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$\beta-$expansion of unity and transcendence in the p-adic field

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Abstract. The aim of this paper is to study the $\beta-$expansion of p-adic numbers. In particular, we obtain an upper bound of strings of consecutive zeros in the $\beta-$expansion of unity for an algebraic number β in $\mathbb{Q}_p$ such that $|\beta|_p > 1$.

1. Introduction

A real number is most classically represented by its continued fraction expansions or by its representations in some integer bases. By a special representation we can generalize standard representations in an integer base to a real base β, this special representation is called $\beta-$expansion which was introduced by by A. Rényi [7] in 1957. Let β be a real number such that $\beta > 1$. Similarly to the case of integral bases, it is possible to define the β-expansion of a real number $x \in [0, 1]$ as the sequence $(x_i)_{i\geq 1}$ with values in $\{0, 1, \ldots, [\beta]\}$ produced by the β-transformation $T_\beta : x \longrightarrow \beta x \pmod 1$ as follows :

$$for\ all\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)],\ and\ so\ x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

Let's mention that an expansion of real number is finite if $(x_i)_{i\geq 1}$ is eventually 0. It's periodic if $p \geq 1$ and $m \geq 1$ exists and verifying $x_k = x_{k+p}$, for all $k \geq m$.

Furthermore, the β-expansion of 1 plays a crucial role in our theory and appeared in several works especially in the study of the classification of algebraic numbers $\beta > 1$. Let's recall that numbers β such that their β-expansion of 1 is ultimately periodic are called Parry numbers and those such that their β-expansion of 1 is finite are called simple Parry numbers. These families of numbers were introduced by W. Parry in [6], its elements were initially called β-numbers and it is easy to check that these elements are algebraic integer numbers. Moreover, these numbers afforded interesting results, for example, it is well known that if β is a Pisot number (an algebraic integer > 1 whose conjugates have modulus strictly less than one), then β is a Parry number. In the same context, D. Boyd have proved in [3] that if β is a Salem number (an algebraic integer > 1 whose conjugates have modulus ≤ 1 and at least one of them has a modulus equal to 1) of degree 4, then β is a Parry number. Unfortunately, there is not a complete characterization of Parry or simple Parry numbers untill now.

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Particularly, many works have been devoted to the study of the occurences of consecutive $0's$ in the β-expansion of 1 and the classification of algebraic numbers β. So, there are many important results along these lines. For instance, in an old result, more precisely in 1965 Mahler has interested in the gaps between the non-zero digits in the β-expansion of 1 and he proved the following theorem in [4]:

Theorem 1.1. *Let $\beta > 1$ be an algebraic number such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence in the following sense:*
There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that:

$$1 = s_0 \leqslant m_1 < s_1 \leqslant m_2 < s_2 \leqslant \ldots \leqslant m_n < s_n \leqslant m_{n+1} < s_{n+1} \leqslant \ldots$$

with$(s_n - m_n) \geqslant 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geqslant 1$. Then,

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) < \infty$$

After that, in 2006 Verger-Gaugry extended Mahler's and Güting's approximation theorems by proving that the gaps in the β-expansion of 1 are shown to exhibit a gappiness bounded through the use of a version of Liouville's inequality in the following:

Theorem 1.2. *[9] Let $\beta > 1$ be an algebraic number and $M(\beta)$ be its Mahler measure such such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence in the following sense: There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that:*

$$1 = s_0 \leqslant m_1 < s_1 \leqslant m_2 < s_2 \leqslant \ldots \leqslant m_n < s_n \leqslant m_{n+1} < s_{n+1} \leqslant \ldots$$

with$(s_n - m_n) \geqslant 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geqslant 1$. Then,

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) < \frac{log(M(\beta))}{log(\beta)}$$

In a natural way, this result provides a new classification of algebraic numbers $\beta > 1$. Later, in 2007 Adamczewski and Bugeaud [1] improved the previous theorem and they established the following result:

Theorem 1.3. *Let $\beta > 1$ be an algebraic number. Then with the above notation,*

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) < \frac{log(M(\beta))}{log(\beta)} - 1$$

