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Wiener-type invariant conditions for k-leaf-connected graphs

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Abstract. For any integer $k \geq 2$, a graph G is called k-leaf-connected if $|V(G)| \geq k + 1$ and given any subset $S \subseteq V(G)$ with $|S| = k$, G always has a spanning tree T such that S is precisely the set of leaves of T. Obviously, a graph is 2-leaf-connected if and only if it is Hamilton-connected. The Wiener-type invariant of a connected graph G are defined as $W_f = \sum_{u,v \in V(G)} f(d_G(u, v))$, where $f(x)$ is a nonnegative function on the distance $d_G(u, v)$. In this paper, we present best possible Wiener-type invariant conditions to guarantee a graph to be k-leaf-connected, which not only improves the result of Ao et al. (2023), but also extends the result of Zhou et al. (2019). As applications, sufficient conditions for a graph to be k-leaf-connected in terms of the distance (distance signless Laplacian) spectral radius of G are also obtained.

1. Introduction

Let G be a simple, undirected and connected graphs with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are denoted by $|V(G)| = n$ and $|E(G)|$, respectively. For any vertex $v \in V(G)$, we denote by $d_G(v)$ the degree of vertex v in G, and by $(d_1, d_2, \ldots, d_n)$ the degree sequence of G with $d_1 \leq d_2 \leq \cdots \leq d_n$. We use $\delta(G)$ (or δ) to denote the minimum degree of G. For any $u, v \in V(G)$, let $d_G(u, v)$ be the distance between vertices u and v in G. We denote by $\omega(G)$ (or ω) the clique number of G. Let G_1 and G_2 be two vertex-disjoint graphs. We use $G_1 + G_2$ to denote the disjoint union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph obtained from $G_1 + G_2$ by adding all possible edges between them.

Fix an integer $l \geq 0$, the l-closure of a graph G is the graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least l until no such pair exists. Denote by $C_l(G)$ the l-closure of G. Then we have

$$d_{C_l(G)}(u) + d_{C_l(G)}(v) \leq l - 1$$

for every pair of nonadjacent vertices u and v of $C_l(G)$.

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A graph G is called Hamilton-connected if every two vertices of G are connected by a Hamiltonian path. For any integer $k \geq 2$, a graph G is called k-leaf-connected if $|V(G)| \geq k+1$ and given any subset $S \subseteq V(G)$ with $|S| = k$, G always has a spanning tree T such that S is precisely the set of leaves of T. Obviously, a graph is 2-leaf-connected if and only if it is Hamilton-connected.

The Wiener-type invariant of a connected graph G are defined as

$$W_f(G) = \sum_{u,v \in V(G)} f(d_G(u,v)),$$

where $f(x)$ is a nonnegative function on the distance $d_G(u,v)$. When $f(x) = x$, $W_x = W(G) = \sum_{u,v\in V(G)} d_G(u,v)$ is the Wiener index [22]. When $f(x) = \frac{1}{x}$, $W_{\frac{1}{x}} = H(G) = \sum_{u,v\in V(G)} \frac{1}{d_G(u,v)}$ is the Harary index [14]. When $f(x) = \frac{x^2+x}{2}$, $W_{\frac{x^2+x}{2}} = HW(G) = \sum_{u,v\in V(G)} \frac{d_G^2(u,v)+d_G(u,v)}{2}$ is the hyper-Wiener index [19]. When $f(x) = x^\lambda$, $W_{x^\lambda} = MW(G) = \sum_{u,v\in V(G)} d_G^\lambda(u,v)$ is the modified Wiener index [8], where $\lambda \neq 0$ is a real number.

The distance matrix $D(G) = (d_{ij})_{n\times n}$ of G is the matrix with (i,j)-entry $d_{ij} = d_G(v_i, v_j)$. For any vertex $v \in V(G)$, the transmission of v, denoted by $Tr(v)$, is the sum of distances from v to all the other vertices of G, i.e., $Tr(v) = \sum_{u\in V(G)} d_G(u,v)$. Let $Tr(G)$ be the diagonal matrix of the vertex transmissions in G, and let $QD(G) = Tr(G) + D(G)$ be the distance signless Laplacian matrix of G. The largest eigenvalue of $D(G)$ and $QD(G)$, denoted by $\rho_D(G)$ and $\rho_Q(G)$, are called the distance spectral radius of G and the distance signless Laplacian spectral radius of G, respectively.

Gurgel and Wakabayashi [11] first proved the following sufficient degree sequence condition for a graph to be k-leaf-connected.

Theorem 1.1 (Gurgel and Wakabayashi [11]). *Let k and n be two integers such that $2 \leq k \leq n-3$. Let G be a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose there is no integer i with $k \leq i \leq \frac{n+k-2}{2}$ such that $d_{i-k+1} \leq i$ and $d_{n-i} \leq n-i+k-2$. Then G is k-leaf-connected.*

In the same paper, they also proposed sufficient conditions based on the minimum degree, the degree sum and the size to assure a graph to be k-leaf-connected. Egawa et al. [7] improved the degree sum condition of Gurgel and Wakabayashi [11]. Ao et al. [1] presented a new sufficient condition based on the size for a graph to be k-leaf-connected. Subsequently, Wu et al. [21] proved a sufficient condition for a graph to be k-leaf-connected in terms of the number of r-cliques, which generalized the result of Ao et al. [1]. For a graph to be k-leaf-connected, one can refer to [3–5, 17, 18, 20, 23].

