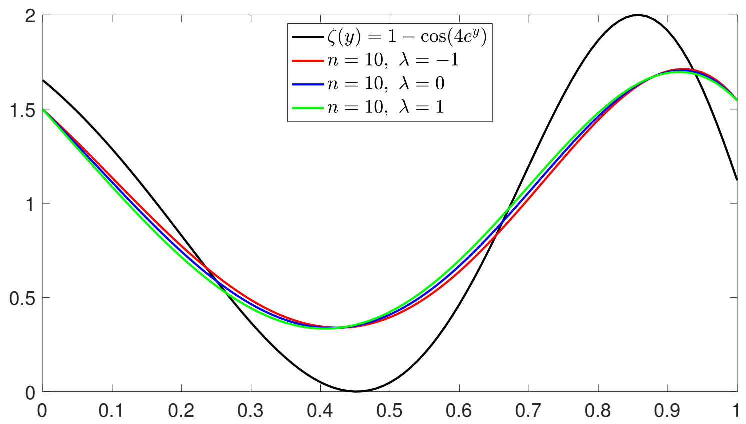

Figure 1: The convergence of $K_{10}^{-1}(\zeta;y)$, $K_{10}^{0}(\zeta;y)$, $K_{10}^{1}(\zeta;y)$ to $\zeta(y)$

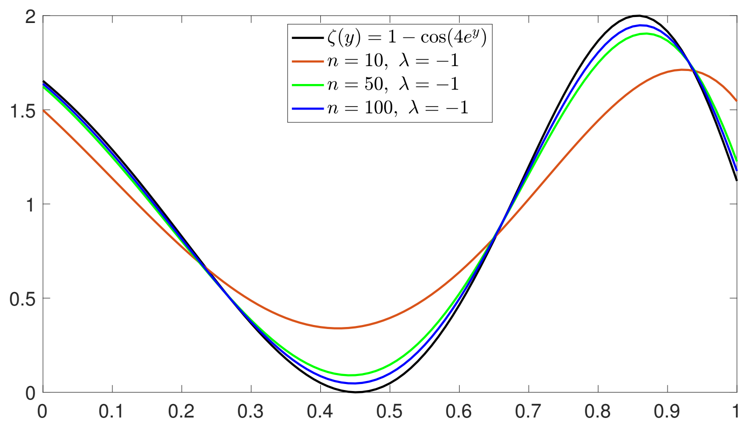

Figure 2: The convergence of $K_{10}^{-1}(\zeta;y)$, $K_{50}^{-1}(\zeta;y)$, $K_{100}^{-1}(\zeta;y)$ to $\zeta(y)$

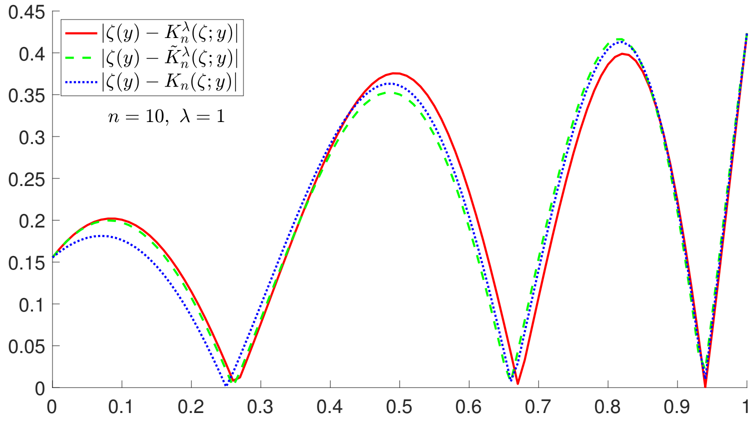

Figure 3: Comparison of errors for $K_{10}^1(\zeta)$, $\widetilde{K}_{10}^1(\zeta)$ and $K_{10}(\zeta)$ to ζ

Example 5.2. *Taking function* $\zeta(y) = \frac{3y^6-2}{y^5+1}$, *Figure 4 and Figure 5 respectively show the approximation performance of the operator when n is fixed and* λ *takes different values, and the approximation error representation when n takes* 10 *and* λ *takes* 1.