In the same context, Allouche and Cosnard in [2] proved that there exists a smallest $q \in]1, 2[$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{+\infty} \delta_n q^{-n}$ where $\delta_n \in \{0, 1\}$. In addition, for this smallest q, the coefficient δ_n is equal to 0 (respectively, 1) if the sum of the binary digits of n is even (respectively, odd). This constant q is named Komornik-Loreti constant. Since the strings of zeros and 1's in the sequence δ_n are known and uniformly bounded, the constant q satisfies

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) = 1.$$

However, authors in [2] have shown that q is a transcendental number.
Therefore, the β-expansion of unity and transcendence in the real case is an interesting topic that has been studied by various authors. For this, there are a lot of results concerning this topic which motivates as to study in this work the analogous of this concept in the field of p-adic numbers and also to introduce the $\beta-$expansion over this field which is currently a popular area of research.
Let's recall that, the β-expansion of 1 and transcendence of p-adic numbers have not been studied yet.
This paper is organized as follows: In section 2, we start by introducing $\mathbb{Q}_p$, the field of p-adic numbers. After that, we give the suitable definition of Pisot-Chabauty numbers as well as the analogous to Pisot

numbers. In section 3, we study the β-expansion algorithm for p-adic numbers and we review some basic properties and notations necessary in our work. The last section is devoted to prove that if β is an algebraic number of algebraic degree $d \geq 2$ such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence then the quotient of gaps in the string of 0 in the sequence $(a_i)_{i\geq 1}$ is bounded. Consequently, if the β-expansion of unity has unbounded quotient of gaps, then β is transcendental. In a natural way, this result provides a family of transcendental numbers β.

2. Field of *p*-adic numbers

In order to introduce $\mathbb{Q}_p$ in an harmonious way, we start by presenting the following sets:
Let p be a prime and $\mathbb{A}_p = \{mp^n, m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{p}]$.

Recall that $\begin{cases} \bullet\ \mathbb{A}_p \subset \mathbb{Q} \textit{ is a principal ring.} \\ \bullet\ \textit{The unit group of } \mathbb{A}_p \textit{ is } \{\pm p^k, k \in \mathbb{Z}\}. \\ \bullet\ \textit{The field of fractions of } \mathbb{A}_p \textit{ is } \mathbb{Q}. \end{cases}$
Particularly, we denote by $\mathbb{A}'_p = \mathbb{A}_p \cap [0, 1)$.
Now, let's define the p-adic valuation:

$$\begin{array}{rcl} v_p : \mathbb{A}_p & \longrightarrow & \mathbb{Z} \bigcup \{\infty\} \\ x & \longmapsto & \begin{cases} max\{n \in \mathbb{Z} : p^n \textit{ divides } x\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \end{array}$$

which fulfills the following properties:
- $v_p(0) = \infty$,
- $v_p(xy) = v_p(x) + v_p(y)$,
- $v_p(x + y) \geq min\{v_p(x), v_p(y)\}$ *with* $v_p(x + y) = min\{v_p(x), v_p(y)\}$, *if* $v_p(x) \neq v_p(y)$.

Then $v_p(.)$ is an exponential valuation on $\mathbb{A}_p$. Consequently, the $p-adic$ absolute value $|.|_p$ is defined by

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Thus $|.|_p$ is a non Archimedean absolute value on $\mathbb{A}_p$ which verifies the strict triangular inequality

$$|x + y|_p \leq max\{|x|_p,\ |y|_p\} \quad with$$

$$|x + y|_p = max\{|x|_p,\ |y|_p\} \quad if \ \ |x|_p \neq |y|_p.$$

In the same direction, we denote by $|.|_\infty$ the Archimedean absolute value.
Now, the completion of $\mathbb{A}_p$ with respect to $|.|_p$ is the field of p-adic numbers $\mathbb{Q}_p$, therefore we have

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

We mention that each element $x \in \mathbb{Q}_p$ $(x \neq 0)$ admits a unique p-adic expansion of the form

$$x = \sum_{n=n_0}^{\infty} x_n p^n, \ \ such\ that\ \ n_0 \in \mathbb{Z},\ \ x_{n_0} \neq 0\ \ and\ x_n \in \{0, \ldots, p-1\} \ \ (\star)$$

From expansions of the form mentioned in $(\star)$, we will use the notation

$$x = \ldots p_2 p_1 p_0 . p_{-1} \ldots p_{n_0}.$$

Definition 2.1. *Each $x \in \mathbb{Q}_p$ of the form mentioned above in ($\star$) has a unique Artin decomposition*

$$x = [x]_p + \{x\}_p$$

where

$$[x]_p = \sum_{n\geq 0} x_n p^n \text{ and } \{x\}_p = \sum_{n<0} x_n p^n.$$

The number $[x]_p \in \mathbb{Z}_p$ is called $p-$adic integer part and $\{x\}_p \in \mathbb{A}_p \cap [0,1)$ is called $p-$adic fractional part of x.