Let $\mathbf{NLC}_n = \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1, K_5 \vee 5K_1, K_4 \vee (K_{1,4} + K_1), K_3 \vee K_{2,5}, K_4 \vee (K_2 + 3K_1), K_4 \vee 4K_1, K_3 \vee (K_{1,3} + K_1), K_2 \vee K_{2,4}\}$. Ao et al. [2] and Zhou et al. [25] presented the following sufficient conditions for a connected graph to be k-leaf-connected and Hamilton-connected in terms of the Wiener-type invariant of G, respectively.

Theorem 1.2 (Ao et al. [2] and Zhou et al. [25]). *Let G be a connected graph of order n and minimum degree $\delta \geq k+1$, where $2 \leq k \leq n-4$. Each of the following holds.*
(i) If $W_f(G) \leq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - (2k+5)[f(2) - f(1)]$ for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, then G is k-leaf-connected unless $G \in \mathbf{NLC}_n$.
(ii) If $W_f(G) \geq \frac{f(1)}{2}n^2 - [\frac{5}{2}f(1) - 2f(2)]n + (2k+5)[f(1) - f(2)]$ for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is k-leaf-connected unless $G \in \mathbf{NLC}_n$.

The following sufficient conditions for a graph to be k-leaf-connected in terms of the Wiener-type invariant of G in this paper extends and improves the result of Theorem 1.2.

Theorem 1.3. *Let G be a connected graph of order n and minimum degree $\delta \geq k+1$, where $n \geq k+17$ and $k \geq 2$. Each of the following holds.*
(i) If $W_f(G) \leq \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2) - f(1)]$ for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \vee (K_{n-k-2} + K_2), K_3 \vee (K_{n-5} + 2K_1), K_4 \vee (K_{n-7} + 3K_1)\}$.
(ii) If $W_f(G) \geq \frac{f(1)}{2}n^2 - [\frac{7}{2}f(1) - 3f(2)]n + (3k+11)[f(1) - f(2)]$ for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \vee (K_{n-k-2} + K_2), K_3 \vee (K_{n-5} + 2K_1), K_4 \vee (K_{n-7} + 3K_1)\}$.

When we choose $f(x) = x, \frac{1}{x}, \frac{x^2+x}{2}$ and x^λ ($\lambda \neq 0$) in Theorem 1.3, we immediately deduce the following corollary directly.

Corollary 1.4. *Let G be a connected graph of order n and minimum degree $\delta \geq k+1$, where $n \geq k+17$ and $k \geq 2$. If one of the following holds,*
(i) $W(G) \leq \frac{1}{2}n^2 + \frac{5}{2}n - 3k - 11$,
(ii) $H(G) \geq \frac{1}{2}n^2 - \frac{5}{2}n + \frac{3}{2}k + \frac{11}{2}$,
(iii) $HW(G) \leq \frac{1}{2}n^2 + \frac{11}{2}n - 6k - 22$,
(iv) $MW(G) \leq \frac{1}{2}n^2 + (3 \cdot 2^\lambda - \frac{7}{2})n - (3k+11)(2^\lambda - 1)$ *for* $\lambda > 0$,
(v) $MW(G) \geq \frac{1}{2}n^2 + (\frac{7}{2} - 3 \cdot 2^\lambda)n + (3k+11)(1 - 2^\lambda)$ *for* $\lambda < 0$,
then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \vee (K_{n-k-2} + K_2), K_3 \vee (K_{n-5} + 2K_1), K_4 \vee (K_{n-7} + 3K_1)\}$.

By setting $k = 2$ in Corollary 1.4, our result improves the corresponding results on Hamilton-connected graphs [15, 25].

Determining whether a given graph is k-leaf-connected is NP-complete. However, the problem of computation of eigenvalues of graphs is solvable in polynomial time, and hence it is very important and interesting to put forward spectral sufficient condition for a graph to be k-leaf-connected. As applications of our Corollary 1.4, sufficient conditions for a graph to be k-leaf-connected in terms of the distance (distance signless Laplacian) spectral radius of G are also obtained.

2. Preliminary lemmas

We first present a preliminary result about the relationship between the distance (distance signless Laplacian) spectral radius of a graph and its spanning graph, which is a corollary of the Perron-Frobenius theorem.

Lemma 2.1 (Godsil [9], Minc [16]). *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then*

$$\rho_D(G + uv) < \rho_D(G) \text{ and } \rho_Q(G + uv) < \rho_Q(G).$$

Let M be the following $n \times n$ matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix $R(M)$ of the matrix M (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is equitable if each block $M_{i,j}$ of M has constant row sum.

Lemma 2.2 (Brouwer and Haemers [6], Godsil and Royle [10], Haemers [12]). *Let M be a real symmetric matrix, and $R(M)$ be its equitable quotient matrix. Then the eigenvalues of the quotient matrix $R(M)$ are eigenvalues of M. Furthermore, if M is a nonnegative and irreducible, then the spectral radius of the quotient matrix $R(M)$ equals to the spectral radius of M.*

Lemma 2.3 (Indual [13]). *Let G be a connected graph on n vertices. Then*

$$\rho_D(G) \geq \frac{2W(G)}{n},$$

with equality if and only if the row sums of $D(G)$ are all equal.

