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Uniformly resolvable decompositions of λ-fold complete multipartite graph into 4-star

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Abstract. Let $\lambda K_u[g]$ be the λ-fold complete multipartite graph with u parts of size g. A $(K_{1,n}, \lambda)$-resolvable group divisible design (RGDD) of type g^u is a $K_{1,n}$-decomposition of the graph $\lambda K_u[g]$ into parallel classes each of which is a partition of the vertex set. A $(K_{1,n}, \lambda)$-frame of type g^u is a $K_{1,n}$-decomposition of $\lambda K_u[g]$ into partial parallel classes each of which is a partition of the vertex set except for those vertices in one of the u parts. In this paper, we completely solve the existence of a $(K_{1,4}, \lambda)$-frame and a $(K_{1,4}, \lambda)$-RGDD of type g^u for any admissible parameters g, u and λ.

In this paper, we will focus on a problem of graph decomposition. We denote the vertex set and edge set (or edge-multiset) of a graph G (or multigraph) by $V(G)$ and $E(G)$, respectively. Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph G is a set of subgraphs (*blocks*) of G whose edge sets partition $E(G)$, and each subgraph is isomorphic to a graph from $\mathcal{H}$. When $\mathcal{H} = \{H\}$, we write $\mathcal{H}$-decomposition as H-decomposition for brevity. A *parallel class* of a graph G is a set of subgraphs whose vertex sets partition $V(G)$. A parallel class is called *uniform* if each block of the parallel class is isomorphic to the same graph. An $\mathcal{H}$-decomposition of a graph G is called (uniformly) *resolvable* if the blocks can be partitioned into (uniform) parallel classes.

A graph G is called a *complete u-partite graph* denoted by $K[m_1, m_2, \ldots, m_u]$ if $V(G)$ can be partitioned into *u parts* (called *groups*) M_i, $1 \le i \le u$, such that two vertices of G, say x and y, are adjacent if and only if $x \in M_i$ and $y \in M_j$ with $i \ne j$. We use $\lambda K[m_1, m_2, \ldots, m_u]$ for the λ-fold of the complete u-partite graph with m_i vertices in the group M_i. When $\lambda = 1$, we usually omit λ in the notation. We denote the complete u-partite graph with u parts of size g by $K_u[g]$ and by K_v the complete graph on v vertices. There are many results on uniformly resolvable $\mathcal{H}$-decompositions of K_v, especially on uniformly resolvable $\mathcal{H}$-decompositions with $\mathcal{H} = \{G_1, G_2\}$, see [1, 9, 10, 12–16].

A (resolvable) $\mathcal{H}$-decomposition of $\lambda K[m_1, m_2, \ldots, m_u]$ is called a (resolvable) *group divisible design*, denoted by $(\mathcal{H}, \lambda)$-(R)GDD. The *type* of an $(\mathcal{H}, \lambda)$-GDD is the multiset of group sizes $|M_i|$, $1 \le i \le u$, and we usually use the "exponential" notation for its description: type $g_1^{n_1} g_2^{n_2} \ldots g_s^{n_s}$ denotes n_i occurrences of g_i for $1 \le i \le s$ in the multiset. If $\mathcal{F} = \{H\}$, we denote it by (H, λ)-GDD. Let L be a set of positive integers. A *pairwise balanced design*, denoted by (L, λ, v)-PBD, is a $(\{K_k : k \in L\}, \lambda)$-GDD of type 1^v.

For brevity, we use $(a; b_1, b_2, \ldots, b_k)$ to denote the k-star $K_{1,k}$ with vertex set $\{a, b_1, b_2, \ldots, b_k\}$ and edge set $\{\{a, b_i\} \mid 1 \le i \le k\}$. Tarsi has solved the existence of a $(K_{1,k}, \lambda)$-GDD of type 1^n in [18]. There are some

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known results on the existence of $K_{1,3}$-RGDDs. For instance, $(K_{1,3}, 1)$-RGDDs of types 2^4 and 4^4 have been constructed in [11], and the existence of a $(K_{1,3}, 1)$-RGDD of type 12^u for any $u \geq 2$ has been solved in [1].

A set of subgraphs of a complete multipartite graph covering all vertices except those belonging to one part M is said to be a *partial parallel class* missing M. A partition of an $(\mathcal{H}, \lambda)$-GDD of type g^u into partial parallel classes is said to be an $(\mathcal{H}, \lambda)$-*frame* of type g^u. Frames are important combinatorial structures used in graph decompositions. The existence of a (K_4, λ)-frame of type g^u has been completely solved in [4, 6–8, 17, 19, 20]. Chen and Cao have proved the existence of a $(K_{1,3}, \lambda)$-frame of type g^u in [2]. It is not difficult to get the following necessary conditions for the existence of two designs.

Theorem 0.1. *The necessary conditions for the existence of a* $(K_{1,n}, \lambda)$*-frame of type* g^u *are* $\lambda g(n+1) \equiv 0 \pmod{2n}$, $g(u-1) \equiv 0 \pmod{n+1}$, $u \geq 3$, *and* $g \equiv 0 \pmod{n+1}$ *when* $u = 3$.

Theorem 0.2. *The necessary conditions for the existence of a* $(K_{1,n}, \lambda)$-RGDD *of type* g^u *are* $\lambda g(u-1) \equiv 0 \pmod{2n}$, $gu \equiv 0 \pmod{n+1}$, $u \geq 2$, *and* $g \equiv 0 \pmod{n+1}$ *when* $u = 2$.

In this paper, we focus on two designs related to the 4-star $K_{1,4}$ and prove the following main results.

Theorem 0.3. *There exists a* $(K_{1,4}, \lambda)$*-frame of type* g^u *if and only if* $\lambda g \equiv 0 \pmod 8$, $g(u-1) \equiv 0 \pmod 5$, $u \geq 3$ *and* $g \equiv 0 \pmod 5$ *when* $u = 3$.

Theorem 0.4. *A* $(K_{1,4}, \lambda)$-RGDD *of type* g^u *exists if and only if* $\lambda g(u-1) \equiv 0 \pmod 8$, $gu \equiv 0 \pmod 5$, $u \geq 2$, *and* $g \equiv 0 \pmod 5$ *when* $u = 2$.

1. The existence of $(K_{1,4}, \lambda)$-frames

Now we state some basic recursive constructions for a $(K_{1,n}, \lambda)$-frame. Similar proofs of these constructions can be found in [2].

Construction 1.1. *If there exists a* $(K_{1,n}, \lambda)$*-frame of type* $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$*, then there is a* $(K_{1,n}, \lambda)$*-frame of type* $(mg_1)^{u_1}(mg_2)^{u_2} \ldots (mg_t)^{u_t}$ *for any* $m \geq 1$.

Construction 1.2. *If there exist a* $(\{K_k : k \in L\}, 1)$-GDD *of type* $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$ *and a* $(K_{1,n}, \lambda)$*-frame of type* m^k *for each* $k \in L$*, then there exists a* $(K_{1,n}, \lambda)$*-frame of type* $(mg_1)^{u_1}(mg_2)^{u_2} \ldots (mg_t)^{u_t}$.

Construction 1.3. *If there is a* $(K_{1,n}, \lambda)$-RGDD *of type* g^2*, then there exists a* $(K_{1,n}, \lambda)$*-frame of type* g^{2u+1} *for any* $u \geq 1$.

Construction 1.4. *If there exist a* $(K_{1,n}, \lambda)$*-frame of type* $(m_1 g)^{u_1}(m_2 g)^{u_2} \ldots (m_t g)^{u_t}$ *and a* $(K_{1,n}, \lambda)$*-frame of type* $g^{m_i+\varepsilon}$ *for any* $1 \leq i \leq t$*, then there exists a* $(K_{1,n}, \lambda)$*-frame of type* $g^{\sum_{i=1}^{t} m_i u_i + \varepsilon}$*, where* $\varepsilon = 0, 1$.

1.1. $(K_{1,4}, 1)$-*frames*

First, we give a direct construction about n-star.

Lemma 1.5. *Let* $n \geq 4$ *be even. There exists a* $(K_{1,n}, 1)$*-frame of type* $(2n)^{n+2}$.

Proof: Let the vertex set be $\mathbb{Z}_{2n(n+2)}$, and let the groups be $G_u = \{u + v(n+2) \mid 0 \leq v \leq 2n-1\}$, $0 \leq u \leq n+1$. The required $n+1$ partial parallel classes with respect to the group G_u are $\{Q_u^i = \{S_i + l + u \mid l \in (n+2)\mathbb{Z}_{2n(n+2)}\} \mid 1 \leq i \leq n+1\}$, where $S_i = (i; i + c_{i1}, \ldots, i + c_{in})$, $1 \leq i \leq n+1$, and

$$c_{ij} = (n+2)(i-1) + j - i + 1,\ 1 \leq i \leq \tfrac{n}{2},\ i \leq j \leq n,$$

$$c_{ij} = (n+2)(i-1) + j - i,\ 2 \leq i \leq \tfrac{n+2}{2},\ 1 \leq j < i,$$

$$c_{ij} = n(n+2) - c_{n+2-i, n+1-j},\ i = \tfrac{n+2}{2},\ \tfrac{n+2}{2} \leq j \leq n \text{ or } \tfrac{n+4}{2} \leq i \leq n+1,\ 1 \leq j \leq n.$$

For each $1 \leq i \leq n+1$, the $n+1$ integers i, $i + c_{ij}$, $1 \leq j \leq n$, are all distinct modulo $n+2$. Then each Q_u^i is a partial parallel class. □

We provide a construction about a $(K_{1,n}, 1)$-RGDD of type $(2n(n+1))^2$. Note that another solution for the case $n = 4$ is provided in [10].

Lemma 1.6. *Let $n \geq 4$ be even. There exists a $(K_{1,n}, 1)$-RGDD of type $(2n(n+1))^2$.*

Proof: Let the vertex set be $\mathbb{Z}_{4n(n+1)}$, and let the groups be $\{u + 2v \mid 0 \leq v \leq 2n(n+1) - 1\}$, $u = 0, 1$. The required $(n+1)^2$ parallel classes can be generated from $n+1$ parallel classes $\{P_i = \{S_i + l, T_i + l \mid l \in 2(n+1)\mathbb{Z}_{4n(n+1)}\} \mid 0 \leq i \leq n\}$, by $+2s \pmod{4n(n+1)}$, $0 \leq s \leq n$, where $S_i = (s_{i0}; s_{i1}, \ldots, s_{in})$, $T_i = (t_{i0}; t_{i1}, \ldots, t_{in})$, $0 \leq i \leq n$, and

$$s_{i0} = 0;\ s_{ij} = 2ni + 2j - 1,\ 0 \leq i \leq n,\ 1 \leq j \leq n,$$

$$t_{i0} = 2n + 1 - 2i;\ t_{ij} = t_{i0} + c_{ij},\ 0 \leq i \leq n,\ 1 \leq j \leq n,$$

$$c_{ij} = 2(n+1)i + 2j + 1,\ 0 \leq i \leq \tfrac{n-2}{2},\ i < j \leq n,$$

$$c_{ij} = 2(n+1)i + 2j - 2n - 3,\ 1 \leq i \leq \tfrac{n}{2},\ 1 \leq j \leq i,$$

$$c_{ij} = 2n(n+1) - c_{n-i,n+1-j},\ i = \tfrac{n}{2},\ \tfrac{n+2}{2} \leq j \leq n \text{ or } \tfrac{n+2}{2} \leq i \leq n,\ 1 \leq j \leq n.$$

For each $0 \leq i \leq n$, since the $2(n+1)$ integers s_{ij}, t_{ij}, $0 \leq j \leq n$, are all distinct modulo $2(n+1)$, each P_i is a parallel class. The proof is complete. □

Lemma 1.7. *There exists a $(K_{1,4}, 1)$-frame of type 40^u for $u \geq 3$.*

Proof: For two values $u = 3, 5$, there exists a $(K_{1,4}, 1)$-RGDD of type 40^2 by Lemma 1.6. Apply Construction 1.3 to get the required $(K_{1,4}, 1)$-frame of type 40^u.

For $u = 4, 6, 8$, let the vertex set be $\mathbb{Z}_{40u}$, and let the groups be $G_i = \{i + uj \mid 0 \leq j \leq 39\}$, $0 \leq i \leq u-1$. The required 25 partial parallel classes with respect to the group G_i can be generated from 5 partial parallel classes $\{Q_{ik} = \{B + l + i \mid B \in C_k, |C_k| = u-1, l \in (5u)\mathbb{Z}_{40u}\} \mid 1 \leq k \leq 5\}$ by $+us \pmod{40u}$, $0 \leq s \leq 4$. The blocks in each C_k are listed below respectively.