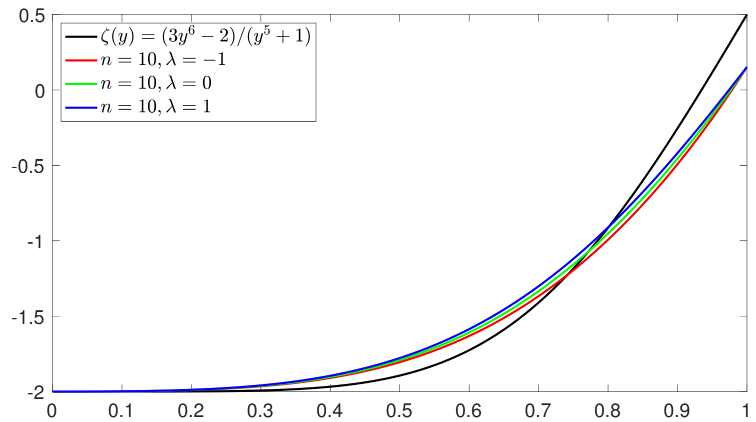

Figure 4: The convergence of $K_{10}^{-1}(\zeta; y)$, $K_{10}^{0}(\zeta; y)$ and $K_{10}^{1}(\zeta; y)$ to $\zeta(y)$

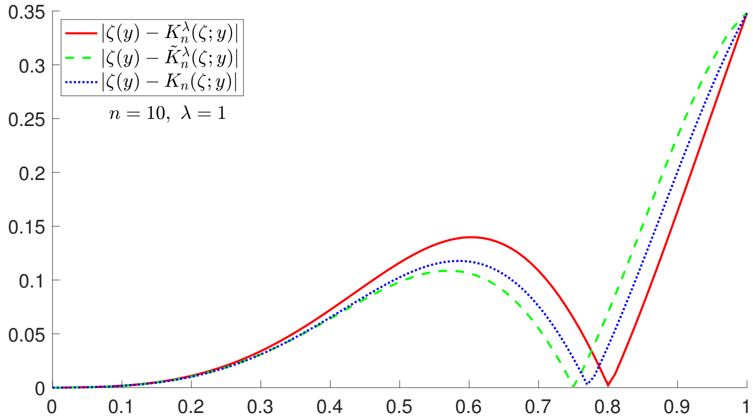

Figure 5: Comparison of errors for $K_{10}^{1}(\zeta)$, $\widetilde{K}_{10}^{1}(\zeta)$ and $K_{10}(\zeta)$ to ζ

Example 5.3. *We take the piecewise continuous function* $\zeta_3(y) = \begin{cases} 4y, \ 0 \le y \le 0.2, \\ 4(1+y)/3, \ 0.2 < y \le 0.5, \\ 4(2-y)/3, \ 0.5 < y \le 0.8, \\ 8(1-y), \ 0.8 < y \le 1 \end{cases}$ *as the test function, Figure 6 and Figure 7 respectively show the approximation effect of the operator to the function* $\zeta_3(y)$ *when* λ *is fixed and* n *takes different values, and the approximation error of the operator to the function* $\zeta_3(y)$ *when* $n = 10$ *and* $\lambda = 1$.

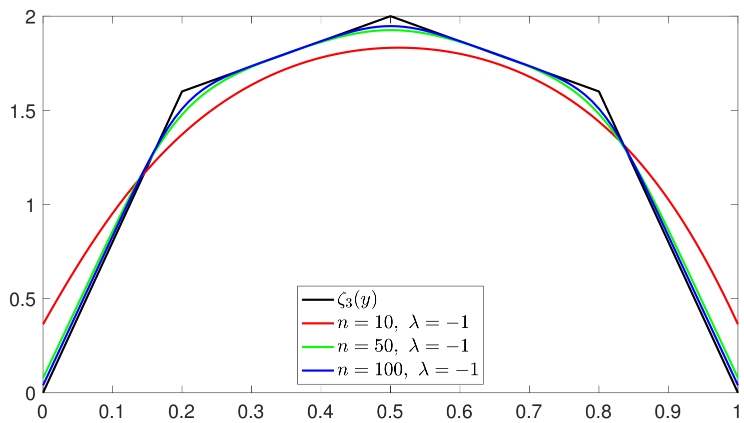

Figure 6: The convergence of $K_{10}^{-1}(\zeta_3; y)$, $K_{50}^{-1}(\zeta_3; y)$ and $K_{100}^{-1}(\zeta_3; y)$ to $\zeta_3(y)$