Lemma 2.4 (Xing, Zhou and Li [24]). *Let G be a connected graph on n vertices. Then*

$$\rho_Q(G) \geq \frac{4W(G)}{n},$$

with equality if and only if the row sums of $QD(G)$ are all equal.

3. Proofs of Theorem 1.3

In this section we prove Theorem 1.3 by using the following closure theory.

Lemma 3.1 (Gurgel and Wakabayashi [11]). *Let G be a graph and k be an integer with $2 \le k \le n-1$. Then G is k-leaf-connected if and only if the $(n+k-1)$-closure $C_{n+k-1}(G)$ of G is k-leaf-connected.*

Proof of Theorem 1.3. (i) Let $f(x)$ be a monotonically increasing function on $x \in [1, n-1]$, and let G be not k-leaf-connected graph, where $n \ge k+17$, $\delta \ge k+1$ and $k \ge 2$. Let $H = C_{n+k-1}(G)$. By Lemma 3.1, H is not k-leaf-connected. Note that $G \subseteq H$. It is easy to see that $W_f(H) \le W_f(G)$. By the assumption of Theorem 1.3, we have

$$W_f(H) \le \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2) - f(1)].$$

Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of H with $d_1 \le d_2 \le \cdots \le d_n$. By Theorem 1.1, there exists an integer i with $k \le i \le \frac{n+k-2}{2}$ such that $d_{i-k+1} \le i$ and $d_{n-i} \le n-i+k-2$. Then

$$\begin{aligned}
W_f(H) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(d_H(v_i, v_j)) \\
&\ge \frac{1}{2}\sum_{i=1}^{n}[f(1)d_H(v_i) + f(2)(n-1-d_H(v_i))] \\
&= \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\sum_{i=1}^{n} d_H(v_i) \\
&= \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\left(\sum_{j=1}^{i-k+1} d_j + \sum_{j=i-k+2}^{n-i} d_j + \sum_{j=n-i+1}^{n} d_j\right) \\
&\ge \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}[(i-k+1)i + (n-2i+k-1)(n-i+k-2) + i(n-1)] \\
&= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)] - \frac{f(2)-f(1)}{2}g_1(i),
\end{aligned}$$

where $g_1(i) = 3i^2 - (2n+4k-5)i + (2k+4)n + k^2 - 9k - 20$. Since $W_f(H) \le \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)]$, we have $g_1(i) \ge 0$. Note that $k+1 \le \delta \le d_{i-k+1} \le i \le \frac{n+k-2}{2}$. We shall divide the proof into the following three cases.

Case 1. $k+3 \le i \le \frac{n+k-2}{2}$.

Since $g_1''(i) = 6 > 0$, then $g_1(i)$ is a convex function on i. For $n \ge k+17$, we have

$$g_1(k+3) = -2n + 2k + 22 < 0,$$

$$\text{and} \quad g_1\left(\frac{n+k-2}{2}\right) = -\frac{n^2}{4} + \frac{k+11}{2}n - \frac{k^2}{4} - \frac{11k}{2} - 22 < 0.$$

This implies that $g_1(i) < 0$, a contradiction.

Case 2. $i = k+2$.

Then the corresponding degree sequence of H is

$$\underbrace{d_1 \le d_2 \le d_3 \le k+2}_{V_1},\ \underbrace{d_4 \le d_5 \le \cdots \le d_{n-k-2} \le n-4}_{V_2},\ \underbrace{d_{n-k-1} \le d_{n-k} \le \cdots \le d_n \le n-1}_{V_3}.$$

According to the above degree sequence, we divide $V(H)$ into three parts: V_1, V_2 and V_3.

Claim 1. There is no vertex of degree less than $\frac{n+k-1}{2}$ in V_2.

Proof. Suppose that there exists a vertex of degree less than $\frac{n+k-1}{2}$ in V_2. Then

$$\begin{aligned}
W_f(H) &\geq \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\sum_{i=1}^{n} d_H(v_i) \\
&= \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\left(\sum_{j=1}^{3} d_j + \sum_{j=4}^{n-k-2} d_j + \sum_{j=n-k-1}^{n} d_j\right) \\
&\geq \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\left[3(k+2) + (n-k-6)(n-4) + (k+2)(n-1) + \frac{n+k-1}{2}\right] \\
&= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)] + \frac{(n-k-11)[f(2)-f(1)]}{4} \\
&> W_f(H),
\end{aligned}$$

a contradiction, since $n \geq k+17$. □

By Claim 1, it follows that $d_H(u) + d_H(v) \geq n+k-1$ for any two different vertices $u, v \in V_2 \cup V_3$. Note that H is $(n+k-1)$-closed. Then $V_2 \cup V_3$ is a clique of H, and hence $\omega(H) \geq |V_2 \cup V_3| \geq (n-k-5) + (k+2) = n-3$.