$u=4$:	C_1	(1; 2, 3, 6, 7)	(5; 14, 15, 18, 19)	(10; 13, 17, 29, 31)	
	C_2	(1; 18, 19, 23, 26)	(2; 13, 17, 25, 31)	(9; 35, 47, 50, 54)	
	C_3	(1; 31, 34, 35, 38)	(2; 29, 33, 37, 59)	(5; 47, 63, 66, 70)	
	C_4	(1; 47, 50, 51, 54)	(2; 45, 49, 53, 75)	(17; 79, 83, 86, 98)	
	C_5	(1; 55, 71, 78, 86)	(9; 83, 87, 102, 110)	(14; 53, 77, 85, 119)	
$u=6$:	C_1	(1; 2, 3, 4, 5)	(7; 14, 15, 16, 17)	(8; 13, 19, 21, 22)	
		(9; 25, 26, 28, 29)	(10; 41, 50, 53, 57)		
	C_2	(1; 16, 22, 23, 26)	(2; 25, 28, 29, 34)	(3; 37, 38, 40, 41)	
		(5; 44, 49, 50, 51)	(13; 69, 75, 77, 87)		
	C_3	(1; 29, 34, 50, 51)	(2; 43, 53, 55, 57)	(3; 68, 70, 71, 74)	
		(5; 75, 82, 86, 88)	(9; 67, 106, 107, 109)		
	C_4	(1; 53, 58, 62, 64)	(3; 76, 79, 82, 85)	(5; 74, 80, 97, 98)	
		(9; 89, 100, 103, 131)	(17; 105, 116, 141, 147)		
	C_5	(1; 86, 87, 88, 104)	(2; 97, 103, 106, 130)	(4; 33, 109, 110, 119)	
		(8; 115, 125, 129, 137)	(22; 81, 131, 135, 173)		
$u=8$:	C_1	(1; 2, 3, 4, 5)	(6; 11, 12, 13, 15)	(7; 17, 18, 19, 20)	(9; 23, 26, 27, 28)
		(10; 25, 30, 31, 33)	(14; 36, 39, 61, 69)	(34; 62, 75, 77, 78)	
	C_2	(1; 27, 28, 30, 31)	(2; 33, 35, 36, 37)	(3; 39, 45, 49, 52)	(4; 54, 55, 57, 58)
		(6; 51, 63, 65, 66)	(7; 59, 69, 74, 78)	(10; 93, 100, 101, 102)	
	C_3	(1; 38, 39, 59, 62)	(2; 65, 67, 68, 70)	(3; 73, 76, 77, 84)	(5; 74, 87, 89, 90)
		(6; 92, 93, 95, 100)	(11; 106, 109, 111, 134)	(17; 138, 141, 143, 155)	
	C_4	(1; 76, 77, 78, 79)	(2; 95, 99, 101, 103)	(3; 105, 106, 108, 109)	
		(4; 111, 113, 114, 115)	(5; 127, 130, 132, 134)	(6; 137, 138, 140, 147)	
		(9; 142, 171, 173, 190)			
	C_5	(1; 109, 114, 115, 116)	(2; 118, 119, 132, 137)	(3; 140, 143, 145, 146)	
		(4; 150, 151, 153, 165)	(13; 131, 166, 167, 214)	(18; 169, 175, 188, 259)	
		(22; 61, 170, 187, 197)			

Let $L = \{3, 4, 5, 6, 8\}$, for all other values of u with $u \geq 3$ and $u \notin L$, there exists an $(L, 1, u)$-PBD from [3] which is actually a $(\{K_k : k \in L\}, 1)$-GDD of type 1^u. Apply Construction 1.2 with a $(L, 1, u)$-PBD and a $(K_{1,4}, 1)$-frame of type 40^k for each $k \in L$ constructed above to obtain the $(K_{1,4}, 1)$-frame of type 40^u for $u \geq 3$ and $u \notin L$. □

Lemma 1.8. *There exists a $(K_{1,4}, 1)$-frame of type 8^u for $u \equiv 1 \pmod 5$ and $u \geq 6$.*

Proof: For $u = 6$, the conclusion comes from Lemma 1.5.

For $u = 11$, let the vertex set be $\mathbb{Z}_{88}$, and let the groups be $G_i = \{i+11j \mid 0 \le j \le 7\}$, $0 \le i \le 10$. The required 5 partial parallel classes with respect to the group G_i are $\{Q_{ik} = \{B+l+i \mid B \in C_k, |C_k| = 2, l \in 11\mathbb{Z}_{88}\} \mid 1 \le k \le 5\}$. The blocks in each C_k are listed below respectively.

C_1	(1; 2, 3, 4, 5)	(6; 18, 19, 20, 21)	C_2	(1; 6, 7, 8, 9)	(2; 21, 25, 26, 27)
C_3	(1; 10, 17, 18, 19)	(2; 31, 36, 37, 38)	C_4	(1; 21, 27, 28, 29)	(2; 41, 42, 47, 48)
C_5	(1; 31, 32, 39, 57)	(4; 14, 41, 51, 71)			

For $u \ge 16$, we begin with a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-1}{5}}$ by Lemma 1.7 and apply Construction 1.4 with $\varepsilon = 1$ to get the required $(K_{1,4}, 1)$-frame of type 8^u, where the input design a $(K_{1,4}, 1)$-frame of type 8^6. □

Theorem 1.9. *There exists a $(K_{1,4}, 1)$-frame of type g^u if and only if $g \equiv 0 \pmod 8$, $g(u-1) \equiv 0 \pmod 5$, $u \ge 3$ and $g \equiv 0 \pmod 5$ when $u = 3$.*

Proof: The necessary condition is obvious by Theorem 0.1. We distinguish the sufficient conditions into the following two cases.

1. $g \equiv 0 \pmod{40}$ and $u \ge 3$.

There exists a $K_{1,4}$-frame of type 40^u by Lemma 1.7. Then apply Construction 1.1 with $m = g/40$ to get the required design.

2. $g \equiv 8, 16, 24, 32 \pmod{40}$ and $u \equiv 1 \pmod 5$, $u \ge 6$.

A $K_{1,4}$-frame of type 8^u exists by Lemma 1.8. Then we apply Construction 1.1 with $m = g/8$ to get a $K_{1,4}$-frame of type g^u. □

1.2. $(K_{1,4}, 2)$-frames

Lemma 1.10. *There exists a $(K_{1,4}, 2)$-RGDD of type 20^2.*

Proof: Let the vertex set be $\mathbb{Z}_{40}$, and let the groups be $\{2u+v \mid 0 \le u \le 19\}$, $v = 0, 1$. For the required 25 parallel classes, 20 of which can be generated from a parallel class $\{\{(0; 1, 3, 5, 7), (2; 9, 11, 13, 15), (17; 6, 8, 12, 30), (19; 4, 16, 18, 34)\} + 20h \mid h = 0, 1\}$ by $+i \pmod{40}$, $0 \le i \le 19$. The last 5 parallel classes can be generated from a parallel class $\{(0; 17, 19, 21, 23) + 5l \mid 0 \le l \le 7\}$ by $+j \pmod{40}$, $0 \le j \le 4$. □

Lemma 1.11. *There exists a $(K_{1,4}, 2)$-frame of type 20^u for $u \ge 3$.*

Proof: For $u = 3, 5$, there exists a $(K_{1,4}, 2)$-RGDD of type 20^2 by Lemma 1.10. Apply Construction 1.3 to get the required $(K_{1,4}, 2)$-frame of type 20^u.

For $u = 4, 6, 8$, let the vertex set be $\mathbb{Z}_{20u}$, and let the groups be $G_i = \{i + uj \mid 0 \le j \le 19\}$, $0 \le i \le u-1$. The required 25 partial parallel classes with respect to the group G_i are $\{Q^l_{ik} = \{i + ul + Q_k\} \mid 1 \le k \le 5, 0 \le l \le 4\}$, where each $Q_k = \{B + 5ut \mid B \in C_k, |C_k| = u-1, 0 \le t \le 3\}$ is a partial parallel class with respect to G_0. The blocks in each C_k are listed below.

$u = 4$:	C_1	(1; 2, 63, 66, 67)	(5; 10, 34, 71, 15)	(18; 9, 39, 37, 53)	
	C_2	(1; 3, 55, 14, 71)	(2; 19, 9, 53, 5)	(7; 10, 38, 26, 57)	
	C_3	(1; 14, 18, 35, 47)	(2; 17, 43, 9, 25)	(13; 46, 51, 50, 39)	
	C_4	(1; 23, 70, 26, 79)	(2; 29, 27, 25, 51)	(15; 53, 77, 58, 74)	
	C_5	(1; 31, 46, 7, 10)	(2; 29, 43, 35, 13)	(19; 14, 18, 25, 77)	
$u = 6$:	C_1	(1; 34, 3, 35, 2)	(11; 7, 8, 10, 39)	(15; 77, 29, 76, 26)	(43; 82, 51, 23, 80)
		(44; 58, 87, 85, 79)			
	C_2	(2; 109, 77, 97, 105)	(13; 118, 46, 111, 44)	(27; 4, 59, 35, 86)	(31; 69, 8, 11, 100)
		(33; 85, 112, 80, 113)			
	C_3	(1; 111, 82, 50, 17)	(2; 58, 115, 113, 27)	(3; 59, 19, 97, 95)	(4; 98, 69, 13, 71)
		(45; 74, 116, 100, 76)			
	C_4	(1; 58, 94, 119, 38)	(3; 80, 47, 49, 106)	(2; 101, 103, 37, 70)	(35; 82, 75, 85, 81)
		(83; 39, 86, 104, 117)			
	C_5	(1; 52, 16, 23, 28)	(2; 103, 65, 77, 9)	(4; 86, 111, 71, 14)	(8; 19, 70, 37, 3)
		(25; 20, 29, 57, 75)			
$u = 8$:	C_1	(1; 22, 110, 45, 101)	(28; 126, 151, 2, 26)	(39; 157, 129, 18, 84)	(67; 54, 97, 76, 100)
		(93; 145, 35, 33, 143)	(95; 118, 34, 139, 123)	(149; 130, 127, 131, 12)	
	C_2	(3; 113, 74, 49, 105)	(53; 79, 137, 150, 52)	(62; 19, 55, 103, 149)	(100; 27, 31, 26, 6)
		(124; 2, 41, 21, 155)	(127; 138, 117, 116, 68)	(130; 118, 5, 94, 51)	
	C_3	(13; 60, 33, 74, 102)	(17; 155, 109, 70, 147)	(47; 129, 94, 139, 12)	(77; 106, 79, 23, 108)
		(84; 55, 45, 101, 158)	(105; 71, 46, 130, 51)	(121; 138, 3, 76, 82)	
	C_4	(18; 73, 137, 93, 55)	(51; 61, 50, 106, 100)	(121; 159, 118, 124, 85)	(59; 74, 86, 87, 2)
		(68; 49, 54, 3, 111)	(143; 147, 77, 150, 155)	(145; 69, 62, 36, 92)	
	C_5	(11; 2, 6, 78, 138)	(39; 130, 19, 53, 153)	(68; 147, 62, 117, 41)	(76; 71, 146, 89, 154)
		(109; 103, 75, 14, 84)	(110; 25, 17, 12, 125)	(123; 141, 15, 60, 127)	

Let $L = \{3, 4, 5, 6, 8\}$, for all other values of u with $u \geq 3$ and $u \notin L$, apply Construction 1.2 with a $(L, 1, u)$-PBD from [3] and a $(K_{1,4}, 2)$-frame of type 20^k for each $k \in L$ constructed above to obtain the conclusion. □

Lemma 1.12. *There exists a* $(K_{1,4}, 2)$*-frame of type* 4^u *for* $u \equiv 1 \pmod 5$ *and* $u \geq 6$.

Proof: For $u = 6, 11$, let the vertex set be $\mathbb{Z}_{4u}$, and let the groups be $G_i = \{i, i+u, i+2u, i+3u\}$, $0 \leq i \leq u-1$. The required 5 partial parallel classes with respect to the group G_i are $\{Q_{ik} = \{B + i + uj \mid B \in C_k, |C_k| = \frac{u-1}{5}, 0 \leq j \leq 3\} \mid 1 \leq k \leq 5\}$. The blocks in each C_k are listed below respectively.