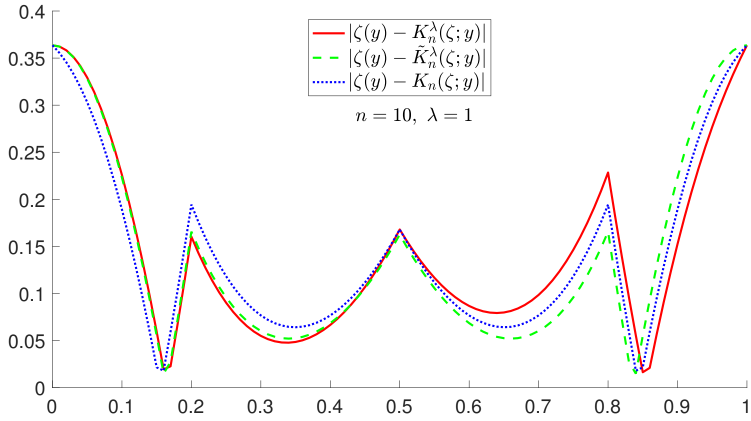

Figure 7: Comparison of errors for $K_{10}^1(\zeta_3)$, $\widetilde{K}_{10}^1(\zeta_3)$ and $K_{10}(\zeta_3)$ to ζ_3

Example 5.4. *Now, we take the test function $\zeta(y) = (y - 0.5)\sin(\pi y)$, Figure 8 shows the convergence of the operator with different values of n when $\lambda = -1$, and Figure 9 shows the error graph of the operator with respect to the approximated function when $n = 10$ and $\lambda = 1$.*

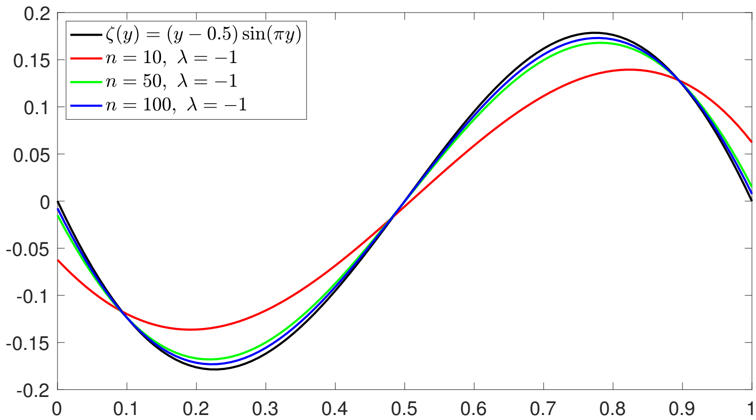

Figure 8: The convergence of $K_{10}^{-1}(\zeta;y)$, $K_{50}^{-1}(\zeta;y)$ and $K_{100}^{-1}(\zeta;y)$ to $\zeta(y)$

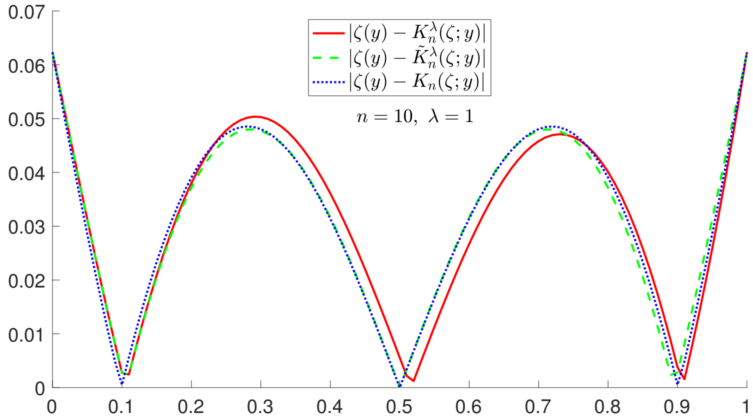

Figure 9: Comparison of errors for $K_{10}^{1}(\zeta)$, $\widetilde{K}_{10}^{1}(\zeta)$ and $K_{10}(\zeta)$ to ζ