If $\omega(H) \geq n-1$, then H contains an $(n-1)$-clique, and hence for any two vertices $u, v \in V(H)$, we always have $d_H(u) + d_H(v) \geq (n-2) + (k+1) = n+k-1$. If there exists two vertices $uv \notin E(H)$, then $d_H(u) + d_H(v) \leq n+k-2$ since H is an $(n+k-1)$-closed graph, a contradiction. Hence any two vertices of H are adjacent. That is, $H \cong K_n$. Then H is k-leaf-connected as a contradiction.

If $\omega(H) = n-2$, then $d_3 \geq n-3$. Note that $d_3 \leq k+2$. Then $n \leq k+5$, which contradicts $n \geq k+17$. Thus, we have

$$\omega(H) = n-3.$$

Next we will characterize the structure of H. Let $C = V_2 \cup V_3$. Note that $|C| = n-3$. Then C is a maximum clique of H, and $V(H) = V_1 \cup C$. Notice that $k+1 \leq \delta \leq d_G(v) \leq d_H(v) \leq k+2$ for each $v \in V_1$. Let $V_1 = \{v_1, v_2, v_3\}$ and $V_1^* = \{v_i \in V_1 \mid d_H(v_i) = k+2\}$.

Claim 2. $|V_1^*| \geq 2$.

Proof. Suppose, to the contrary, that $|V_1^*| \leq 1$. Note that $k+1 \leq d_H(v_i) \leq k+2$ for any $v_i \in V_1$. Then

$$\begin{aligned}
W_f(H) &\geq \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\sum_{i=1}^{n} d_H(v_i) \\
&= \frac{n(n-1)}{2}f(2) - [f(2)-f(1)]e(H) \\
&\geq \frac{n(n-1)}{2}f(2) - [f(2)-f(1)]\left[e(C) + \sum_{i=1}^{3} d_H(v_i)\right] \\
&\geq \frac{n(n-1)}{2}f(2) - [f(2)-f(1)]\left[\binom{n-3}{2} + 2(k+1) + (k+2)\right] \\
&= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+10)[f(2)-f(1)] \\
&= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)] + [f(2)-f(1)] \\
&> W_f(H),
\end{aligned}$$

a contradiction. □

Define $C^* = \{v \in C \mid N_H(v) \cap V_1 \neq \emptyset\}$.

Claim 3. $|C^*| = k+2$.

Proof. By the definition of C^*, we know that $d_H(v) \geq n-3$ for each $v \in C^*$. Then $d_H(v)+d_H(v_i) \geq (n-3)+(k+2) = n+k-1$ for any $v \in C^*$ and $v_i \in V_1^*$. Note that H is $(n+k-1)$-closed. It follows that each vertex of C^* is adjacent to each vertex of V_1^*. Combining Claim 2, we have $d_H(v) \geq d_C(v) + |V_1^*| \geq (n-4)+2 = n-2$ for each $v \in C^*$. Therefore, $d_H(v) + d_H(v_i) \geq (n-2)+(k+1) = n+k-1$ for any $v \in C^*$ and $v_i \in V_1$. Then each vertex of V_1 is adjacent to each vertex of C^*, which implies that $|C^*| \leq d_H(v_i) \leq k+2$, where $v_i \in V_1$.

On the other hand, let $e(V_1, C)$ denote the number of edges between V_1 and C. Notice that $e(V_1, C) = e(V_1, C^*) = |V_1||C^*| = 3|C^*|$ and $e(V_1) = \frac{1}{2}(\sum_{v_i \in V_1} d_G(v_i) - 3|C^*|) \leq \frac{3(k+2-|C^*|)}{2}$. Then

$$\begin{aligned} W_f(H) &\geq \frac{n(n-1)}{2}f(2) - [f(2)-f(1)]e(H) \\ &= \frac{n(n-1)}{2}f(2) - [f(2)-f(1)][e(C)+e(V_1,C^*)+e(V_1)] \\ &\geq \frac{n(n-1)}{2}f(2) - [f(2)-f(1)]\left[\binom{n-3}{2} + \frac{3(k+2+|C^*|)}{2}\right]. \end{aligned}$$

Since $W_f(H) \leq \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)]$, we have $|C^*| \geq k+2$. Therefore, $|C^*| = k+2$. □

Recall that $d_H(v_i) \leq k+2$ for each $v_i \in V_1$. According to Claim 3, V_1 is an independent set. This implies that $H \cong K_{k+2} \vee (K_{n-k-5} + 3K_1)$. It is easy to check that $K_{k+2} \vee (K_{n-k-5} + 3K_1)$ is k-leaf-connected for $k \geq 3$, a contradiction. However, one can check that $K_4 \vee (K_{n-7} + 3K_1)$ is not 2-leaf-connected. Therefore, $H \cong K_4 \vee (K_{n-7} + 3K_1)$.

Case 3. $i = k+1$.

Then the degree sequence of H is given by

$$\underbrace{d_1 = d_2 = k+1}_{V_1},\ \underbrace{d_3 \leq d_4 \leq \cdots \leq d_{n-k-1} \leq n-3}_{V_2},\ \underbrace{d_{n-k} \leq d_{n-k+1} \leq \cdots \leq d_n \leq n-1}_{V_3}.$$

Claim 4. There are at most three vertices of degree less than $\frac{n+k-1}{2}$ in V_2.