$u = 6$:	C_1	(2; 9, 13, 11, 4)	C_2	(1; 23, 21, 22, 2)	C_3	(1; 22, 11, 21, 14)	C_4	(2; 1, 9, 10, 11)
	C_5	(3; 8, 11, 13, 22)						

$u = 11$:	C_1	(1; 2, 3, 4, 5)	(6; 7, 8, 9, 10)	C_2	(1; 6, 7, 8, 9)	(2; 10, 12, 14, 15)
	C_3	(1; 6, 7, 8, 10)	(2; 12, 14, 15, 16)	C_4	(1; 10, 15, 16, 17)	(2; 18, 19, 20, 21)
	C_5	(1; 16, 18, 19, 20)	(4; 24, 25, 27, 28)			

For $u \geq 16$, we begin with a $(K_{1,4}, 2)$-frame of type $20^{\frac{u-1}{5}}$ by Lemma 1.11 and apply Construction 1.4 with $\varepsilon = 1$ to get the required $(K_{1,4}, 2)$-frame of type 4^u, where the input design a $(K_{1,4}, 2)$-frame of type 4^6. □

Theorem 1.13. *There exists a* $(K_{1,4}, 2)$*-frame of type* g^u *if and only if* $g \equiv 0 \pmod 4$, $g(u-1) \equiv 0 \pmod 5$, $u \geq 3$ *and* $g \equiv 0 \pmod 5$ *when* $u = 3$.

Proof: The necessary condition is obvious by Theorem 0.1. We distinguish the sufficient conditions into the following two cases.

1. $g \equiv 0 \pmod{20}$ and $u \geq 3$.

There exists a $(K_{1,4}, 2)$-frame of type 20^u by Lemma 1.11. Then apply Construction 1.1 with $m = g/20$ to get the required design.

2. $g \equiv 4, 8, 12, 16 \pmod{20}$ and $u \equiv 1 \pmod 5$, $u \geq 6$.

A $(K_{1,4}, 2)$-frame of type 4^u exists by Lemma 1.12. Then we apply Construction 1.1 with $m = g/4$ to get a $(K_{1,4}, 2)$-frame of type g^u. □

1.3. $(K_{1,4}, 4)$*-frames*

Lemma 1.14. *There exists a* $(K_{1,4}, 4)$-RGDD *of type* 10^2.

Proof: Let the vertex set be $\mathbb{Z}_{20}$, and let the groups be $\{2u+i \mid 0 \le u \le 9\}$, $i = 0, 1$. For the required 25 parallel classes, 20 of which can be generated from a parallel class $\{(0; 1, 3, 5, 7), (2; 9, 11, 13, 15), (17; 6, 8, 10, 12), (19; 4, 14, 16, 18)\}$ by $+1 \pmod{20}$. The last 5 parallel classes can be generated from a parallel class $\{(0; 1, 3, 17, 19) + 5l \mid 0 \le l \le 3\}$ by $+j \pmod{20}$, $0 \le j \le 4$. □

Lemma 1.15. *There exists a* $(K_{1,4}, 4)$*-frame of type* 10^u *for* $u \ge 3$.

Proof: For $u = 3, 5$, there exists a $(K_{1,4}, 4)$-RGDD of type 10^2 by Lemma 1.14. Apply Construction 1.3 to get the required $(K_{1,4}, 4)$-frame of type 10^u.

For $u = 4, 6, 8$, let the vertex set be $\mathbb{Z}_{10u}$, and let the groups be $G_i = \{i + uj \mid 0 \le j \le 9\}$, $0 \le i \le u-1$. The required 25 partial parallel classes with respect to the group G_i are $\{Q^l_{ik} = \{i + ul + Q_k\} \mid 1 \le k \le 5, 0 \le l \le 4\}$, where each $Q_k = \{B + 5ut \mid B \in C_k, |C_k| = u-1, t = 0, 1\}$ is a partial parallel class with respect to G_0. The blocks in each C_k are listed below.

$u=4$:	C_1	(1; 2, 3, 6, 7)	(5; 10, 11, 14, 15)	(18; 9, 13, 17, 19)	
	C_2	(1; 2, 3, 6, 7)	(5; 11, 14, 15, 18)	(10; 13, 17, 19, 29)	
	C_3	(1; 3, 11, 14, 15)	(2; 5, 9, 13, 19)	(7; 10, 17, 18, 26)	
	C_4	(1; 3, 14, 15, 18)	(2; 5, 9, 13, 27)	(17; 6, 10, 31, 39)	
	C_5	(1; 14, 15, 18, 19)	(2; 17, 23, 25, 27)	(11; 26, 29, 30, 33)	
$u=6$:	C_1	(1; 34, 3, 2, 35)	(7; 10, 9, 41, 8)	(13; 47, 16, 15, 14)	(19; 22, 23, 21, 20)
		(29; 25, 58, 57, 56)			
	C_2	(3; 19, 20, 17, 14)	(31; 8, 11, 9, 40)	(32; 13, 7, 45, 46)	(34; 51, 55, 26, 53)
		(57; 28, 29, 5, 52)			
	C_3	(1; 11, 8, 39, 10)	(2; 37, 15, 16, 13)	(3; 47, 49, 20, 14)	(34; 51, 26, 55, 23)
		(57; 29, 35, 28, 52)			
	C_4	(2; 21, 22, 15, 47)	(3; 43, 19, 16, 53)	(5; 44, 58, 56, 37)	(20; 27, 4, 55, 59)
		(31; 40, 9, 8, 41)			
	C_5	(1; 16, 8, 4, 51)	(2; 39, 47, 7, 22)	(3; 26, 59, 58, 44)	(10; 35, 41, 43, 25)
		(23; 15, 19, 50, 57)			
$u=8$:	C_1	(1; 53, 5, 43, 31)	(22; 65, 28, 19, 58)	(27; 77, 74, 10, 42)	(47; 54, 12, 11, 73)
		(57; 78, 69, 66, 36)	(61; 23, 6, 39, 75)	(70; 20, 15, 44, 49)	
	C_2	(3; 34, 9, 2, 53)	(18; 4, 57, 54, 5)	(20; 22, 7, 77, 61)	(26; 12, 71, 75, 68)
		(33; 29, 50, 23, 55)	(39; 46, 65, 27, 19)	(76; 70, 51, 38, 1)	
	C_3	(2; 25, 49, 15, 7)	(17; 43, 6, 63, 69)	(35; 53, 28, 36, 26)	(41; 58, 37, 19, 12)
		(44; 78, 71, 61, 62)	(45; 34, 30, 33, 11)	(60; 54, 50, 67, 39)	
	C_4	(5; 42, 46, 63, 52)	(14; 25, 34, 66, 10)	(22; 61, 7, 77, 3)	(38; 41, 35, 51, 33)
		(39; 49, 68, 70, 19)	(58; 60, 76, 29, 15)	(67; 53, 31, 44, 57)	
	C_5	(3; 5, 1, 74, 12)	(29; 14, 10, 17, 28)	(30; 53, 65, 75, 77)	(33; 67, 6, 44, 76)
		(42; 47, 23, 15, 61)	(62; 9, 11, 31, 59)	(78; 60, 26, 58, 79)	

Let $L = \{3, 4, 5, 6, 8\}$, for all other values of u with $u \ge 3$ and $u \notin L$, apply Construction 1.2 with a $(L, 1, u)$-PBD from [3] and a $(K_{1,4}, 4)$-frame of type 10^k for each $k \in L$ constructed above to get the conclusion. □

Lemma 1.16. *There exists a* $(K_{1,4}, 4)$*-frame of type* 2^u *for* $u \equiv 1 \pmod{5}$ *and* $u \ge 6$.

Proof: For $u = 6, 11$, let the vertex set be $\mathbb{Z}_{2u}$, and let the groups be $G_i = \{i, i+u\}$, $0 \le i \le u-1$. The required 5 partial parallel classes with respect to the group G_i are $\{Q_{ik} = \{B+i+uj \mid B \in C_k, |C_k| = \frac{u-1}{5}, j = 0, 1\} \mid 1 \le k \le 5\}$. The blocks in each C_k are listed below respectively.

$u=6$:	C_1	(1; 11, 2, 9, 4)	C_2	(1; 8, 4, 3, 5)	C_3	(2; 1, 4, 5, 9)	C_4	(2; 1, 3, 4, 5)	C_5	(3; 7, 8, 10, 11)

$u=11$:	C_1	(1; 2, 3, 4, 5)	(6; 7, 8, 9, 10)	C_2	(1; 6, 7, 9, 8)	(2; 10, 14, 15, 16)
	C_3	(1; 2, 3, 4, 5)	(6; 7, 8, 9, 10)	C_4	(1; 6, 7, 8, 9)	(3; 10, 13, 16, 15)
	C_5	(1; 6, 7, 8, 14)	(4; 9, 10, 13, 16)			

For $u \ge 16$, we begin with a $(K_{1,4}, 4)$-frame of type $10^{\frac{u-1}{5}}$ by Lemma 1.15 and apply Construction 1.4 with $\varepsilon = 1$ to get the required $(K_{1,4}, 4)$-frame of type 2^u, where the input design a $(K_{1,4}, 4)$-frame of type 2^6. □

Theorem 1.17. *There exists a* $(K_{1,4}, 4)$*-frame of type* g^u *if and only if* $g \equiv 0 \pmod{2}$, $g(u-1) \equiv 0 \pmod{5}$, $u \ge 3$ *and* $g \equiv 0 \pmod{5}$ *when* $u = 3$.

Proof: The necessary condition is obvious by Theorem 0.1. We distinguish the sufficient conditions into the following 2 cases.

1. $g \equiv 0 \pmod{10}$ and $u \geq 3$.

There exists a $(K_{1,4}, 4)$-frame of type 10^u by Lemma 1.15. Then apply Construction 1.1 with $m = g/10$ to get the required design.

2. $g \equiv 2, 4, 6, 8 \pmod{10}$ and $u \equiv 1 \pmod 5$, $u \geq 6$.

A $(K_{1,4}, 4)$-frame of type 2^u exists by Lemma 1.16. Then we apply Construction 1.1 with $m = g/2$ to get a $(K_{1,4}, 4)$-frame of type g^u. □

1.4. $(K_{1,4}, 8)$-*frames*

Lemma 1.18. *There exists a* $(K_{1,4}, 8)$-RGDD *of type* 5^2.

Proof: Let the vertex set be $\mathbb{Z}_{10}$ and the groups be $\{i, 2+i, 4+i, 6+i, 8+i\}$, $i = 0, 1$. For the required 25 parallel classes, 20 of which can be generated from two parallel classes $\{(0; 1, 3, 5, 7), (9; 2, 4, 6, 8)\}$ and $\{(0; 1, 3, 5, 9), (7; 2, 4, 6, 8)\}$ by $+1 \pmod{10}$. The last 5 parallel classes can be generated from a parallel class $\{(0; 1, 3, 7, 9), (5; 6, 8, 2, 4)\}$ by $+j \pmod{10}$, $0 \leq j \leq 4$. □

Lemma 1.19. *There exists a* $(K_{1,4}, 8)$-*frame of type* 5^u *for* $u \geq 3$.

Proof: For $u = 3, 5$, there exists a $(K_{1,4}, 8)$-RGDD of type 5^2 by Lemma 1.18. Apply Construction 1.3 to get the required $(K_{1,4}, 8)$-frame of type 5^u.

For $u = 4, 6, 8$, let the vertex set be $\mathbb{Z}_{5u}$, and let the groups be $G_i = \{i + uj \mid 0 \leq j \leq 4\}$, $0 \leq i \leq u-1$. The required 25 partial parallel classes with respect to the group G_i can be generated from 5 partial parallel classes $\{Q_{ik} = \{i + Q_k\} \mid 1 \leq k \leq 5\}$ by $+u \pmod{5u}$, where Q_k is a partial parallel class with respect to G_0. The blocks in each Q_k are listed below.