Moreover, we can also extend v_p in $\mathbb{Q}_p$ as follows:

If $x = \sum_{n=n_0}^{\infty} x_n p^n$, where $n_0 \in \mathbb{Z}$, $x_{n_0} \neq 0$, $x_n \in \{0, \dots, p-1\}$, we define $v_p(x)$ by:

$$v_p(x) = \begin{cases} n_0 & if\ x \neq 0 \\ \infty & if\ x = 0. \end{cases}$$

Furthermore, $\mathbb{Q}_p$ is equivalent to the fraction field of the p-adic integers $\mathbb{Z}_p$ where

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p; |x|_p \leq 1\}.$$

Therefore, it easily follows that

$$\mathbb{Z} = \mathbb{A}_p \cap \mathbb{Z}_p = \{x \in \mathbb{A}_p; |x|_p \leq 1\} \text{ and } p\mathbb{Z}_p = \{x \in \mathbb{Q}_p; |x|_p < 1\}.$$

Now, we aim to define the Pisot-Chabauty numbers as the analogous to Pisot numbers in the real case. For this, we need some definitions.

Definition 2.2. *An element α is called algebraic over $\mathbb{A}_p$, if there is a polynomial*

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{A}_p[x] \text{ with } a_n \neq 0 \text{ and } P(\alpha) = 0.$$

If P is irreducible over $\mathbb{A}_p$, then P is called a minimal polynomial of α. In addition, if $a_n = p^k$ for some $k \in \mathbb{Z}$, thus α is an algebraic integer. As p^k is a unit of $\mathbb{A}_p$, we can assume without loss of generality, that $a_n = 1$. If $a_0 = p^{k'}$ for some $k' \in \mathbb{Z}$, so α is called an algebraic unit.

It turns out that algebraic elements over $\mathbb{Q}$ are not necessarily contained in $\mathbb{Q}_p$. In our context, we will only need that $|.|_p$ can be extended uniquely from $\mathbb{Q}_p$ to all of its algebraic extensions. This follows from the next theorem, which holds generally in non-archimedean fields.

Theorem 2.3 ([5], Chapter II, Theorem 4.8). *Let K be a field which is complete with respect to $|.|$ and L/K be an algebraic extension of degree m. Thus $|.|$ has a unique extension to L defined by : $|\alpha| = \sqrt[m]{|N_{L/K}(\alpha)|}$ and L is complete with respect to this extension.*

Remark 2.4. *In what follows, for algebraic elements β over $\mathbb{A}_p$ we will denote by $\beta_1, \dots, \beta_n$ the non-Archimedean conjugates of β and by $\beta_{n+1}, \dots, \beta_{2n}$ the Archimedean conjugates of β (the complex roots of the minimal polynomial of β).*

Finally, we reach to give the definition of Pisot-Chabauty numbers.

Definition 2.5. *A Pisot-Chabauty number (for short PC number) is a p-adic number $\beta \in \mathbb{Q}_p$, such that*

- *$\beta_1 = \beta$ is an algebraic integer over $\mathbb{A}_p$.*
- *$|\beta_1|_p > 1$ for one non-Archimedean conjugate of β.*
- *$|\beta_i|_p \leq 1$ for all non-Archimedean conjugates β_i, $i \in \{2, \dots, n\}$ of β.*
- *$|\beta_i|_\infty < 1$ for all Archimedean conjugates β_i, $i \in \{n+1, \dots, 2n\}$ of β.*