Proof. Assume that there exist four vertices of degree less than $\frac{n+k-1}{2}$ in V_2. Then we have

$$\begin{aligned} W_f(H) &\geq \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\sum_{i=1}^{n} d_H(v_i) \\ &> \frac{n(n-1)}{2}f(2) - \frac{f(2)-f(1)}{2}\left[2(k+1) + (n-k-7)(n-3) + (k+1)(n-1) + 4\cdot\frac{n+k-1}{2}\right] \\ &= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+10)[f(2)-f(1)] \\ &= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)] + [f(2)-f(1)] \\ &> W_f(H), \end{aligned}$$

a contradiction. □

Let $V_2^* = \{v \in V_2 \mid d_H(v) \geq \frac{n+k-1}{2}\}$. By Claim 4, we have $|V_2^*| \geq |V_2| - 3 = n-k-6 > 0$. It is clear that $d_H(u) + d_H(v) \geq n+k-1$ for any $u, v \in V_2^* \cup V_3$. Note that H is an $(n+k-1)$-closed graph. This implies that $V_2^* \cup V_3$ is a clique of H, and hence $\omega(H) \geq |V_2^* \cup V_3| \geq (n-k-6)+(k+1) = n-5$. According to the proof in case 2, we have $\omega(H) \leq n-2$, and hence $n-5 \leq \omega(H) \leq n-2$. Define $C = V_2^* \cup V_3$.

Claim 5. C is a maximum clique of H.

Proof. By the definition of V_2^*, we know that $d_H(u) < \frac{n+k-1}{2} \leq n-9 < n-5$ for any $u \in V_1 \cup (V_2 \backslash V_2^*)$, since $n \geq k+17$. Hence there exists at least one vertex $v \in C$ such that $uv \notin E(H)$ for any $u \in V_1 \cup (V_2 \backslash V_2^*)$, and thus $u \notin C$. This implies that C is a maximum clique of H. □

Claim 6. $d_H(u) \leq n+k-\omega-1$ for each $u \in V_2 \backslash V_2^*$.

Proof. Suppose, to the contrary, that $d_H(u) \geq n+k-\omega$ for some $u \in V_2 \backslash V_2^*$. Then $d_H(u)+d_H(v) \geq (n+k-\omega)+(\omega-1) = n+k-1$ for $u \in V_2 \backslash V_2^*$ and $v \in C$. Note that H is an $(n+k-1)$-closed graph. Then u is adjacent to every vertex of C, and hence $C \cup \{u\}$ is a larger clique, which contradicts Claim 5. □

Note that $|V_2 \backslash V_2^*| = n - |V_1| - |V_2^* \cup V_3| = n-\omega-2$. By Claim 6, we have

$$\sum_{u \in V_2 \backslash V_2^*} d_H(u) \leq (n-\omega-2)(n+k-\omega-1).$$

Then we obtain

$$\begin{aligned}
W_f(H) &\geq \frac{n(n-1)}{2} f(2) - [f(2)-f(1)]e(H) \\
&\geq \frac{n(n-1)}{2} f(2) - [f(2)-f(1)]\left[\sum_{u \in V_1} d_H(u) + \sum_{u \in V_2 \backslash V_2^*} d_H(u) + e(V_2^* \cup V_3)\right] \\
&\geq \frac{n(n-1)}{2} f(2) - [f(2)-f(1)]\left[2(k+1) + (n-\omega-2)(n+k-\omega-1) + \binom{\omega}{2}\right] \\
&= -\frac{3}{2}[f(2)-f(1)]\omega^2 + \left[2[f(2)-f(1)]n + (k-\frac{5}{2})[f(2)-f(1)]\right]\omega - \left[\frac{f(2)}{2} - f(1)\right]n^2 \\
&\quad - \left[[f(2)-f(1)]k - \frac{5}{2}f(2) + 3f(1)\right]n - 4[f(2)-f(1)] \\
&\triangleq g_2(\omega).
\end{aligned}$$

Note that $g_2(\omega)$ is a concave function on ω. If $n-5 \leq \omega(H) \leq n-3$ and $n \geq 10$, then

$$\begin{aligned}
W_f(H) &\geq \min\{g_2(n-5), g_2(n-3)\} = g_2(n-3) \\
&= \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k+11)[f(2)-f(1)] + [f(2)-f(1)] \\
&> W_f(H),
\end{aligned}$$

a contradiction. Therefore, $\omega(H) = n-2$.

Let C be an $(n-2)$-clique of H and F be a subgraph of H induced by $V(H) \backslash C$, and let $V(F) = \{v_1, v_2\}$.

Claim 7. $d_H(v_i) = k+1$ for each $v_i \in V(F)$.

Proof. Suppose there exists a vertex $v_i \in V(F)$ with $d_H(v_i) \geq k+2$. Then $d_H(v_i)+d_H(v) \geq (k+2)+(n-3) = n+k-1$ for any $v \in C$. Recall that $H = C_{n+k-1}(G)$. Then v_i is adjacent to vertex v. Note that v is an arbitrary vertex of C. Hence v_i is adjacent to all vertices of C. This implies that $\omega(H) \geq n-1$, a contradiction. □

Claim 8. $N_H(v_1) \cap C = N_H(v_2) \cap C$.