$u = 4$:	Q_1	(1; 10, 6, 19, 3)	(2; 7, 13, 17, 9)	(5; 18, 15, 11, 14)		
	Q_2	(2; 15, 9, 5, 17)	(13; 7, 11, 6, 3)	(19; 18, 14, 1, 10)		
	Q_3	(1; 15, 3, 6, 7)	(5; 2, 14, 11, 18)	(10; 13, 19, 9, 17)		
	Q_4	(3; 14, 2, 9, 6)	(17; 7, 18, 11, 19)	(15; 13, 10, 1, 5)		
	Q_5	(7; 18, 9, 6, 10)	(2; 1, 3, 5, 15)	(14; 19, 13, 17, 11)		
$u = 6$:	Q_1	(11; 28, 20, 22, 27)	(13; 2, 8, 9, 17)	(26; 10, 21, 25, 29)	(7; 16, 14, 5, 23)	(4; 19, 15, 3, 1)
	Q_2	(22; 2, 15, 26, 1)	(29; 25, 7, 16, 21)	(27; 17, 8, 19, 4)	(9; 10, 11, 20, 23)	(28; 3, 14, 13, 5)
	Q_3	(10; 9, 11, 3, 23)	(25; 29, 8, 28, 21)	(22; 17, 20, 2, 14)	(16; 27, 15, 19, 7)	(4; 1, 26, 5, 13)
	Q_4	(8; 3, 25, 15, 22)	(28; 5, 23, 17, 20)	(16; 14, 19, 7, 27)	(9; 29, 26, 10, 11)	(4; 21, 13, 2, 1)
	Q_5	(25; 17, 20, 27, 15)	(23; 13, 3, 8, 9)	(26; 22, 16, 11, 29)	(5; 1, 21, 14, 28)	(2; 7, 19, 10, 4)
$u = 8$:	Q_1	(23; 26, 33, 34, 35)	(21; 36, 22, 20, 3)	(31; 27, 2, 30, 17)	(9; 6, 29, 18, 37)	
		(10; 12, 39, 38, 15)	(13; 11, 25, 28, 4)	(1; 19, 14, 7, 5)		
	Q_2	(35; 31, 17, 38, 9)	(26; 3, 6, 7, 5)	(29; 11, 22, 12, 4)	(23; 1, 34, 18, 33)	
		(14; 13, 39, 15, 20)	(25; 10, 36, 2, 37)	(21; 28, 27, 19, 30)		
	Q_3	(10; 27, 14, 15, 38)	(31; 33, 26, 1, 11)	(4; 23, 22, 18, 17)	(30; 37, 25, 9, 19)	
		(6; 12, 21, 20, 2)	(36; 13, 5, 39, 29)	(34; 3, 28, 35, 7)		
	Q_4	(35; 4, 37, 12, 17)	(27; 21, 1, 39, 34)	(11; 13, 25, 14, 2)	(30; 36, 10, 3, 7)	
		(18; 29, 15, 28, 22)	(38; 9, 19, 20, 23)	(5; 26, 6, 33, 31)		
	Q_5	(11; 13, 6, 21, 38)	(9; 35, 36, 5, 34)	(10; 15, 14, 23, 17)	(28; 18, 30, 7, 37)	
		(12; 2, 19, 39, 22)	(4; 3, 25, 27, 1)	(26; 29, 20, 33, 31)		

Let $L = \{3, 4, 5, 6, 8\}$, for all other values of u with $u \geq 3$ and $u \notin L$, apply Construction 1.2 with a $(L, 1, u)$-PBD from [3] and a $(K_{1,4}, 8)$-frame of type 5^k for each $k \in L$ constructed above to obtain the conclusion. □

Lemma 1.20. *There exists a* $(K_{1,4}, 8)$-*frame of type* 1^u *for* $u \equiv 1 \pmod 5$ *and* $u \geq 6$.

Proof: For $u = 6, 11$, let the vertex set be $\mathbb{Z}_u$, and let the groups be $G_i = \{i\}$, $0 \leq i \leq u-1$. The required 5 partial parallel classes with respect to the group G_i are $\{Q_{ik} = \{B + i \mid B \in C_k, |C_k| = \frac{u-1}{5}\} \mid 1 \leq k \leq 5\}$. The blocks in each C_k are listed below respectively.

$u = 6$:	C_1	(1; 2, 3, 4, 5)	C_2	(2; 1, 3, 4, 5)	C_3	(3; 1, 2, 4, 5)	C_4	(4; 1, 2, 3, 5)	C_5	(5; 1, 2, 3, 4)

$u = 11$:	C_1	(2; 1, 5, 6, 4)	(3; 7, 8, 10, 9)	C_2	(1; 2, 9, 3, 10)	(5; 6, 8, 7, 4)
	C_3	(6; 9, 10, 7, 8)	(1; 3, 5, 2, 4)	C_4	(1; 5, 3, 2, 6)	(4; 8, 7, 10, 9)
	C_5	(1; 2, 4, 6, 7)	(5; 3, 8, 9, 10)			

For $u \geq 16$, we begin with a $(K_{1,4}, 8)$-frame of type $5^{\frac{u-1}{5}}$ by Lemma 1.19 and apply Construction 1.4 with $\varepsilon = 1$ to get the required $(K_{1,4}, 8)$-frame of type 1^u, where the input design a $(K_{1,4}, 8)$-frame of type 1^6. □

Theorem 1.21. *There exists a* $(K_{1,4}, 8)$*-frame of type* g^u *if and only if* $g(u-1) \equiv 0 \pmod 5$, $u \geq 3$ *and* $g \equiv 0 \pmod 5$ *when* $u = 3$.

Proof: The necessary condition is obvious by Theorem 0.1. We distinguish the sufficient conditions into the following 2 cases.

1. $g \equiv 0 \pmod 5$ and $u \geq 3$.

There exists a $(K_{1,4}, 8)$-frame of type 5^u by Lemma 1.19. Then apply Construction 1.1 with $m = g/5$ to get the required design.

2. $g \equiv 1, 2, 3, 4 \pmod 5$ and $u \equiv 1 \pmod 5$, $u \geq 6$.

A $(K_{1,4}, 8)$-frame of type 1^u exists by Lemma 1.20. Then we apply Construction 1.1 with $m = g$ to get a $(K_{1,4}, 8)$-frame of type g^u. □

1.5. Main results on $(K_{1,4}, \lambda)$*-frames*

Proof of Theorem 0.3: The necessary conditions for the existence of a $(K_{1,4}, \lambda)$-frame of type g^u are clearly established by Theorem 0.1. Now we consider its sufficiency and distinguish into 4 cases.

1. $\lambda \equiv 1 \pmod 2$.

There exists a $K_{1,4}$-frame of type g^u by Theorem 1.9. Repeat each block λ times to get a $(K_{1,4}, \lambda)$-frame of type g^u.

2. $\lambda \equiv 2 \pmod 4$.

A $(K_{1,4}, 2)$-frame of type g^u exists by Theorem 1.13. Repeat each block $\lambda/2$ times to get the conclusion.

3. $\lambda \equiv 4 \pmod 8$.

A $(K_{1,4}, 4)$-frame of type g^u exists by Theorem 1.17. Repeat each block $\lambda/4$ times to get a $(K_{1,4}, \lambda)$-frame of type g^u.

4. $\lambda \equiv 0 \pmod 8$.

There exists a $(K_{1,4}, 8)$-frame of type g^u by Theorem 1.21. Repeat each block $\lambda/8$ times to get the required design. □

2. The existence of $(K_{1,4}, \lambda)$-RGDDs

Now we state some basic recursive constructions for $(K_{1,n}, \lambda)$-RGDDs. Similar proofs of these constructions can be found in [1, 2, 5].

Construction 2.1. *If there exists a* $(K_{1,n}, \lambda)$-RGDD *of type* g^u*, then there is a* $(K_{1,n}, \lambda)$-RGDD *of type* $(mg)^u$ *for any* $m \geq 1$.

Construction 2.2. *If there exist a* $(K_{1,n}, \lambda)$-RGDD *of type* $(gu)^l$ *and a* $(K_{1,n}, \lambda)$-RGDD *of type* g^u*, then there is a* $(K_{1,n}, \lambda)$-RGDD *of type* g^{ul}.

Construction 2.3. *If there exist a* $(K_{1,n}, \lambda)$*-frame of type* $(g(u-1))^l$ *and a* $(K_{1,n}, \lambda)$-RGDD *of type* g^u*, then there exists a* $(K_{1,n}, \lambda)$-RGDD *of type* $g^{l(u-1)+1}$.

Proof: Suppose there is a $(K_{1,n}, \lambda)$-frame of type $(g(u-1))^l$ with the groups G_j, $1 \leq j \leq l$, then there are $\frac{\lambda g(n+1)(u-1)}{2n}$ partial parallel classes missing G_j, $1 \leq j \leq l$, denoted by $\{Q_j^i \mid 1 \leq i \leq \frac{\lambda g(n+1)(u-1)}{2n}\}$. Add g new common vertices to the vertex set of G_j and form a new vertex set G'_j. Then break up G'_j with a $(K_{1,n}, \lambda)$-RGDD of type g^u with the groups $G_j^1, G_j^2, \ldots, G_j^{u-1}, M$, where the g common vertices are viewed as a new group M. It has $\frac{\lambda g(n+1)(u-1)}{2n}$ parallel classes, denoted by $\{P_j^i \mid 1 \leq i \leq \frac{\lambda g(n+1)(u-1)}{2n}\}$. Hence, $Q_j^i \cup P_j^i$ is a parallel class of the required $(K_{1,n}, \lambda)$-RGDD of type $g^{l(u-1)+1}$, $1 \leq i \leq \frac{\lambda g(n+1)(u-1)}{2n}$, $1 \leq j \leq l$. Thus, we get $\frac{\lambda g l(n+1)(u-1)}{2n}$ parallel classes as required. □

Before the following construction, we first introduce a concept. Suppose H is a subgraph of a graph G, we use $G - V(H)$ to denote the subgraph of G obtained by deleting the vertices in $V(H)$ and all edges incident with them, and use $G - E(H)$ to denote a subgraph of G obtained by deleting all edges in $E(H)$.

Definition 2.4. *Let G be a λ-fold complete $(u+l)$-partite graph with $u+l$ groups $M_1, M_2, \ldots, M_{u+l}$ such that $|M_i| = g$ for each $1 \le i \le u+l$. Let H be a λ-fold complete l-partite graph with l groups (called holes) $M_{u+1}, M_{u+2}, \ldots, M_{u+l}$. An incomplete resolvable $(K_{1,n}, \lambda)$-group divisible design of type g^u with l holes, denoted by $(K_{1,n}, \lambda)$-IRGDD of type $g^{(u+l,l)}$, is a resolvable $(K_{1,n}, \lambda)$-decomposition of $G - E(H)$ in which there are $\frac{\lambda g u(n+1)}{2n}$ parallel classes of G and $\frac{\lambda g(n+1)(l-1)}{2n}$ partial parallel classes of $G - V(H)$.*

Lemma 2.5. *There exists a* $(K_{1,4}, 1)$-IRGDD *of type* $1^{(65,25)}$.

Proof: Let the vertex set be $\mathbb{Z}_{40} \cup \{\infty_0, \infty_1, \ldots, \infty_{24}\}$, and let the groups be $\{u\}$, $u \in \mathbb{Z}_{40}$, and $\{\infty_l\}$, $0 \le l \le 24$. The required 25 parallel classes and 15 partial parallel classes can be generated from 5 parallel classes $\{P_i \mid 1 \le i \le 5\}$, and 3 partial parallel classes $\{Q_j \mid 1 \le j \le 3\}$, by +8 (mod 40), respectively. The blocks in each P_i and Q_j are listed below respectively.