Example 5.5. *Finally, for the test function of binary operators, we choose* $\zeta(y,z) = \sin(4y) + \cos(7z)$*, Figure 10 and Figure 11 respectively show the approximation behavior of binary operators on function* $\zeta(y,z)$ *and the partial representation display.*

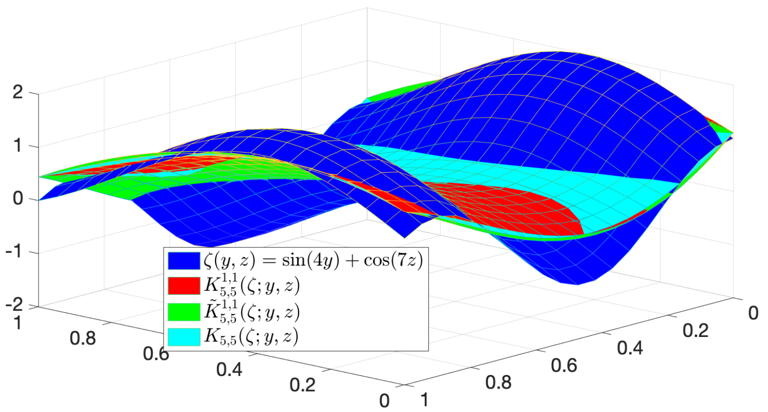

Figure 10: The convergence of $K_{5,5}^{1,1}(\zeta;y,z)$, $\widetilde{K}_{5,5}^{1,1}(\zeta;y,z)$ and $K_{5,5}(\zeta;y,z)$ to $\zeta(y,z)$

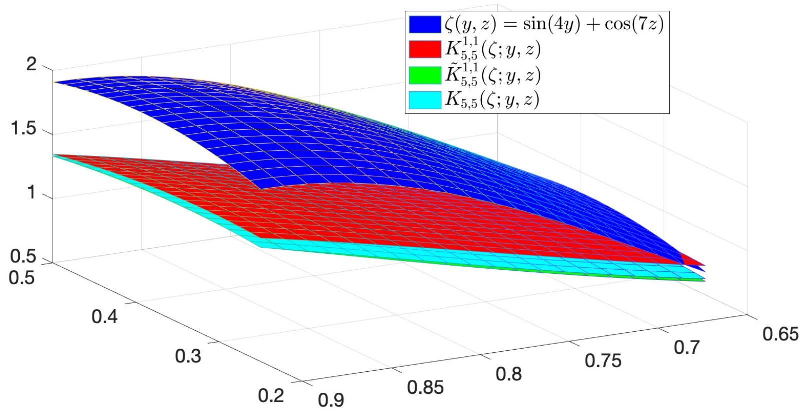

Figure 11: A partial representation of Figure 10

Judging by the performance of above numerical examples, it should be said that the convergence behavior of the newly defined operators and the original λ-Bernstein-Kantorovich type operators has advantages depending on the value of λ, but the newly defined operators will be simpler in form than the original ones.

6. Conclusion

This study explores a kind of modified λ-Bernstein-Kantorovich operators adopting the new λ-Bézier basis. It covers various aspects of the newly defined operators, including approximation properties, convergence rate and Voronovskaja-type asymptotic formula. The adaptability and convergence of the proposed operators are vital aspects of the study and depend on the choice of parameter λ. It explores how different values of λ impacts the performance of operators. The graphs are also used to dream up the performance of operators under various selections of n and λ. Graphs provide a more instinctive understanding of how λ affects the behavior of proposed operators.