Proof. Without loss of generality, assume that a vertex v of C is adjacent to v_1 of F, then $d_H(v) \geq n-2$. Therefore, $d_H(v) + d_H(v_2) \geq (n-2)+(k+1) = n+k-1$. Note that $H = C_{n+k-1}(G)$. Then v is also adjacent to vertex v_2. Hence $N_H(v_1) \cap C = N_H(v_2) \cap C$. □

Let $|N_H(v_i) \cap C| = t$. Note that $|V(F)| = 2$. By Claim 7, we know that $d_H(v_i) = k + 1$. Then $t \geq k$. On the other hand, $t \leq d_H(v_i) = k + 1$. Hence $k \leq t \leq k + 1$. Next, we will discuss the following two cases.

Case 3.1. $t = k$.

Then $H \cong K_k \vee (K_{n-k-2} + K_2)$. It is easy to see that $K_k \vee (K_{n-k-2} + K_2)$ is not k-leaf-connected. Note that $W_f(H) = \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - (2k + 4)[f(2) - f(1)] < \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - (3k + 11)[f(2) - f(1)]$. Hence $H \cong K_k \vee (K_{n-k-2} + K_2)$.

Case 3.2. $t = k + 1$.

Then $H \cong K_{k+1} \vee (K_{n-k-3} + 2K_1)$. One can check that $K_{k+1} \vee (K_{n-k-3} + 2K_1)$ is k-leaf-connected for $k \geq 3$, a contradiction. However, $K_3 \vee (K_{n-5} + 2K_1)$ is not 2-leaf-connected. Notice that $W_f(H) = \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 9[f(2) - f(1)] < \frac{f(1)}{2}n^2 + [3f(2) - \frac{7}{2}f(1)]n - 17[f(2) - f(1)]$. Therefore, $H \cong K_3 \vee (K_{n-5} + 2K_1)$.

By the above proof, we have $H = C_{n+k-1}(G) \in \{K_k \vee (K_{n-k-2} + K_2), K_3 \vee (K_{n-5} + 2K_1), K_4 \vee (K_{n-7} + 3K_1)\}$. The proof of (i) is completed.

(ii) If $f(x)$ is a monotonically decreasing function on $[1, n - 1]$, we can also prove the result similarly. This completes the proof of Theorem 1.3. □

4. Applications of Corollary 1.4

Lemma 4.1. *Let $H \cong K_k \vee (K_{n-k-2} + K_2)$.*
(i) If $n \geq k + 10$, then $\rho_D(H) < n + 5 - \frac{6k+22}{n}$.
(ii) If $n \geq k + 4$, then $\rho_Q(H) > 2n + 10 - \frac{12k+44}{n}$.

Proof. (i) Let $R(D)$ be an equitable quotient matrix of the distance matrix $D(H)$ with respect to the partition $(V(K_k), V(K_{n-k-2}), V(K_2))$. One can see that

$$R(D) = \begin{pmatrix} k-1 & n-k-2 & 2 \\ k & n-k-3 & 4 \\ k & 2n-2k-4 & 1 \end{pmatrix}.$$

Then the characteristic polynomial $P_{R(D)}(x) = x^3 - (n-3)x^2 - (8n - 6k - 15)x + (2k - 7)n - 2k^2 + 2k + 13$. By Lemma 2.2, we know that $\rho_D(H) = \lambda_1(R(D))$ is the largest root of the equation $P_{R(D)}(x) = 0$. Let $P'_{R(D)}(x) = 3x^2 - 2(n-3)x - 8n + 6k + 15 = 0$. We can solve this equation to obtain that

$$x_1 = \frac{n - 3 - \sqrt{n^2 + 18n - 18k - 36}}{3} \text{ and } x_2 = \frac{n - 3 + \sqrt{n^2 + 18n - 18k - 36}}{3}.$$

Then $P_{R(D)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_D(H) = \lambda_1(R(D)) > x_2$ and $n + 5 - \frac{6k+22}{n} > x_2$ for $n \geq k + 2$. By Maple, $P_{R(D)}(n + 5 - \frac{6k+22}{n}) > 0 = P_{R(D)}(\rho_D(H))$ for $n \geq k + 10$. This implies that $\rho_D(H) < n + 5 - \frac{6k+22}{n}$ for $n \geq k + 10$.

(ii) Let $R(QD)$ be an equitable quotient matrix of the distance matrix $QD(H)$ with respect to the partition $(V(K_k), V(K_{n-k-2}), V(K_2))$. Then

$$R(QD) = \begin{pmatrix} n+k-2 & n-k-2 & 2 \\ k & 2n-k-2 & 4 \\ k & 2n-2k-4 & 2n-k-2 \end{pmatrix}.$$

The characteristic polynomial $P_{R(QD)}(x) = x^3 - (5n - k - 6)x^2 + [8n^2 - (3k + 28)n + 12k + 28]x - 4n^3 + (2k + 24)n^2 - (4k + 52)n + 20k + 40$. By Lemma 2.2, we know that $\rho_{QD}(H) = \lambda_1(R(QD))$ is the largest root of the equation $P_{R(QD)}(x) = 0$. Let $P'_{R(QD)}(x) = 3x^2 - 2(5n - k - 6)x + 8n^2 - (3k + 28)n + 12k + 28 = 0$. We can solve that

$$x_1 = \frac{5n - k - 6 - \sqrt{n^2 - (k - 24)n + k^2 - 24k - 48}}{3} \text{ and } x_2 = \frac{5n - k - 6 + \sqrt{n^2 - (k - 24)n + k^2 - 24k - 48}}{3}.$$