P_1	$(0;14,15,\infty_7,\infty_6)$	$(1;16,17,\infty_{11},\infty_{12})$	$(2;10,11,\infty_{17},\infty_8)$	$(3;12,21,\infty_{22},\infty_{13})$
	$(4;23,\infty_9,\infty_5,\infty_{18})$	$(5;29,\infty_{10},\infty_{23},\infty_{14})$	$(6;36,\infty_{15},\infty_{16},\infty_{19})$	$(7;22,\infty_{20},\infty_{21},\infty_{24})$
	$(\infty_0;8,9,18,19)$	$(\infty_1;26,27,20,13)$	$(\infty_2;28,37,30,31)$	$(\infty_3;38,39,24,25)$
	$(\infty_4;32,33,34,35)$			
P_2	$(0;16,17,\infty_{12},\infty_{11})$	$(1;18,19,\infty_{16},\infty_{17})$	$(2;12,22,\infty_{22},\infty_{13})$	$(3;28,29,\infty_2,\infty_{18})$
	$(4;31,\infty_{14},\infty_{10},\infty_{23})$	$(5;14,\infty_{15},\infty_3,\infty_{19})$	$(6;13,\infty_{20},\infty_{21},\infty_{24})$	$(7;23,\infty_0,\infty_1,\infty_4)$
	$(\infty_5;8,9,10,11)$	$(\infty_6;26,27,20,21)$	$(\infty_7;36,37,30,15)$	$(\infty_8;38,39,24,25)$
	$(\infty_9;32,33,34,35)$			
P_3	$(0;18,21,\infty_{17},\infty_{16})$	$(1;23,28,\infty_{21},\infty_{22})$	$(2;8,9,\infty_2,\infty_{18})$	$(3;11,30,\infty_7,\infty_{23})$
	$(4;12,\infty_{19},\infty_{15},\infty_3)$	$(5;39,\infty_{20},\infty_8,\infty_{24})$	$(14;29,\infty_0,\infty_1,\infty_4)$	$(7;38,\infty_5,\infty_6,\infty_9)$
	$(\infty_{10};16,17,10,19)$	$(\infty_{11};26,27,20,13)$	$(\infty_{12};36,37,6,15)$	$(\infty_{13};22,31,24,25)$
	$(\infty_{14};32,33,34,35)$			
P_4	$(0;22,23,\infty_{22},\infty_{21})$	$(1;21,29,\infty_1,\infty_2)$	$(2;28,35,\infty_7,\infty_{23})$	$(3;8,9,\infty_{12},\infty_3)$
	$(4;10,\infty_{24},\infty_{20},\infty_8)$	$(13;31,\infty_0,\infty_{13},\infty_4)$	$(6;12,\infty_5,\infty_6,\infty_9)$	$(7;30,\infty_{10},\infty_{11},\infty_{14})$
	$(\infty_{15};16,17,18,11)$	$(\infty_{16};26,19,20,5)$	$(\infty_{17};36,37,14,15)$	$(\infty_{18};38,39,24,25)$
	$(\infty_{19};32,33,34,27)$			
P_5	$(0;28,29,\infty_2,\infty_1)$	$(1;30,31,\infty_6,\infty_7)$	$(2;23,37,\infty_{12},\infty_3)$	$(3;36,38,\infty_{17},\infty_8)$
	$(4;9,\infty_4,\infty_0,\infty_{13})$	$(5;18,\infty_5,\infty_{18},\infty_9)$	$(6;16,\infty_{10},\infty_{11},\infty_{14})$	$(7;19,\infty_{15},\infty_{16},\infty_{19})$
	$(\infty_{20};8,17,10,11)$	$(\infty_{21};26,27,12,13)$	$(\infty_{22};20,21,14,15)$	$(\infty_{23};22,39,24,25)$
	$(\infty_{24};32,33,34,35)$			
Q_1	$(0;1,2,3,4)$	$(5;6,7,8,9)$	$(10;11,12,13,14)$	$(15;16,17,18,19)$
	$(20;21,22,23,24)$	$(25;26,27,28,29)$	$(30;31,32,33,34)$	$(35;36,37,38,39)$
Q_2	$(0;5,6,7,8)$	$(1;9,10,11,12)$	$(2;13,14,15,16)$	$(3;17,18,19,20)$
	$(4;21,22,24,26)$	$(23;28,31,32,34)$	$(29;35,36,37,39)$	$(38;25,27,30,33)$
Q_3	$(0;9,10,11,12)$	$(1;7,8,13,15)$	$(2;17,18,19,21)$	$(3;16,22,23,24)$
	$(4;20,25,29,30)$	$(5;31,32,35,36)$	$(14;26,33,37,38)$	$(39;6,27,28,34)$

□

Lemma 2.6. *There exists a* $(K_{1,4}, 1)$-IRGDD *of type* $4^{(15,5)}$.

Proof: Let the vertex set be $\mathbb{Z}_{40} \cup \{\infty_0, \infty_1, \ldots, \infty_{19}\}$, and let the groups be $\{u, 10+u, 20+u, 30+u\}$, $0 \le u \le 9$, and $\{\infty_l, \infty_{5+l}, \infty_{10+l}, \infty_{15+l}\}$, $0 \le l \le 4$. The required 25 parallel classes and 10 partial parallel classes can be generated from 5 parallel classes $\{P_i \mid 1 \le i \le 5\}$, and 2 partial parallel classes $\{Q_j \mid 1 \le j \le 2\}$, by +8 (mod 40), respectively. The blocks in each P_i and Q_j are listed below respectively.

P_1	$(0;9,11,\infty_5,\infty_6)$	$(1;8,14,\infty_9,\infty_{10})$	$(2;10,17,\infty_{14},\infty_{15})$	$(3;12,16,\infty_{18},\infty_{19})$
	$(4;15,18,\infty_4,\infty_7)$	$(5;19,20,\infty_8,\infty_{11})$	$(6;13,21,\infty_{12},\infty_{13})$	$(7;22,23,\infty_{16},\infty_{17})$
	$(\infty_0;24,25,26,27)$	$(\infty_1;34,35,28,29)$	$(\infty_2;36,37,38,39)$	$(\infty_3;30,31,32,33)$
P_2	$(0;14,15,\infty_9,\infty_{10})$	$(1;16,17,\infty_{13},\infty_{14})$	$(2;8,9,\infty_{18},\infty_{19})$	$(3;10,11,\infty_2,\infty_3)$
	$(4;12,19,\infty_8,\infty_{11})$	$(5;18,21,\infty_{12},\infty_{15})$	$(6;20,23,\infty_{16},\infty_{17})$	$(7;13,38,\infty_0,\infty_1)$
	$(\infty_4;24,25,26,27)$	$(\infty_5;34,35,28,29)$	$(\infty_6;36,37,30,39)$	$(\infty_7;22,31,32,33)$
P_3	$(0;16,17,\infty_{13},\infty_{14})$	$(1;15,18,\infty_{17},\infty_{18})$	$(2;19,20,\infty_2,\infty_3)$	$(3;8,9,\infty_6,\infty_7)$
	$(4;10,11,\infty_{12},\infty_{15})$	$(5;12,14,\infty_{16},\infty_{19})$	$(6;22,39,\infty_0,\infty_1)$	$(7;21,29,\infty_4,\infty_5)$
	$(\infty_8;24,25,26,27)$	$(\infty_9;34,35,28,13)$	$(\infty_{10};36,37,30,31)$	$(\infty_{11};23,38,32,33)$
P_4	$(0;18,19,\infty_{17},\infty_{18})$	$(1;20,22,\infty_1,\infty_2)$	$(2;21,23,\infty_6,\infty_7)$	$(3;14,25,\infty_{10},\infty_{11})$
	$(4;9,31,\infty_{16},\infty_{19})$	$(5;10,11,\infty_0,\infty_3)$	$(6;24,29,\infty_4,\infty_5)$	$(7;12,16,\infty_8,\infty_9)$
	$(\infty_{12};8,17,26,27)$	$(\infty_{13};34,35,28,13)$	$(\infty_{14};36,37,38,39)$	$(\infty_{15};15,30,32,33)$
P_5	$(0;21,23,\infty_1,\infty_2)$	$(1;28,29,\infty_5,\infty_6)$	$(2;18,30,\infty_{10},\infty_{11})$	$(3;22,31,\infty_{14},\infty_{15})$
	$(4;16,20,\infty_0,\infty_3)$	$(13;24,35,\infty_4,\infty_7)$	$(6;17,19,\infty_8,\infty_9)$	$(15;26,33,\infty_{12},\infty_{13})$
	$(\infty_{16};8,9,10,11)$	$(\infty_{17};34,27,36,37)$	$(\infty_{18};5,7,12,14)$	$(\infty_{19};25,32,38,39)$
Q_1	$(0;1,2,3,4)$	$(5;6,7,8,9)$	$(10;11,12,13,14)$	$(15;16,17,18,19)$
	$(20;21,22,23,24)$	$(25;26,27,28,29)$	$(30;31,32,33,34)$	$(35;36,37,38,39)$
Q_2	$(0;5,6,7,8)$	$(1;9,10,12,13)$	$(2;11,14,15,16)$	$(3;17,18,19,20)$
	$(4;21,22,23,32)$	$(30;25,35,36,38)$	$(37;24,26,28,29)$	$(39;27,31,33,34)$

□

Construction 2.7. *Suppose there exist a* $(K_{1,n},\lambda)$*-frame of type* $(gu)^t$*, a* $(K_{1,n},\lambda)$-IRGDD *of type* $g^{(u+l,l)}$*, and a* $(K_{1,n},\lambda)$-RGDD *of type* g^{u+l}*, then there exists a* $(K_{1,n},\lambda)$-RGDD *of type* g^{ut+l}*.*

Proof: We start with a $(K_{1,n},\lambda)$-frame of type $(gu)^t$ with the groups G_j, $1 \le j \le t$. There are $\frac{\lambda gu(n+1)}{2n}$ partial parallel classes missing G_j, denoted by $\{Q_j^i \mid 1 \le i \le \frac{\lambda gu(n+1)}{2n}\}$. Add gl new common vertices to the vertex set of each G_j and form a new vertex set G_j'.

For $1 \le j \le t-1$, break up G_j' with a $(K_{1,n},\lambda)$-IRGDD of type $g^{(u+l,l)}$ with the groups $G_j^1, G_j^2, \ldots, G_j^u$, $M^1, M^2, \ldots, M^l$, where the gl common vertices are viewed as l holes $M^1, M^2, \ldots, M^l$. It has $\frac{\lambda gu(n+1)}{2n}$ parallel classes (denoted by $\{R_j^i \mid 1 \le i \le \frac{\lambda gu(n+1)}{2n}\}$) and $\frac{\lambda g(n+1)(l-1)}{2n}$ partial parallel classes (denoted by $\{S_j^i \mid 1 \le i \le \frac{\lambda g(n+1)(l-1)}{2n}\}$).

For the last set G_t', we break up it with a $(K_{1,n},\lambda)$-RGDD of type g^{u+l} with the groups $G_t^1, G_t^2, \ldots, G_t^u$, $M^1, M^2, \ldots, M^l$. Its $\frac{\lambda g(n+1)(u+l-1)}{2n}$ parallel classes are denoted by $\{R_t^i \mid 1 \le i \le \frac{\lambda g(n+1)(u+l-1)}{2n}\}$.

Let $F_j^i = R_j^i \cup Q_j^i$, $1 \le i \le \frac{\lambda gu(n+1)}{2n}$, $1 \le j \le t$, and let $T_k = R_t^{\frac{\lambda gu(n+1)}{2n}+k} \cup (\cup_{j=1}^{t-1} S_j^k)$, $1 \le k \le \frac{\lambda g(n+1)(l-1)}{2n}$. It is easy to see F_j^i and T_k are parallel classes of the required $(K_{1,n},\lambda)$-RGDD of type g^{ut+l}. □

Construction 2.8. *Suppose there exist a* $(K_{1,n},\lambda)$-IRGDD *of type* $1^{(u+l,l)}$ *and a* $(K_{1,n},\lambda)$-RGDD *of type* 1^l*, then there exists a* $(K_{1,n},\lambda)$-RGDD *of type* 1^{u+l}*.*

Proof: We start with $(K_{1,n},\lambda)$-IRGDD of type $1^{(u+l,l)}$ whose $\alpha = \frac{\lambda u(n+1)}{2n}$ parallel classes are denoted by $\{P_i \mid 1 \le i \le \alpha\}$, and whose $\beta = \frac{\lambda(n+1)(l-1)}{2n}$ partial parallel classes are denoted by $\{Q_j \mid 1 \le j \le \beta\}$. And $(K_{1,n},\lambda)$-RGDD of type 1^l with β parallel classes denoted by $\{P_j' \mid 1 \le j \le \beta\}$. Let $A_j = Q_j \cup P_j'$, $1 \le j \le \beta$. Then both A_j and P_i are parallel classes on the whole vertex set, and they form a $(K_{1,n},\lambda)$-RGDD of type 1^{u+l}. □

2.1. $(K_{1,4},1)$-RGDDs

Lemma 2.9. *There exists a* $(K_{1,4},1)$-RGDD *of type* 5^u *for* $u \equiv 1 \pmod 8$ *and* $u \ge 9$*.*

Proof: For $u = 9$, let the vertex set be $\mathbb{Z}_{45}$, and let the groups be $\{i+9j \mid 0 \le j \le 4\}$, $0 \le i \le 8$. Let $C_1 = (0;1,2,3,4)$ and $C_2 = (0;6,7,8,14)$. For $j = 1,2$, each C_j can generate a parallel class P_j by $+5 \pmod{45}$. P_j can generate 5 parallel classes by $+r \pmod{45}$, $0 \le r \le 4$. Thus, we get 10 parallel classes. The other 15 parallel classes can be generated from a parallel class $\{(0;5,10,11,12), (1;6,13,14,16), (2;7,15,17,18), (3;19,20,24,37), (4;21,25,26,36)\ (8;27,29,30,33), (9;28,31,39,44), (22;32,38,41,42), (23;34,35,40,43)\}$ by $+3 \pmod{45}$.