3. $\beta-$expansion in the field $\mathbb{Q}_p$

Similarly to the classical β-expansions for the real numbers, we introduce the β-expansions for p-adic numbers. For this, let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and $x \in \mathbb{Z}_p$. A representation in base β (or β-representation) of x is a sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{A'}_p = \mathbb{A}_p \cap [0, 1)$, such

$$x = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of x is called the β-expansion of x and noted by $d_\beta(x) = (d_i)_{i\geq 1}$ with values in $\mathbb{A}_{\beta,p} = [0, 1) \cap \{x \in \mathbb{A}_p : |x|_p \leq |\beta|_p\}$ produced by the β-transformation $T : \mathbb{Z}_p \to \mathbb{Z}_p$, which is given by the mapping $z \mapsto [\beta z]_p$. For $k \geq 0$, let's define

$$T^0(x) = x \text{ and } T^k(x) = T(T^{k-1}(x)).$$

So, $d_k = \{\beta T^{k-1}(x)\}_p$ for all $k \geq 1$. An equivalent definition of the β-expansion can be obtained by using a greedy algorithm. This algorithm proceeds as follows :

$$r_0 = x;\ d_k = \{\beta r_{k-1}\}_p \text{ and } r_k = \lfloor \beta r_{k-1} \rfloor_p \text{ for all } k \geq 1.$$

The β-expansion of x will be noted as $d_\beta(x) = (d_k)_{k\geq 1}$.

Now, let $x \in \mathbb{Q}_p$ with $|x|_p > 1$. Thus there is a unique $k \in \mathbb{N}$ such that $|\beta|_p^k \leq |x|_p < |\beta|_p^{k+1}$. We can represent x by shifting $d_\beta(\beta^{-(k+1)}x)$ by k digits to the left. Therefore, if $d_\beta(x) = 0 \bullet d_1 d_2 d_3 \ldots$, then $d_\beta(\beta x) = d_1 \bullet d_2 d_3 \ldots$. Thereby, if $d_\beta(x) = d_1 d_2 d_3 \ldots d_k \bullet d_{k+1} d_{k+2} \ldots$. We denote x by

$$x = [x]_\beta + \{x\}_\beta$$

with

$$[x]_\beta = \sum_{1\leq i\leq k} d_i \beta^i \ \textit{and}\ \{x\}_\beta = \sum_{i\geq k+1} \frac{d_i}{\beta^i}.$$

The number $[x]_\beta$ is called a p-adic β-integer part of x and the number $\{x\}_\beta$ is called a p-adic β-fractional part of x.

Moreover, we mention that $d_\beta(x)$ is finite if and only if there is a $k \geq 0$ with $T^k(x) = 0$, $d_\beta(x)$ is ultimately periodic if and only if there is some smallest $n \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) when $T^{n+s}(x) = T^n(x)$, namely the period length. In a special case, where $n = 0$, $d_\beta(x)$ is purely periodic.

Afterwards, we will use the following notations :

$$Fin(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \textit{ is finite}\} \text{ and } Per(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \textit{ is eventually periodic}\}.$$

Hence, it is easy to check that

$$Fin(\beta) \subset Per(\beta).$$

Through the use of the previous sets and the *PC* numbers, K.Scheicher, V. F. Sirvent and P. Surer established the following theorem ([8]):

Theorem 3.1. *Let β be a PC number. Then $Per(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.*

Furthermore, analogously to the notion of β-numbers in the real case we define the beta-p-adic numbers in $\mathbb{Q}_p$ as follows:

Definition 3.2. *Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$. β is called a beta-p-adic number if $1 \in Per(\beta)$ and is called a simple beta-p-adic number if $1 \in Fin(\beta)$.*

4. Results

Lemma 4.1. *Let $x \in \mathbb{Q}_+$ such that $x < p^n$ with $n \in \mathbb{N}$. Then $|x|_p > p^{-n}$.*

Proof:
Since $x < p^n$ then the p-adique expansion of x will be written as follows:

$$x = \sum_{i=n_0}^{\infty} a_i p^i, \text{ such that } n_0 < n.$$

Otherwise x will be greater than p^n. □

Lemma 4.2. *Let $(x_i) \in \mathbb{A}'_p$ for all $1 \le i \le m$ where $m \in \mathbb{N}^*$. Then*

$$|\sum_{i=1}^{m} x_i|_p > p^{-([log_p(m)]+1)}$$

Proof:
By assumption, we have $(x_i)_{1\le i\le m} \in \mathbb{A}'_p$ which implies that $0 \le x_i < 1$. Therefore

$$\begin{aligned} \sum_{i=1}^{m} x_i &< m = p^{log_p(m)} \\ &< p^{[log_p(m)]+1}. \end{aligned}$$