Then $P_{R(QD)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_{QD}(H) = \lambda_1(R(QD)) > x_2$ and $2n + 10 - \frac{12k+44}{n} > x_2$ for $n \geq k + 4$. By direct calculation, $P_{R(QD)}(2n + 10 - \frac{12k+44}{n}) < 0 = P_{R(QD)}(\rho_{QD}(H))$ for $n \geq k + 3$. It follows that $\rho_{QD}(H) > 2n + 10 - \frac{12k+44}{n}$ for $n \geq k + 4$. □

Lemma 4.2. *Let* $H \cong K_3 \vee (K_{n-5} + 2K_1)$.
(i) If $n \geq 10$, *then* $\rho_D(H) < n + 5 - \frac{34}{n}$.
(ii) If $n \geq 6$, *then* $\rho_Q(H) > 2n + 10 - \frac{68}{n}$.

Proof. (i) Let $R(D)$ be an equitable quotient matrix of the distance matrix $D(H)$ with respect to the partition $(V(K_3), V(K_{n-5}), V(2K_1))$. It is easy to see that

$$R(D) = \begin{pmatrix} 2 & n-5 & 2 \\ 3 & n-6 & 4 \\ 3 & 2n-10 & 2 \end{pmatrix}.$$

Then the characteristic polynomial $P_{R(D)}(x) = x^3 - (n-2)x^2 - (7n-29)x - 2$. By Lemma 2.2, we know that $\rho_D(H) = \lambda_1(R(D))$ is the largest root of the equation $P_{R(D)}(x) = 0$. Let $P'_{R(D)}(x) = 3x^2 - 2(n-2)x - 7n + 29 = 0$. We can solve that

$$x_1 = \frac{n - 2 - \sqrt{n^2 + 17n - 83}}{3} \text{ and } x_2 = \frac{n - 2 + \sqrt{n^2 + 17n - 83}}{3}.$$

Then $P_{R(D)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_D(H) = \lambda_1(R(D)) > x_2$ and $n + 5 - \frac{34}{n} > x_2$ for $n \geq 5$. By simple calculation, $P_{R(D)}(n + 5 - \frac{34}{n}) > 0 = P_{R(D)}(\rho_D(H))$ for $n \geq 10$. This means that $\rho_D(H) < n + 5 - \frac{34}{n}$ for $n \geq 10$.

(ii) Let $R(QD)$ be an equitable quotient matrix of the distance signless Laplacian matrix $QD(H)$ with respect to the partition $(V(K_3), V(K_{n-5}), V(2K_1))$. Then we have

$$R(QD) = \begin{pmatrix} n+1 & n-5 & 2 \\ 3 & 2n-5 & 4 \\ 3 & 2n-10 & 2n-3 \end{pmatrix}.$$

The characteristic polynomial $P_{R(QD)}(x) = x^3 - (5n-7)x^2 + (8n^2 - 31n + 56)x - 4n^3 + 26n^2 - 82n + 80$. By Lemma 2.2, we know that $\rho_Q(H) = \lambda_1(R(QD))$ is the largest root of the equation $P_{R(QD)}(x) = 0$. Let $P'_{R(QD)}(x) = 3x^2 - 2(5n-7)x + 8n^2 - 31n + 56 = 0$. We can solve that

$$x_1 = \frac{5n - 7 - \sqrt{n^2 + 23n - 119}}{3} \text{ and } x_2 = \frac{5n - 7 + \sqrt{n^2 + 23n - 119}}{3}.$$

Then $P_{R(QD)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_Q(H) = \lambda_1(R(QD)) > x_2$ and $2n + 10 - \frac{68}{n} > x_2$ for $n \geq 6$. Then $P_{R(QD)}(2n + 10 - \frac{68}{n}) < 0 = P_{R(QD)}(\rho_Q(H))$, which implies that $\rho_Q(H) > 2n + 10 - \frac{68}{n}$ for $n \geq 6$. □

Lemma 4.3. *Let* $H \cong K_4 \vee (K_{n-7} + 3K_1)$.
(i) If $n \geq 8$, *then* $\rho_D(H) > n + 5 - \frac{34}{n}$.
(ii) If $n \geq 8$, *then* $\rho_Q(H) > 2n + 10 - \frac{68}{n}$.