For $u = 17$, let the vertex set be $\mathbb{Z}_{17} \times \mathbb{Z}_5$, and let the groups be $\{i\} \times \mathbb{Z}_5$, $i \in \mathbb{Z}_{17}$. For each $1 \le j \le 16$, the block C_j can generate a parallel class by $(+1 \pmod{17}, -)$. The other 34 parallel classes can be generated from two parallel classes P_1 and P_2 by $(+1 \pmod{17}, -)$. P_1, P_2 and C_j are listed below respectively.

P_1	$(7_4;8_3,12_3,0_4,8_4)$	$(0_3;4_3,2_4,5_4,10_4)$	$(8_1;6_1,7_1,4_2,11_3)$	$(13_0;1_0,4_0,10_1,11_4)$
	$(5_0;2_0,15_0,3_1,9_3)$	$(15_3;0_2,1_3,7_3,16_3)$	$(9_1;1_1,4_1,14_2,12_4)$	$(12_1;0_0,6_0,10_0,2_1)$
	$(15_2;8_0,14_0,13_1,16_2)$	$(12_0;11_0,16_0,15_1,7_2)$	$(11_1;0_1,14_1,8_2,12_2)$	$(3_2;6_2,10_2,6_3,10_3)$
	$(13_2;2_2,11_2,1_4,3_4)$	$(1_2;5_2,9_2,2_3,4_4)$	$(3_3;13_3,14_3,14_4,16_4)$	$(9_4;5_3,6_4,13_4,15_4)$
	$(9_0;3_0,7_0,5_1,16_1)$			
P_2	$(15_2;3_2,8_3,11_3,14_3)$	$(16_2;13_3,9_4,13_4,14_4)$	$(7_0;1_2,4_2,3_3,5_3)$	$(0_4;3_0,6_2,8_2,13_2)$
	$(10_4;5_0,16_0,2_4,12_4)$	$(3_4;0_0,8_0,12_0,6_1)$	$(15_4;6_0,9_0,9_2,14_2)$	$(5_4;1_0,0_1,9_1,10_1)$
	$(6_3;11_0,8_1,11_1,12_1)$	$(3_1;16_1,2_2,11_2,12_2)$	$(16_4;7_1,15_1,1_3,4_4)$	$(2_0;5_2,7_2,10_2,16_3)$
	$(6_4;4_0,10_0,2_1,4_1)$	$(4_3;13_0,14_0,15_0,5_1)$	$(10_3;12_3,15_3,1_4,7_4)$	$(13_1;0_3,2_3,9_3,11_4)$
	$(8_4;1_1,14_1,0_2,7_3)$			

C_1	$(0_0;8_1,9_2,5_3,16_4)$	C_2	$(0_0;5_1,6_2,3_3,10_4)$	C_3	$(0_0;4_1,4_2,2_3,7_4)$	C_4	$(0_0;1_1,2_2,1_3,1_4)$
C_5	$(0_1;8_0,7_2,7_3,16_4)$	C_6	$(0_1;7_0,6_2,5_3,10_4)$	C_7	$(0_1;6_0,4_2,2_3,8_4)$	C_8	$(0_1;1_0,3_2,1_3,6_4)$
C_9	$(0_2;7_0,7_1,6_3,16_4)$	C_{10}	$(0_2;4_0,6_1,5_3,13_4)$	C_{11}	$(0_2;2_0,5_1,4_3,12_4)$	C_{12}	$(0_2;1_0,2_1,2_3,2_4)$
C_{13}	$(0_3;8_0,9_1,9_2,9_4)$	C_{14}	$(0_3;7_0,8_1,8_2,7_4)$	C_{15}	$(0_3;6_0,7_1,6_2,6_4)$	C_{16}	$(0_3;1_0,3_1,5_2,3_4)$

For $u \ge 25$, a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-1}{8}}$ exists by Theorem 1.9 and a $(K_{1,4}, 1)$-RGDD of type 5^9 constructed above, we get the conclusion by applying Construction 2.3. □

Lemma 2.10. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 1^u *for* $u \equiv 25 \pmod{40}$ *and* $u \ge 25$.

Proof: For $u = 25$, let the vertex set be $\mathbb{Z}_5 \times \mathbb{Z}_5$, and let the groups be $\{i\}$, $i \in \mathbb{Z}_5 \times \mathbb{Z}_5$. The block $C_1 = (0_0; 1_0, 2_0, 3_1, 4_2)$ can generate a parallel class P_1 by $(-, +1 \pmod 5)$. The blocks $C_2 = (0_0; 0_1, 0_2, 3_3, 1_4)$ and $C_3 = (0_0; 1_1, 1_2, 2_3, 3_4)$ can generate two parallel classes P_2 and P_3 by $(+1 \pmod 5, -)$. We can get 5 parallel classes from P_1 by $(+1 \pmod 5, -)$ and 10 parallel classes from P_2 and P_3 by $(-, +1 \pmod 5)$. We get the required 15 parallel classes.

For $u = 65$, there exist a $(K_{1,4}, 1)$-IRGDD of type $1^{(65,25)}$ by Lemma 2.5 and a $(K_{1,4}, 1)$-RGDD of type 1^{25} constructed above, we get a $(K_{1,4}, 1)$-RGDD of type 1^{65} by using Construction 2.8.

For $u = 105$, let the vertex set be $\mathbb{Z}_{21} \times \mathbb{Z}_5$, and let the groups be $\{i\}$, $i \in \mathbb{Z}_{21} \times \mathbb{Z}_5$. For each $1 \le j \le 44$, the block C_j can generate a parallel class by $(+1 \pmod{21}, -)$. The other 21 parallel classes can be generated from a parallel classes P by $(+1 \pmod{21}, -)$. The blocks in P and C_j are listed below respectively.

P	$(0_0;3_0,2_0,1_0,0_1)$	$(1_1;3_1,19_1,2_1,0_2)$	$(1_2;3_2,2_2,19_2,8_3)$	$(1_3;2_3,13_3,3_3,17_4)$
	$(1_4;2_4,7_4,4_4,4_0)$	$(5_0;10_0,11_0,9_0,19_3)$	$(5_1;10_1,9_1,20_1,3_4)$	$(5_2;10_2,9_2,16_2,20_0)$
	$(6_3;11_3,12_3,10_3,8_1)$	$(6_4;15_4,8_4,14_4,8_2)$	$(6_0;18_0,17_0,11_1,11_2)$	$(6_1;18_1,16_1,4_2,16_3)$
	$(6_2;12_2,18_2,18_3,10_4)$	$(20_3;9_3,17_3,5_4,12_0)$	$(11_4;16_4,18_4,13_0,4_1)$	$(7_0;15_0,14_0,5_3,12_4)$
	$(7_1;14_1,15_1,19_4,16_0)$	$(7_2;14_2,20_2,8_0,12_1)$	$(7_3;0_3,15_3,13_1,15_2)$	$(9_4;20_4,13_4,13_2,4_3)$
	$(19_0;17_1,17_2,14_3,0_4)$			

C_1	$(0_1;0_2,0_3,0_4,1_0)$	C_2	$(0_2;0_3,0_4,0_0,3_1)$	C_3	$(0_3;0_4,0_0,1_1,1_2)$
C_4	$(0_4;0_0,1_1,1_2,1_3)$	C_5	$(0_0;1_1,1_2,1_3,1_4)$	C_6	$(0_1;1_2,1_3,1_4,3_0)$
C_7	$(0_2;1_3,1_4,3_0,4_1)$	C_8	$(0_3;1_4,1_0,3_1,2_2)$	C_9	$(0_4;1_0,3_1,3_2,2_3)$
C_{10}	$(0_0;2_1,2_2,2_3,3_4)$	C_{11}	$(0_1;2_2,2_3,2_4,4_0)$	C_{12}	$(0_2;2_3,2_4,4_0,6_1)$
C_{13}	$(0_3;2_4,3_0,4_1,3_2)$	C_{14}	$(0_4;4_0,4_1,5_2,3_3)$	C_{15}	$(0_0;3_1,3_2,3_3,4_4)$
C_{16}	$(0_1;3_2,3_3,3_4,5_0)$	C_{17}	$(0_2;3_3,3_4,5_0,7_1)$	C_{18}	$(0_3;3_4,4_0,5_1,4_2)$
C_{19}	$(0_4;5_0,5_1,6_2,4_3)$	C_{20}	$(0_0;4_1,4_2,4_3,6_4)$	C_{21}	$(0_1;4_2,4_3,4_4,6_0)$
C_{22}	$(0_2;4_3,5_4,6_0,8_1)$	C_{23}	$(0_3;4_4,6_0,7_1,5_2)$	C_{24}	$(0_4;6_0,6_1,7_2,6_3)$
C_{25}	$(0_0;6_1,7_2,5_3,7_4)$	C_{26}	$(0_1;5_2,5_3,5_4,7_0)$	C_{27}	$(0_2;5_3,6_4,7_0,9_1)$
C_{28}	$(0_3;7_4,8_0,8_1,6_2)$	C_{29}	$(0_4;7_0,7_1,8_2,7_3)$	C_{30}	$(0_0;7_1,8_2,6_3,8_4)$
C_{31}	$(0_1;6_2,6_3,6_4,8_0)$	C_{32}	$(0_2;6_3,7_4,8_0,10_1)$	C_{33}	$(0_3;8_4,9_0,9_1,7_2)$
C_{34}	$(0_4;8_0,8_1,9_2,8_3)$	C_{35}	$(0_0;8_1,9_2,7_3,9_4)$	C_{36}	$(0_1;7_2,7_3,8_4,10_0)$
C_{37}	$(0_2;8_3,8_4,9_0,11_1)$	C_{38}	$(0_3;9_4,10_0,10_1,10_2)$	C_{39}	$(0_4;9_0,10_1,10_2,9_3)$
C_{40}	$(0_0;9_1,10_2,9_3,10_4)$	C_{41}	$(0_1;8_2,8_3,9_4,11_0)$	C_{42}	$(0_2;9_3,9_4,10_0,12_1)$
C_{43}	$(0_3;10_4,11_0,12_1,11_2)$	C_{44}	$(0_4;10_0,11_1,11_2,10_3)$		

For $u \ge 145$, a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-25}{40}}$ exists by Theorem 1.9, a $(K_{1,4}, 1)$-IRGDD of type $1^{(65,25)}$ exists by Lemma 2.5, and a $(K_{1,4}, 1)$-RGDD of type 1^{25} which is constructed above. Then apply Construction 2.7 to get the required design. □

Lemma 2.11. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 2^u *for* $u \equiv 5 \pmod{20}$ *and* $u \ge 5$.

Proof: For $u = 5$, let the vertex set be $\mathbb{Z}_{10}$, and let the groups be $\{i, i+5\}$, $0 \le i \le 4$. The required 5 parallel classes are $\{(0; 1, 2, 3, 4) + i, (6; 7, 8, 9, 10) + i\}$.

For $u \ge 25$, a $(K_{1,4}, 1)$-frame of type $8^{\frac{u-1}{5}}$ exists by Theorem 1.9 and a $(K_{1,4}, 1)$-RGDD of type 2^5 is constructed above, we get the conclusion by applying Construction 2.3. □

Lemma 2.12. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 10^u *for* $u \equiv 1 \pmod 4$ *and* $u \ge 5$.

Proof: For $u = 5$, apply Construction 2.1 with $m = 5$ and a $(K_{1,4}, 1)$-RGDD of type 2^5 which exists by Lemma 2.11 to obtain the conclusion.

For $u = 9$, apply Construction 2.1 with $m = 2$ and a $(K_{1,4}, 1)$-RGDD of type 5^9 which exists by Lemma 2.9 to obtain the required design.

