Filomat 39:5 (2025), 1735–1745
https://doi.org/10.2298/FIL2505735N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Maximum signless Laplacian Estrada index of tetracyclic graphs

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Abstract. In this study, we aim to determine the unique tetracyclic graph that maximizes the signless Laplacian Estrada index (*SLEE*) among all tetracyclic graphs. The *SLEE* of a graph Ω is defined as the sum of the exponentials of its eigenvalues, expressed as follows:

$$SLEE(\Omega) = \sum_{i=1}^{n} e^{s_i},$$

where $s_1, s_2, ..., s_n$ are the eigenvalues of the signless Laplacian matrix of Ω. By identifying this unique tetracyclic graph, we desire to understand the specific structural characteristics that contribute to the maximum *SLEE* within the class of tetracyclic graphs.

1. Introduction

Consider the connected simple graph $\Omega = (V, E)$ where $|V(\Omega)| = n$ and $|E(\Omega)| = m$. A graph Ω is called a c-cyclic graph if $m = n + c - 1$. In particular, Ω is referred to as a tetracyclic graph if $c = 4$. Let $d_\Omega(v) = |N_\Omega(v)|$ be the degree of vertex v in Ω if $N_\Omega(v)$ stands for the neighbor set of vertex v in Ω. A vertex of degree 1 is called a pendant vertex. A pendant edge is the incidence of an edge with a pendant vertex.

Let $A(\Omega) = \left[a_{ij}\right]$ be the adjacency matrix of Ω and let $D(\Omega) = diag\,(d_1, d_2, ..., d_n)$ be the degree matrix whose diagonal elements are the vertex degrees. The typical Laplacian matrix of Ω is represented by $L(\Omega) = D(\Omega) - A(\Omega)$. The signless Laplacian matrix is represented by $S(\Omega) = D(\Omega) + A(\Omega)$ where $A(\Omega)$, $L(\Omega)$ and $S(\Omega)$ are real symmetric matrices. Their eigenvalues are hence real numbers. For $A(\Omega)$, $L(\Omega)$ and $S(\Omega)$, we represent the eigenvalues as $\lambda_1, \lambda_2, ..., \lambda_n$, $\mu_1, \mu_2, ..., \mu_n$ and $s_1, s_2, ..., s_n$, respectively. The largest eigenvalue of signless Laplacian matrix is called the signless Laplacian spectral radius of Ω.

2020 *Mathematics Subject Classification*. Primary 05C35; Secondary 05C38, 05C50.
Keywords. Tetracyclic graph, signless Laplacian Estrada index, extremal graphs, semi-edge walk.
Received: 06 September 2024; Revised: 17 December 2024; Accepted: 03 January 2025
Communicated by Paola Bonacini
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The Estrada index of a graph Ω is defined as $EE(\Omega) = \sum_{i=1}^{n} e^{\lambda_i}$. It was initially proposed as a measure of the degree folding of a protein [9], however, it has since been found in a variety of biochemical and complex networks related issues. The Laplacian Estrada index shortly extended in [17], and Ayyasamy et al. [2] described the signless Laplacian Estrada index (abbreviated *SLEE*) of a graph Ω as

$$SLEE(\Omega) = \sum_{i=1}^{n} e^{s_i}.$$

Additionally, they established some bounds for *SLEE*. In order to define the extremal graph, an upper bound for *SLEE* based on the vertex connectivity of a graph was constructed by Binthiya et al. [4]. In [7], Ellahi et al. determined the unique graph with maximum *SLEE* among graphs with diameter, number of bridges, vertex connectivity, pendant vertices, and edge connectivity. Additionally, they characterized the unicyclic graphs with the first two smallest and largest *SLEE* in [8]. Wang et al. in [15] characterized the bicyclic graphs and determined the maximal *SLEE* of both the bipartite bicyclic graphs and bicyclic graphs. Moreover, Ellahi et al. [11] demonstrated that, out of all tricyclic graphs, only two graphs have the maximum *SLEE*.

Motivated by these research efforts, we aim to extend the study to tetracyclic graphs and characterize those with the maximum *SLEE*. Building upon the existing knowledge of extremal graphs and maximizing *SLEE* in various graph classes, we seek to identify and elucidate the unique tetracyclic graphs that achieve the highest *SLEE* values, further contributing to the understanding of this intriguing graph parameter.

2. Preliminaries

In this section, we begin with some definitions and notations that were utilized in our research, and then we restate some results that were proven in references [5], [7]. Following that, we provide an auxiliary lemma that is required in order to accomplish the objectives of this paper.

Recall that $T_k(\Omega)$ represents the k^{th} signless Laplacian spectral moment of a graph Ω, expressed as $T_k(\Omega) = \sum_{i=1}^{n} s_i^k$. Let S^k represents the k^{th} power of the matrix $S(\Omega)$. Accordingly, based on the definition of $T_k(\Omega)$, it is evident that $T_k(\Omega)$ = The trace of the matrix S^k. Put differently, $T_k(\Omega) = Tr\left(S^k\right)$. As a result, $SLEE(\Omega)$, given by the Taylor expansion of the exponential function e^{s_i} and the definition of $SLEE(\Omega)$, can be expressed as $SLEE(\Omega) = \sum_{k=0}^{\infty} \frac{T_k(\Omega)}{k!}$.

This equation inspires us to consider the concept of signless Laplacian spectral moments of graphs as a means to compare their *SLEE* values. To further explore this concept, we require a closely related notion that captures the essence of signless Laplacian spectral moments. The following definition and proposition offer this suitable notion and outline its close relation to the signless Laplacian spectral moments of a graph.