Proof. (i) Let $R(D)$ be an equitable quotient matrix of the distance matrix $D(H)$ with respect to the partition $(V(K_4), V(K_{n-7}), V(3K_1))$. One can see that

$$R(D) = \begin{pmatrix} 3 & n-7 & 3 \\ 4 & n-8 & 6 \\ 4 & 2n-14 & 4 \end{pmatrix}.$$

Then the characteristic polynomial $P_{R(D)}(x) = x^3-(n-1)x^2-(9n-56)x+4n-28$. By Lemma 2.2, we know that $\rho_D(H) = \lambda_1(R(D))$ is the largest root of the equation $P_{R(D)}(x) = 0$. Let $P'_{R(D)}(x) = 3x^2 - 2(n-1)x - 9n + 56 = 0$. We can solve that

$$x_1 = \frac{n-1-\sqrt{n^2+25n-167}}{3} \text{ and } x_2 = \frac{n-1+\sqrt{n^2+25n-167}}{3}.$$

Then $P_{R(D)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_D(H) = \lambda_1(R(D)) > x_2$ and $n+5-\frac{34}{n} > x_2$. By Maple, $P_{R(D)}(n+5-\frac{34}{n}) < 0 = P_{R(D)}(\rho_D(H))$ for $n \geq 8$. This implies that $\rho_D(H) > n+5-\frac{34}{n}$ for $n \geq 8$.

(ii) Let $R(QD)$ be an equitable quotient matrix of the distance matrix $QD(H)$ with respect to the partition $(V(K_4), V(K_{n-7}), V(3K_1))$. Then

$$R(QD) = \begin{pmatrix} n+2 & n-7 & 3 \\ 4 & 2n-6 & 6 \\ 4 & 2n-14 & 2n-2 \end{pmatrix}.$$

The characteristic polynomial $P_{R(QD)}(x) = x^3 - (5n-6)x^2 + (8n^2 - 32n + 96)x - 4n^3 + 28n^2 - 128n + 128$. By Lemma 2.2, we know that $\rho_Q(H) = \lambda_1(R(QD))$ is the largest root of the equation $P_{R(QD)}(x) = 0$. Let $P'_{R(QD)}(x) = 3x^2 - 2(5n-6)x + 8n^2 - 32n + 96 = 0$. We can solve that

$$x_1 = \frac{5n-6-\sqrt{n^2+36n-252}}{3} \text{ and } x_2 = \frac{5n-6+\sqrt{n^2+36n-252}}{3}.$$

Then $P_{R(QD)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho_Q(H) = \lambda_1(R(QD)) > x_2$ and $2n+10-\frac{68}{n} > x_2$. We Know that $P_{R(QD)}(2n+10-\frac{68}{n}) < 0 = P_{R(QD)}(\rho_Q(H))$ for $n \geq 8$. It follows that $\rho_Q(H) > 2n+10-\frac{68}{n}$ for $n \geq 8$. □

Using Corollary 1.4, we will provide sufficient conditions for a graph to be k-leaf-connected in terms of the distance (distance signless Laplacian) spectral radius of G.

Theorem 4.4. *Let G be a connected graph of order n and minimum degree $\delta \geq k+1$, where $n \geq k+17$ and $k \geq 2$. Each of the following holds.*
(i) If $\rho_D(G) \leq n+5-\frac{6k+22}{n}$, then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \vee (K_{n-k-2}+K_2), K_3 \vee (K_{n-5}+2K_1)\}$.
(ii) If $\rho_Q(G) \leq 2n+10-\frac{12k+44}{n}$, then G is k-leaf-connected.

Proof. Suppose, to the contrary, that G is not k-leaf-connected.

(i) By Lemma 2.3 and the assumption of Theorem 4.4, we have

$$\frac{2W(G)}{n} \leq \rho_D(G) \leq n+5-\frac{6k+22}{n}.$$

Then $W(G) \leq \frac{1}{2}n^2 + \frac{5}{2}n - 3k - 11$. Let $H = C_{n+k-1}(G)$. By Corollary 1.4, we have $H \in \{K_k \vee (K_{n-k-2}+K_2), K_3 \vee (K_{n-5}+2K_1), K_4 \vee (K_{n-7}+3K_1)\}$. Assume that $H \cong K_4 \vee (K_{n-7}+3K_1)$. According to Lemma 2.1 and (i) of Lemma 4.3, we have

$$\rho_D(G) > \rho_D(H) > n+5-\frac{34}{n},$$

which contradicts the assumption. For $H \in \{K_k \vee (K_{n-k-2}+K_2), K_3 \vee (K_{n-5}+2K_1)\}$ and $n \geq k+17$, by (i) of Lemmas 4.1 and 4.2, we can not compare completely $\rho_D(G)$ with $n+5-\frac{6k+22}{n}$. For the brevity of discussion, we have $C_{n+k-1}(G) = H \in \{K_k \vee (K_{n-k-2}+K_2), K_3 \vee (K_{n-5}+2K_1)\}$.

(ii) Note that $\rho_Q(G) \leq 2n+10-\frac{12k+44}{n}$. Combining Lemma 2.4, we obtain

$$W(G) \leq \frac{1}{2}n^2 + \frac{5}{2}n - 3k - 11.$$

Let $H = C_{n+k-1}(G)$. By Corollary 1.4, we have $H \in \{K_k \vee (K_{n-k-2} + K_2), K_3 \vee (K_{n-5} + 2K_1), K_4 \vee (K_{n-7} + 3K_1)\}$. By Lemmas 2.1, 4.1, 4.2 and 4.3, we have

$$\rho_Q(G) > \rho_Q(H) > 2n + 10 - \frac{12k + 44}{n},$$

a contradiction. This completes the proof of Theorem 4.4. □

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