For $u \ge 13$, a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-1}{4}}$ exists by Theorem 1.9 and a $(K_{1,4}, 1)$-RGDD of type 10^5 is constructed above, we get the conclusion by using Construction 2.3. □

Lemma 2.13. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 20^u *for* $u \equiv 1 \pmod 2$ *and* $u \ge 3$.

Proof: For $u = 3$, the conclusion comes from [10].

For $u = 5$, apply Construction 2.1 with $m = 10$ and a $(K_{1,4}, 1)$-RGDD of type 2^5 which exists by Lemma 2.11 to obtain the conclusion.

For $u \ge 7$, there exist a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-1}{2}}$ by Theorem 1.9 and a $(K_{1,4}, 1)$-RGDD of type 20^3 from [10], we get the conclusion by using Construction 2.3. □

Lemma 2.14. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 4^u *for* $u \equiv 5 \pmod{10}$ *and* $u \ge 5$.

Proof: For $u = 5$, apply Construction 2.1 with $m = 2$ and a $(K_{1,4}, 1)$-RGDD of type 2^5 which exists by Lemma 2.11 to obtain the conclusion.

For $u = 15$, let the vertex set be $\mathbb{Z}_{60}$, and let the groups be $\{i + 15j \mid 0 \le j \le 3\}$, $0 \le i \le 14$. The block $(0; 4, 23, 31, 32)$ can generate a parallel class P_1 by $+5 \pmod{60}$. The block set $\{(7; 20, 21, 23, 29), (5; 10, 11, 12, 13), (19; 39, 44, 52, 58), (0; 1, 2, 3, 24), (8; 26, 27, 34, 25), (6; 15, 16, 17, 18)\}$ can generate a parallel class P_2 by $+30 \pmod{60}$. We can get 5 parallel classes from P_1 by $+r \pmod{60}$, $0 \le r \le 4$, and 30 parallel classes from P_2 by $+s \pmod{60}$, $0 \le s \le 29$. Thus, we get the required 35 parallel classes.

For $u = 25$, apply Construction 2.1 with $m = 4$ and a $(K_{1,4}, 1)$-RGDD of type 1^{25} which exists by Lemma 2.10 to obtain the conclusion.

For $u \ge 35$, a $(K_{1,4}, 1)$-frame of type $40^{\frac{u-5}{10}}$ exists by Theorem 1.9, a $(K_{1,4}, 1)$-IRGDD of type $4^{(15,5)}$ exists by Lemma 2.6, and a $(K_{1,4}, 1)$-RGDD of type 4^{15} which is constructed above. Then apply Construction 2.7 to get the required design. □

Lemma 2.15. *There exists a* $(K_{1,4}, 1)$-RGDD *of type* 8^u *for* $u \equiv 0 \pmod 5$ *and* $u \ge 5$.

Proof: For $u = 5$, apply Construction 2.1 with $m = 4$ and a $(K_{1,4}, 1)$-RGDD of type 2^5 which exists by Lemma 2.11 to obtain the conclusion.

For $u = 10$, let the vertex set be $\mathbb{Z}_{80}$, and let the groups be $\{i + 10j \mid 0 \le j \le 7\}$, $0 \le i \le 9$. For each $1 \le l \le 4$, the block set C_l can generate a parallel class P_l by $+10 \pmod{80}$. Each P_l can generate 10 parallel classes by $+r \pmod{80}$, $0 \le r \le 9$. The block $(0; 36, 37, 38, 39)$ can generate a parallel class P_5 by $+5 \pmod{80}$. P_5 can generate 5 parallel classes by $+s \pmod{80}$, $0 \le s \le 4$. The blocks in C_l are listed below respectively.

C_1	$(0; 1, 2, 3, 4)$	$(5; 16, 17, 18, 19)$	C_2	$(0; 5, 6, 7, 8)$	$(1; 19, 22, 23, 24)$
C_3	$(0; 9, 15, 16, 17)$	$(1; 28, 32, 33, 34)$	C_4	$(0; 19, 24, 25, 26)$	$(3; 31, 32, 37, 38)$

For $u \ge 15$, there exist a $(K_{1,4}, 1)$-RGDD of type $40^{\frac{u}{5}}$ from [10] and a $(K_{1,4}, 1)$-RGDD of type 8^5 which is constructed above, we get the conclusion by using Construction 2.2. □

Theorem 2.16. *A* $K_{1,4}$-RGDD *of type* g^u *exists if and only if* $g(u-1) \equiv 0 \pmod 8$, $gu \equiv 0 \pmod 5$, $u \ge 2$, *and* $g \equiv 0 \pmod 5$ *when* $u = 2$.

Proof: The necessary condition is obvious by Theorem 0.2. We distinguish the sufficient conditions into the following 8 cases.

1. $g \equiv 0 \pmod{40}$ and $u \geq 2$.

There exists a $K_{1,4}$-RGDD of type 40^u from [10]. Then apply Construction 2.1 with $m = g/40$ to get the required design.

2. $g \equiv 4, 12, 28, 36 \pmod{40}$ and $u \equiv 5 \pmod{10}$, $u \geq 5$.

A $K_{1,4}$-RGDD of type 4^u exists by Lemma 2.14. Then we apply Construction 2.1 with $m = g/4$ to get a $K_{1,4}$-RGDD of type g^u.

3. $g \equiv 8, 16, 24, 32 \pmod{40}$ and $u \equiv 0 \pmod{5}$, $u \geq 5$.

Similarly, we can use Construction 2.1 with $m = g/8$ and a $K_{1,4}$-RGDD of type 8^u by Lemma 2.15 to obtain the required design.

4. $g \equiv 20 \pmod{40}$ and $u \equiv 1 \pmod{2}$, $u \geq 3$.

We apply Construction 2.1 with $m = g/20$ and a $K_{1,4}$-RGDD of type 20^u by Lemma 2.13 to get a $K_{1,4}$-RGDD of type g^u.

5. $g \equiv 10 \pmod{20}$ and $u \equiv 1 \pmod{4}$, $u \geq 5$.

A $K_{1,4}$-RGDD of type 10^u exists by Lemma 2.12. Then we apply Construction 2.1 with $m = g/10$ to get a $K_{1,4}$-RGDD of type g^u.

6. $g \equiv 2, 6, 14, 18 \pmod{20}$ and $u \equiv 5 \pmod{20}$, $u \geq 5$.

We apply Construction 2.1 with $m = g/2$ and a $K_{1,4}$-RGDD of type 2^u by Lemma 2.11 to get a $K_{1,4}$-RGDD of type g^u.

7. $g \equiv 5 \pmod{10}$ and $u \equiv 1 \pmod{8}$, $u \geq 9$.

Similarly, we can use Construction 2.1 with $m = g/5$ and a $K_{1,4}$-RGDD of type 5^u by Lemma 2.9 to obtain the required design.

8. $g \equiv 1, 3, 7, 9 \pmod{10}$ and $u \equiv 25 \pmod{40}$, $u \geq 25$.

A $K_{1,4}$-RGDD of type 1^u exists by Lemma 2.10. Then we apply Construction 2.1 with $m = g$ to get a $K_{1,4}$-RGDD of type g^u. □

2.2. $(K_{1,4}, 2)$-RGDDs

Lemma 2.17. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 20^u *for* $u \geq 2$.

Proof: For $u \equiv 1 \pmod{2}$, there exists a $K_{1,4}$-RGDD of type 20^u by Theorem 2.16. Repeat each block two times to get a $(K_{1,4}, 2)$-RGDD of type 20^u.

For $u \equiv 0 \pmod{2}$, we first construct a $(K_{1,4}, 2)$-RGDD of type 20^2. Let the vertex set be $\mathbb{Z}_{40}$, and let the groups be $\{i + 2j \mid 0 \leq j \leq 19\}$, $i = 0, 1$. The block set $\{(0; 1, 3, 5, 7), (2; 9, 11, 13, 15), (17; 6, 8, 12, 30), (19; 4, 16, 18, 34)\}$ can generate a parallel class P_1 by $+20 \pmod{40}$. P_1 can generate 20 parallel classes by $+r \pmod{40}$, $0 \leq r \leq 19$. The block $(0; 17, 19, 21, 23)$ can generate a parallel class P_2 by $+5 \pmod{40}$. P_2 can generate 5 parallel classes by $+s \pmod{40}$, $0 \leq s \leq 4$.

When $u \geq 4$, we can obtain a $(K_{1,4}, 2)$-RGDD of type $40^{\frac{u}{2}}$ by repeating each block of a $K_{1,4}$-RGDD of type $40^{\frac{u}{2}}$ (Theorem 2.16) two times. Then apply Construction 2.2 with a $(K_{1,4}, 2)$-RGDD of type 20^2 constructed above to get the required design. □

Lemma 2.18. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 1^u *for* $u \equiv 5 \pmod{20}$ *and* $u \geq 5$.

Proof: For $u = 5$, let the vertex set be $\mathbb{Z}_5$, and the groups be $\{i\}$, $i \in \mathbb{Z}_5$. The required parallel classes are $(0; 1, 2, 3, 4) + i$, $0 \leq i \leq 4$.

For $u \geq 25$, there exist a $(K_{1,4}, 2)$-frame of type $4^{\frac{u-1}{4}}$ by Theorem 1.13 and a $(K_{1,4}, 2)$-RGDD of type 1^5, we get the conclusion by using Construction 2.3. □

Lemma 2.19. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 10^u *for* $u \equiv 1 \pmod{2}$ *and* $u \geq 3$.

Proof: For $u = 3$, let the vertex set be $\mathbb{Z}_{30}$, and let the groups be $\{i + 3j \mid 0 \le j \le 9\}$, $0 \le i \le 2$. The block set $\{(25; 24, 15, 23, 20), (6; 28, 17, 26, 22), (14; 1, 12, 18, 19)\}$ can generate a parallel class P_1 by $+15 \pmod{30}$. P_1 can generate 15 parallel classes by $+r \pmod{30}$, $0 \le r \le 14$. The blocks $(7; 18, 6, 15, 14)$ and $(0; 4, 7, 13, 16)$ can generate 2 parallel classes P_2 and P_3 by $+5 \pmod{30}$. Each P_l $(l = 2, 3)$ can generate 5 parallel classes by $+s \pmod{30}$, $0 \le s \le 4$.

For $u = 5$, there exists a $K_{1,4}$-RGDD of type 10^5 by Theorem 2.16. Repeat each block two times to get the required design.

For $u \ge 7$, there exist a $(K_{1,4}, 2)$-frame of type $20^{\frac{u-1}{2}}$ by Theorem 1.13 and a $(K_{1,4}, 2)$-RGDD of type 10^3, we get the conclusion by using Construction 2.3. □

Lemma 2.20. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 2^u *for* $u \equiv 5 \pmod{10}$ *and* $u \ge 5$.

Proof: For $u = 5$, there exists a $K_{1,4}$-RGDD of type 2^5 by Theorem 2.16. Repeat each block two times to get the required design.

For $u \ge 15$, there is a $(K_{1,4}, 2)$-RGDD of type $10^{\frac{u}{5}}$ by Lemma 2.19 and a $(K_{1,4}, 2)$-RGDD of type 2^5 which is constructed above, we get the conclusion by using Construction 2.2. □

Lemma 2.21. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 4^u *for* $u \equiv 0 \pmod{5}$ *and* $u \ge 5$.

Proof: For $u = 5$, there exists a $K_{1,4}$-RGDD of type 4^5 by Theorem 2.16. Repeat each block two times to get the required design.

For $u \ge 10$, there exist a $(K_{1,4}, 2)$-RGDD of type $20^{\frac{u}{5}}$ by Lemma 2.17 and a $(K_{1,4}, 2)$-RGDD of type 4^5, we get the conclusion by using Construction 2.2. □

Lemma 2.22. *There exists a* $(K_{1,4}, 2)$-RGDD *of type* 5^u *for* $u \equiv 1 \pmod{4}$ *and* $u \ge 5$.

Proof: For $u = 5$, there exists a $(K_{1,4}, 2)$-RGDD of type 1^5 by Lemma 2.18. Then apply Construction 2.1 with $m = 5$ to get the conclusion.

For $u = 9$, there exists a $K_{1,4}$-RGDD of type 5^9 by Theorem 2.16. Repeat each block two times to get the required design.