Consequently, according to Lemma 4.1 we infer that $|\sum_{i=1}^{m} x_i|_p > p^{-([log_p(m)]+1)}$. □

Theorem 4.3. *(Classical Theorem of symmetric polynomials)*
Let $Q \in \mathbb{A}_p[x][y]$ and $F(y^{(1)}, y^{(2)}, ..., y^{(d)}) = Q(y^{(1)})Q(y^{(2)})...Q(y^{(d)})$.
Then, there exists a polynomial T with d variables and coefficients in $\mathbb{A}_p[x]$ such that

$$F(y^{(1)}, y^{(2)}, ..., y^{(d)}) = T(\sigma_1, \sigma_2, ..., \sigma_d)$$

where:

$$\begin{cases} \sigma_1 = \sum_{i=1}^{d} y^{(i)} \\ \sigma_2 = \sum_{1\le i<j\le d} y^{(i)} y^{(j)} \\ \sigma_3 = \sum_{1\le i<j<k\le d} y^{(i)} y^{(j)} y^{(k)} \\ \vdots \\ \sigma_d = y^{(1)} y^{(2)} ... y^{(d)} \end{cases}$$

Let's mention also that the total degree of T is lower or equal to the degree of Q.

Lemma 4.4.
Let β be an algebraic integer with minimal polynomial $P_\beta(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$, where $A_i \in \mathbb{A}_p[x]$ for all $0 \le i \le d$. Let $m \ge d$ and $K(y) = y^m B_m + B_{m-1}y^{m-1} + \cdots + B_0$ with $B_i \in \mathbb{A}_p[x]$ for all $0 \le i \le m$. Then

$$|K(\beta)|_p > p^{-([log_p((m_n)^d)]+1)}.$$

Proof:
Let $K(y) = y^m B_m + B_{m-1}y^{m-1} + \cdots + B_0$ a polynomial of degree $m \geq d$.
Since $\beta^d = -A_{d-1}\beta^{d-1} - \cdots - A_0$, there exist $C_{(i,s)} \in \mathbb{A}_p[x]$ such that

$$\beta^{d+s-1} = C_{(d-1,s)}\beta^{d-1} + \cdots + C_{(0,s)} \text{ for all } s \geq 1.$$

Let now $\beta = \beta^{(1)}$ and $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of β. For $s = m - d + 1$, there exist $D_i \in \mathbb{A}_p[x]$ such that

$$K(\beta^{(j)}) = D_{d-1}(\beta^{(j)})^{d-1} + \cdots + D_0 \text{ for all } 1 \leq j \leq d.$$

By Theorem 4.3, there exists a polynomial T with d variables and coefficients in $\mathbb{A}'_p[X]$ such that

$$\prod_{j=1}^{d} K(\beta^{(j)}) = T(\sigma_1, \sigma_2, ..., \sigma_d)$$

with $|\sigma_i|_p = |\sum_{1\leq j_1<j_2<...\leq d} \beta^{(j_1)}\beta^{(j_2)}...\beta^{(j_i)}|_p = |A_{d-i}|_p \in \mathbb{A}'_p$ for all $1 \leq i \leq d$ and the total degree of T is lower or equal to d, which involves that the polynomial K contains at most $(m)^d$ monomials of the form $\sum_{i=1}^{d} D_i\sigma_i^{\alpha_i}$ with $\sum_{i=1}^{d} \alpha_i \leq m$ and without loss of generality we can assume that $D_i \in \mathbb{A}'_p$. Therefore, K contains at most $(m)^d$ monomials in $\mathbb{A}'_p$. Hence by Lemma 4.2, we get

$$|K(\beta)|_p > p^{-([log_p((m)^d)]+1)}.$$

□

Theorem 4.5. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ an algebraic integer of algebraic degree $d \geq 2$ such that $d_\beta(1) = (a_i)_{i\geq1}$ is an infinite and lacunary sequence in the following sense: There exists two sequences $(m_n)_{n\geq1}$ and $(s_n)_{n\geq0}$ such that:*

$$1 = s_0 \leqslant m_1 < s_1 \leqslant m_2 < s_2 \leqslant ... \leqslant m_n < s_n \leqslant m_{n+1} < s_{n+1} \leqslant ...$$

with$(s_n - m_n) \geqslant 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geqslant 1$. Then,

$$\limsup_{n\to+\infty} \frac{v_p(\beta)(m_n - s_n)}{[log_p((m_n)^d)] + 1} \leqslant 1$$