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References

[1] T. Acar, M. Bodur, E. Isikli, Bivariate Bernstein Chlodovsky Operators Preserving Exponential Functions and Their Convergence Properties., Numer. Func. Anal. Opt. 45(1) (2024) 16-37.
[2] A. M. Acu, N. Manav, S. Sofonea, Approximation properties of λ-Kantorovich operators, J. Inequal. Appl. (2018) 2018: 202.
[3] G. Agrawal, V. Gupta, Modified Lupaş-Kantorovich operators with Pólya distribution, Rocky Mt. J. Math. 52(6) (2022) 1909-1919.
[4] K. J. Ansari, F. Özger, Z. Ödemiş Özger, Numerical and theoretical approximation results for Schurer-Stancu operators with shape parameter λ, Comp. Appl. Math. (2022) 41: 181.
[5] R. Aslan, A. İzgi, Approximation by One and Two Variables of the Bernstein-Schurer-Type Operators and Associated GBS Operators on Symmetrical Mobile Interval, J. Funct. Space. (2021) 9979286.
[6] R. Aslan, Some approximation results on λ-Szász-Mirakjan-Kantorovich operators, Fundamental J. Math. Appl. 4(3) (2021) 150-158.
[7] R. Aslan, Approximation properties of univariate and bivariate new class λ-Bernstein-Kantorovich operators and its associated GBS operators, Comp. Appl. Math. 42(1) (2023) 34.