For $u \ge 13$, there exist a $(K_{1,4}, 2)$-frame of type $20^{\frac{u-1}{4}}$ by Theorem 1.13 and a $(K_{1,4}, 2)$-RGDD of type 5^5, we get the conclusion by using Construction 2.3. □

Theorem 2.23. *A* $(K_{1,4}, 2)$-RGDD *of type* g^u *exists if and only if* $g(u-1) \equiv 0 \pmod{4}$, $gu \equiv 0 \pmod{5}$, $u \ge 2$, *and* $g \equiv 0 \pmod{5}$ *when* $u = 2$.

Proof: The necessary conditions for the existence of a $(K_{1,4}, 2)$-RGDD of type g^u are clearly established by Theorem 0.2. Now we consider its sufficiency and distinguish into the following 6 cases.

1. $g \equiv 0 \pmod{20}$ and $u \ge 2$.

We use Construction 2.1 with $m = g/20$ and a $(K_{1,4}, 2)$-RGDD of type 20^u by Lemma 2.17 to obtain the required design.

2. $g \equiv 2, 6, 14, 18 \pmod{20}$ and $u \equiv 5 \pmod{10}$, $u \ge 5$.

A $(K_{1,4}, 2)$-RGDD of type 2^u exists by Lemma 2.20. We apply Construction 2.1 with $m = g/2$ to obtain a $(K_{1,4}, 2)$-RGDD of type g^u.

3. $g \equiv 4, 8, 12, 16 \pmod{20}$ and $u \equiv 0 \pmod{5}$, $u \ge 5$.

Similarly, we can use Construction 2.1 with $m = g/4$ and a $(K_{1,4}, 2)$-RGDD of type 4^u by Lemma 2.21 to obtain the required design.

4. $g \equiv 10 \pmod{20}$ and $u \equiv 1 \pmod{2}$, $u \ge 3$.

A $(K_{1,4}, 2)$-RGDD of type 10^u exists by Lemma 2.19. We apply Construction 2.1 with $m = g/10$ to obtain a $(K_{1,4}, 2)$-RGDD of type g^u.

5. $g \equiv 5 \pmod{10}$ and $u \equiv 1 \pmod{4}$, $u \ge 5$.

We apply Construction 2.1 with $m = g/5$ and a $(K_{1,4}, 2)$-RGDD of type 5^u by Lemma 2.22 to get a $(K_{1,4}, 2)$-RGDD of type g^u.

6. $g \equiv 1, 3, 7, 9 \pmod{10}$ and $u \equiv 5 \pmod{20}$, $u \ge 5$.

Similarly, we can use Construction 2.1 with $m = g$ and a $(K_{1,4}, 2)$-RGDD of type 1^u by Lemma 2.18 to obtain the required design. □

2.3. $(K_{1,4},4)$*-RGDDs*

Lemma 2.24. *There exists a* $(K_{1,4},4)$-RGDD *of type* 10^u *for* $u \geq 2$.

Proof: For $u \equiv 1 \pmod 2$, there exists a $(K_{1,4},2)$-RGDD of type 10^u by Theorem 2.23. Repeat each block two times to get the required design.

For $u \equiv 0 \pmod 2$, we first construct a $(K_{1,4},4)$-RGDD of type 10^2. Let the vertex set be $\mathbb{Z}_{20}$, and let the groups be $\{i+2j \mid 0 \leq j \leq 9\}, i = 0,1$. The parallel class $P_1 = \{(0;1,3,5,7),(2;9,11,13,15),(17;6,8,10,12),(19;4,14,16,18)\}$ can generate 20 parallel classes by $+1 \pmod{20}$. The block $(0;1,3,17,19)$ can generate a parallel class P_2 by $+5 \pmod{20}$. P_2 can generate 5 parallel classes by $+s \pmod{20}$, $0 \leq s \leq 4$.

When $u \geq 4$, we can obtain a $(K_{1,4},4)$-RGDD of type $20^{\frac{u}{2}}$ by repeating each block of a $(K_{1,4},2)$-RGDD of type $20^{\frac{u}{2}}$ (Theorem 2.23) two times. Then apply Construction 2.2 with a $(K_{1,4},4)$-RGDD of type 10^2 constructed above to get the required design. □

Lemma 2.25. *There exists a* $(K_{1,4},4)$-RGDD *of type* 5^u *for* $u \equiv 1 \pmod 2$ *and* $u \geq 3$.

Proof: For $u = 3$, let the vertex set be $\mathbb{Z}_{30}$, and let the groups be $\{i+3j \mid 0 \leq j \leq 4\}, 0 \leq i \leq 2$. The parallel class $P_1 = \{(0;1,2,4,5),(3;7,8,11,13),(14;6,9,10,12)\}$ can generate 15 parallel classes by $+1 \pmod{15}$. The blocks $(0;1,2,4,8)$ and $(0;1,2,8,14)$ can generate 2 parallel classes P_2 and P_3 by $+5 \pmod{15}$. Each P_l $(l = 2,3)$ can generate 5 parallel classes by $+s \pmod{15}$, $0 \leq s \leq 4$.

For $u = 5$, there exists a $(K_{1,4},2)$-RGDD of type 5^5 by Theorem 2.23. Repeat each block two times to get the required design.

For $u \geq 7$, there exist a $(K_{1,4},4)$-frame of type $10^{\frac{u-1}{2}}$ by Lemma 1.15 and a $(K_{1,4},4)$-RGDD of type 5^3, we get the conclusion by using Construction 2.3. □

Lemma 2.26. *There exists a* $(K_{1,4},4)$-RGDD *of type* 1^u *for* $u \equiv 5 \pmod{10}$ *and* $u \geq 5$.

Proof: For $u = 5$, there exists a $(K_{1,4},2)$-RGDD of type 1^5 by Theorem 2.23. Repeat each block two times to get the required design.

For $u \geq 15$, there exist a $(K_{1,4},4)$-RGDD of type $5^{\frac{u}{5}}$ by Lemma 2.25 and a $(K_{1,4},4)$-RGDD of type 1^5, we get the conclusion by using Construction 2.2. □

Lemma 2.27. *There exists a* $(K_{1,4},4)$-RGDD *of type* 2^u *for* $u \equiv 0 \pmod 5$ *and* $u \geq 5$.

Proof: For $u = 5$, there exists a $K_{1,4}$-RGDD of type 2^5 by Theorem 2.16. Repeat each block four times to get the required design.

For $u \geq 10$, there exist a $(K_{1,4},4)$-RGDD of type $10^{\frac{u}{5}}$ by Lemma 2.24 and a $(K_{1,4},4)$-RGDD of type 2^5, we get the conclusion by using Construction 2.2. □

Theorem 2.28. *A* $(K_{1,4},4)$-RGDD *of type* g^u *exists if and only if* $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod 5$, $u \geq 2$, *and* $g \equiv 0 \pmod 5$ *when* $u = 2$.

Proof: The necessary condition is obvious by Theorem 0.2. We distinguish the sufficient conditions into the following 4 cases.

1. $g \equiv 0 \pmod{10}$ and $u \geq 2$.

There exists a $(K_{1,4},4)$-RGDD of type 10^u by Lemma 2.24. Then apply Construction 2.1 with $m = g/10$ to get the required design.

2. $g \equiv 5 \pmod{10}$ and $u \equiv 1 \pmod 2$, $u \geq 3$.

A $(K_{1,4},4)$-RGDD of type 5^u exists by Lemma 2.25. Then we apply Construction 2.1 with $m = g/5$ to get a $(K_{1,4},4)$-RGDD of type g^u.

3. $g \equiv 1,3,7,9 \pmod{10}$ and $u \equiv 5 \pmod{10}$, $u \geq 5$.

Similarly, we can use Construction 2.1 with $m = g$ and a $(K_{1,4},4)$-RGDD of type 1^u by Lemma 2.26 to obtain the required design.

4. $g \equiv 2,4,6,8 \pmod{10}$ and $u \equiv 0 \pmod 5$, $u \geq 5$.

We apply Construction 2.1 with $m = g/2$ and a $(K_{1,4},4)$-RGDD of type 2^u by Lemma 2.27 to get a $(K_{1,4},4)$-RGDD of type g^u. □

2.4. $(K_{1,4}, 8)$-RGDDs

Lemma 2.29. *There exists a* $(K_{1,4}, 8)$-RGDD *of type* 5^u *for* $u \geq 2$.

Proof: For $u \equiv 1 \pmod 2$, there exists a $(K_{1,4}, 4)$-RGDD of type 5^u by Theorem 2.28. Repeat each block two times to get the required design.

For $u \equiv 0 \pmod 2$, we first construct a $(K_{1,4}, 8)$-RGDD of type 5^2. Let the vertex set be $\mathbb{Z}_{10}$, and let the groups be $\{i + 2j \mid 0 \leq j \leq 4\}$, $i = 0, 1$. Two parallel classes $P_1 = \{(0; 1, 3, 5, 7), (9; 2, 4, 6, 8)\}$ and $P_2 = \{(0; 1, 3, 5, 9), (7; 2, 4, 6, 8)\}$ can generate 20 parallel classes by $+1 \pmod{10}$. The block $(0; 1, 3, 7, 9)$ can generate a parallel class P_3 by $+5 \pmod{10}$. P_3 can generate 5 parallel classes by $+s \pmod{10}$, $0 \leq s \leq 4$. When $u \geq 4$, we can obtain a $(K_{1,4}, 8)$-RGDD of type $10^{\frac{u}{2}}$ by repeating each block of a $(K_{1,4}, 4)$-RGDD of type $10^{\frac{u}{2}}$ (Theorem 2.28) two times. Then apply Construction 2.2 with a $(K_{1,4}, 8)$-RGDD of type 5^2 constructed above to get the required design. □

Lemma 2.30. *There exists a* $(K_{1,4}, 8)$-RGDD *of type* 1^u *for* $u \equiv 0 \pmod 5$ *and* $u \geq 5$.

Proof: For $u = 5$, there exists a $(K_{1,4}, 2)$-RGDD of type 1^5 by Theorem 2.23. Repeat each block four times to get the required design.

For $u \geq 10$, there exist a $(K_{1,4}, 8)$-RGDD of type $5^{\frac{u}{5}}$ by Lemma 2.29 and a $(K_{1,4}, 8)$-RGDD of type 1^5, we get the conclusion by using Construction 2.2. □

Theorem 2.31. *A* $(K_{1,4}, 8)$-RGDD *of type* g^u *exists if and only if* $gu \equiv 0 \pmod 5$, $u \geq 2$, *and* $g \equiv 0 \pmod 5$ *when* $u = 2$.

Proof: The necessary conditions for the existence of $(K_{1,4}, 8)$-RGDD of type g^u are clearly established by Theorem 0.2. Now we consider its sufficiency and distinguish into 2 cases.

1. $g \equiv 0 \pmod 5$ and $u \geq 2$.

We use Construction 2.1 with $m = g/5$ and a $(K_{1,4}, 8)$-RGDD of type 5^u by Lemma 2.29 to obtain the required design.

2. $g \equiv 1, 2, 3, 4 \pmod 5$ and $u \equiv 0 \pmod 5$, $u \geq 5$.

A $(K_{1,4}, 8)$-RGDD of type 1^u exists by Lemma 2.30. We apply Construction 2.1 with $m = g$ to obtain a $(K_{1,4}, 8)$-RGDD of type g^u. □

2.5. Main result on $(K_{1,4}, \lambda)$-RGDDs

Now we prove our main result. By Theorem 0.2, it is easy to see that the 4 cases $\lambda = 1, 2, 4, 8$ are crucial for the whole problem.

Proof of Theorem 0.4: We distinguish 4 cases.

1. $\lambda \equiv 1 \pmod 2$.

There exists a $K_{1,4}$-RGDD of type g^u by Theorem 2.16. Repeat each block λ times to get a $(K_{1,4}, \lambda)$-RGDD of type g^u.

2. $\lambda \equiv 2 \pmod 4$.

A $(K_{1,4}, 2)$-RGDD of type g^u exists by Theorem 2.23. Repeat each block $\lambda/2$ times to get the conclusion.

3. $\lambda \equiv 4 \pmod 8$.

A $(K_{1,4}, 4)$-RGDD of type g^u exists by Theorem 2.28. Repeat each block $\lambda/4$ times to get a $(K_{1,4}, \lambda)$-RGDD of type g^u.

4. $\lambda \equiv 0 \pmod 8$.

There exists a $(K_{1,4}, 8)$-RGDD of type g^u by Theorem 2.31. Repeat each block $\lambda/8$ times to get the required design. □

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