Proof:
We consider the polynomial

$$k_n(y) = -y^{m_n} + a_1 y^{m_{n-1}} + a_2 y^{m_{n-2}} + ... + a_{m_n}$$

It is clear that $k_n(y)$ is a polynomial of degree m_n. Let $P_\beta(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$ be the minimal polynomial of β. Therefore, on the one hand according to Lemma 4.4 we get

$$|k_n(\beta)|_p > p^{-([log_p((m_n)^d)]+1)}. \quad (1)$$

and on the other hand, we have

$$k_n(\beta) = \beta^{m_n}(a_{s_n}\beta^{-s_n} + a_{s_{n-1}}\beta^{-s_n+1} + ...),$$

which involves that

$$\begin{aligned} |k_n(\beta)|_p &= |\beta|_p^{m_n}|a_{s_n}\beta^{-s_n} + a_{s_{n-1}}\beta^{-s_n+1} + ...|_p \\ &\leq |\beta|_p^{m_n}|\beta|_p|\beta|_p^{-s_n} \\ &\leq |\beta|_p^{m_n-s_n+1} \\ &\leq p^{-v_p(\beta)(m_n-s_n+1)}. \quad (2) \end{aligned}$$

Combining (1) and (2), we get

$$p^{-([log_p((m_n)^d)]+1)} \leq |k_n(\beta)|_p \leq p^{-v_p(\beta)(m_n-s_n+1)}.$$

Therefore,

$$\frac{1}{p^{([log_p((m_n)^d)]+1)}} \leq \frac{1}{p^{v_p(\beta)(m_n-s_n+1)}}.$$

So,

$$p^{([log_p((m_n)^d)]+1)} \geq p^{v_p(\beta)(m_n-s_n+1)}.$$

Finally, we obtain that

$$\limsup_{n\to+\infty} \frac{v_p(\beta)(m_n-s_n)}{[log_p((m_n)^d)]+1} \leqslant 1.$$

□

From the previous Theorem, we display this immediate consequence:

Corollary 4.6. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence in the following sense: There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that:*

$$1 = s_0 \leqslant m_1 < s_1 \leqslant m_2 < s_2 \leqslant ... \leqslant m_n < s_n \leqslant m_{n+1} < s_{n+1} \leqslant ...$$

with$(s_n - m_n) \geqslant 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geqslant 1$.

If $\limsup_{n\to+\infty} \dfrac{v_p(\beta)(m_n-s_n)}{[log_p((m_n)^d)]+1} = +\infty$ then β is a transcendental number.

References

[1] B. Adamczewski and Y. Bugeaud: Dynamics for beta-shifts and Diophantine approximation, Ergod. Th. Dynam. Sys, 27 , 1695-1711, (2007).
[2] J.P. Allouche, M. Cosnard: The Komornik-Loreti constant is tanscendental. Amer. Math. Monthly 107, 448-449, (2000).
[3] D. Boyd: On beta-expansions for Pisot numbers, Math. Comp. 65(N214), 841-860, (1996).
[4] K. Mahler: Arithmetic properties of lacunary power series with integral coefficients, J. Aust.Math. Soc, 5, 56-64, (1965).
[5] J. Neukirch: Algebraic number theory, volume 322 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, (1999).
[6] W. Parry: On the beta-expansions of real numbers, Acta.Math. Acad. Sci. Hungar, 11, 401-416, (1960).
[7] A. Rényi: Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung, 8, 477-493, (1975).
[8] K. Scheicher, V.F. Sirvent, P. Surer: *beta*-expansions of p-adic numbers, Ergod. Th. Dynam. Sys, 36, 924-943, (2016).
[9] J.P. Verger-Gaugry. On gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number, Ann. Inst. Fourier (Grenoble), 56 , pp. 2565-2579, (2006).