[8] R. Aslan, Approximation by Szász-Mirakjan-Durrmeyer operators based on shape parameter λ, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 71(2) (2022) 407-421.
[9] R. Aslan, M. Mursaleen, Some approximation results on a class of new type λ-Bernstein polynomials, J. Math. Inequal. 16(2) (2022) 445-462.
[10] R. Aslan, Rate of approximation of blending type modified univariate λ-Schurer-Kantorovich operators, Kuwait J. Sci. 51(1) (2024) 100168.
[11] S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités, Comm. Soc. Math. Charkow Sér. 2 t. 13 (1912) 1-2.
[12] S. Berwal, S. A. Mohiuddine, A. Kajla, A. Alotaibi, Approximation by Riemann-Liouville type fractional α-Bernstein-Kantorovich operators, Math. Meth. Appl. Sci. 47 (2024) 8275-8288.
[13] R. Bojanic, F. Cheng, Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation, J. Math. Anal. Appl. 141 (1989) 136-151.
[14] N. L. Braha, T. Mansour, M. Mursaleen, T. Acar, Convergence of λ Bernstein operators via power series summability method, J. Appl. Math. Comput. 65 (2020) 125-146.
[15] P. L. Butzer, H. Berens, Semi-Groups of Operators and Approximation. New York:Springer, 1967.
[16] Q. -B. Cai, B. -Y. Lian, G. Zhou, Approximation properties of λ-Bernstein operators, J. Inequal. Appl. (2018) 2018: 61.
[17] Q. -B. Cai, On (p, q)-analogue of modified Bernstein-Schurer operators for functions of one and two variables, J. Appl. Math. Comput. 54 (2017) 1-21.
[18] Q. -B. Cai, Ü. D. Kantar, B. Cekim, Approximation properties for the genuine modified Bernstein-Durrmeyer-Stancu operators, Appl. Math. J. Chinese Univ. 35 (2020) 468-478.
[19] Q. -B. Cai, G. Torun, Ü. Dinlemez Kantar, Approximation Properties of Generalized λ-Bernstein-Stancu-Type Operators, J. Math. (2021) 5590439.
[20] Q. -B. Cai, R. Aslan, Note on a new construction of Kantorovich form q-Bernstein operators related to shape parameter λ, Comput. Model. Eng. Sci. 130(3) (2022) 1479-1493.
[21] Q. -B. Cai, M. Sofyalıoğlu, K. Kanat, B. Cekim, Some approximation results for the new modification of Bernstein-Beta operators. AIMS Math. 7(2) (2022) 1831-1844.
[22] Q. -B. Cai, A. Khan, M. S. Mansoori, M. Iliyas, K. Khan, Approximation by λ-Bernstein type operators on triangular domain, Filomat 37(6) (2023) 1941-1958.
[23] R. A. De Vore, G. G. Lorentz, Constructive approximation, Springer, Berlin, 1993.
[24] V. Gupta, Anjali, Kantorovich variant of Stancu operators, Filomat 36(15) (2022) 5107-5117.
[25] M. Iliyas, A. Khan, M. Arif, M. Mursaleen, M. R. Lone, Iterates of q-Bernstein operators on triangular domain with all curved sides, Demonstr. Math. 55 (2022) 891-899.
[26] A. Kajla, S. A. Mohiuddine, A. Alotaibi, Blending-type approximation by Lupaş-Durrmeyer-type operators involving Pólya distribution, Math. Meth. Appl. Sci. 44 (2021) 9407-9418.
[27] L. V. Kantorovich, Sur certain développements suivant les polynômes de la forme de S, Bernstein, I, II, CR Acad. URSS (1930) 563-568.
[28] S. Khan, M. Iliyas, M. Mursaleen, Approximation of Lebesgue integrable functions by Bernstein-Lototsky-Kantorovich operartors, Rendiconti Mat. Palermo Ser. 2 (2022) doi: 10.1007/s12215-022-00747-6.
[29] P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions, Doklady Akademii Nauk SSSR 90 (1953) 961-964.
[30] A. Kumar, Approximation properties of generalized λ-Bernstein-Kantorovich type operators, Rendiconti Mat. Palermo Ser. 2 (2020) doi: 10.1007/s12215-020-00509-2.
[31] S. A. Mohiuddine, T. Acar, A. Alotaibi, Construction of a new family of Bernstein-Kantorovich operators, Math. Method. Appl. Sci. 40(18) (2017) 7749-7759.
[32] A. Alotaibi, Md Nasiruzzaman, S. A. Mohiuddine, On the convergence of Bernstein-Kantorovich-Stancu shifted knots operators involving Schurer parameter, Complex Anal. Oper. Th. 18(1) (2024) 4.
[33] S. A. Mohiuddine, K. K. Singh, A. Alotaibi, On the order of approximation by modified summation-integral-type operators based on two parameters, Demonstr. Math. 56 (2023) 20220182.
[34] M. Mursaleen, A. A. H. Al-Abied, M. A. Salman, Approximation by Stancu-Chlodowsky type λ-Bernstein operators, J. Appl. Anal. 1 (2020) 97-110.
[35] F. Özger, Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators, Filomat 33(11) (2019) 3473-3486.
[36] F. Özger, H. M. Srivastava, S. A. Mohiuddine, Approximation of functions by a new class of generalized Bernstein-Schurer operators, RACSAM 114 (2020) 173.
[37] M. Sofyalıoğlu, K. Kanat, B. Cekim, Parametric generalization of the modified Bernstein operators, Filomat 36(5) (2022) 1699-1709.
[38] L. -T. Su, R. Aslan, F. -S. Zheng, M. Mursaleen, On the Durrmeyer variant of q-Bernstein operators based on the shape parameter λ, J. Inequal. Appl. (2023) 2023: 56.
[39] L. -T. Su, K. Kanat, M. Sofyalioğlu Aksoy, M. Kisakol, Approximation by bivariate Bernstein-Kantorovich-Stancu operators that reproduce exponential functions, J. Inequal. Appl., (2024) 6.
[40] L. -T. Su, E. Kangal, Ü. D. Kantar, Q. -B. Cai, Some statistical and direct approximation properties for a new form of the generalization of q-Bernstein operators with the parameter λ, Axioms 13 (2024) 485.
[41] V. I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, In Doklady Akademii Nauk (Vol. 115, No. 1, pp. 17-19), Russian Academy of Sciences, 1957.
[42] J. Yadav, S. A. Mohiuddine, A. Kajla, A. Alotaibi, α-Bernstein-integral type operators, Bull. Iran. Math. Soc. 49 (2023), Article 59.
[43] İ. Yüksel, Ü. Dinlemez Kantar, B. Altın, On approximation of Baskakov-Durrmeyer type operators of two variables, Politehn.

Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys 78(1) (2016) 123-134.
[44] X. -M. Zeng, F. Cheng, On the rates of approximation of Bernstein type operators, J. Approx. Theory 109(2) (2001) 242-256.
[45] G. Zhou, S. Chen, G. Zhao, Approximation properties of a new kind of λ-Bernstein operators, in press.