Definition 2.1. *[5] A k-length semi-edge walk in Ω, is an alternating sequence $W = y_1e_1y_2e_2...y_ke_ky_{k+1}$ of vertices $y_1, y_2, ..., y_k, y_{k+1}$ and edges $e_1, e_2, ..., e_k$ such that for any $i = 1, 2, 3, ..., k$, the vertices y_i and y_{i+1} are end vertices (which can be non-distinct) of edge e_i. We can define W as a closed semi-edge walk if $y_1 = y_{k+1}$.*

Proposition 2.2. [5] *Let S as the signless Laplacian matrix of a graph Ω. The $(x, y)^{th}$ entry in the matrix S^k is equal to the cardinality of k length semi-edge walks beginning at a vertex x and terminating at vertex y.*

Based on the aforementioned propositions, we can deduce that the cardinality of *k*-length closed semi-edge walk in Ω equals the spectral moment $T_k(\Omega)$.

Let Ω and Γ be two graphs, with $u, v \in V(\Omega)$, and $x, y \in V(\Gamma)$ respectively. Let $S_eW_k(\Omega; u, v)$ be the collection of all *k*-length semi-edge walk in Ω, that begin at u and terminate at v, and $|S_eW_k(\Omega; u, v)| = T_k(\Omega; u, v)$. If $T_k(\Omega; u, v) \leq T_k(\Gamma; x, y)$, for all $k > 0$.

In addition, if $(\Omega; u, v) \prec_s (\Gamma; x, y)$, and there exists some k_0 such that $\left|S_eW_{k_0}(\Omega; u, v)\right| < \left|S_eW_{k_0}(\Gamma; x, y)\right|$. Then we write $(\Omega; u, v) \prec_s (\Gamma; x, y)$.

Let $S_eW_k(\Omega; u, u) = S_eW_k(\Omega; u)$, $T_k(\Omega; u, u) = T_k(\Omega; u)$ and $(\Omega; u, u) = (\Omega; u)$.

We can rewrite the following lemma, which serves as a valuable tool for comparing the *SLEE* values of two graphs, especially when each graph contains a specific subgraph that is isomorphic.

Lemma 2.3. [7] *Let Ω be a graph where $x, y, w_1, w_2, ..., w_r$ be the vertices of Ω. Assume that the subsets of edges $E_y = \{e_1 = yw_1, ..., e_r = yw_r\}$ and $E_x = \{e_1' = xw_1, ..., e_r' = xw_r\}$, that are not in Ω, i.e., $e_i, e_i' \notin E(\Omega)$, for $(i = 1, 2, ..., r)$. Let $\Omega_x \cong \Omega + E_x$ and $\Omega_y \cong \Omega + E_y$. If for each $i = 1, 2, 3, ..., r$, $(\Omega; y) \prec_s (\Omega; x)$ and $(\Omega; w_i, y) \prec_s (\Omega; w_i, x)$, then $SLEE\left(\Omega_y\right) < SLEE(\Omega_x)$.*

Using the aforementioned lemma, we can state that the graph Ω_x is obtained from Ω_y by transferring vertices $w_1, w_2, ..., w_r$ from $N(y)$ to $N(x)$. In this scenario, we refer to the vertices $w_1, w_2, ..., w_r$ as the transferred neighbors, and the graph Ω is termed the transfer route. It is important to note that Ω is a subgraph of both Ω_x and Ω_y.

While Lemma 2.3 serves as a valuable tool, it is often accompanied by various conditions that must be met to apply it successfully. To address this issue, the following lemma offers a special case that allows us to identify the necessary conditions for using Lemma 2.3.

Lemma 2.4. *Let Ω be a graph with $x, y \in V(\Omega)$. If $N(y) \subseteq N(x) \cup \{x\}$, then $(\Omega; y) \prec_s (\Omega; x)$, and $(\Omega; w, y) \prec_s (\Omega; w, x)$ for each $w \in V(\Omega) \setminus \{y\}$. Furthermore, if $d_\Omega(y) < d_\Omega(x)$, then $(\Omega; y) \prec_s (\Omega; x)$, where $d_\Omega(y)$ is the degree of vertex y in the graph Ω.*

Proof. Let Ω be a graph. Assume that $x, y \in V(\Omega)$. When $k \geq 0$, $W \in S_eW_k(\Omega; y)$. Decompose W into three distinct parts, W_1,W_2 and W_3, where W_1 and W_3 should be as long as possible and should only contain the vertex y and the edges yw, where $w \in N(y)\setminus\{x\}$. When W does not contain any other vertex than y, then W_2 and W_3 are empty. By changing the vertex y by x and the edges yw by xw where w is adjacent to y, not to x, we can obtain W_j' from W_j, when $j = 1, 3$. The map $f : S_eW_k(\Omega; y) \to S_eW_k(\Omega; x)$ defined by the rule $f(W_1W_2W_3) = W_1'W_2W_3'$ is injective. This establishes that $(\Omega; y) \prec_s (\Omega; x)$.

Similarly, through the decomposition of each semi-edge walk in $S_eW_k(\Omega; w, y)$ and modifying their concluding segments, it can be inferred that, for each $w \in V(\Omega)$, the relation $(\Omega; w, y) \prec_s (\Omega; w, x)$ holds true.

Since $d_\Omega(y) = \left|S_eW_k(\Omega; y)\right|$, where $d_\Omega(y)$ is the degree of the vertex y in graph Ω, the comparision $d_\Omega(y) < d_\Omega(x)$ leads to the conclusion that $(\Omega; y) \prec_s (\Omega; x)$ □

3. Maximum *SLEE* of tetracyclic graphs

Let $\mathcal{F}_n$ be the class of n-vertex tetracyclic graphs. The unique maximal subgraphs of a tetracyclic graph Ω that have no pendant vertices are defined as base of the graph, denoted by the symbol $B(\Omega)$. In fact, Ω can be obtained from $B(\Omega)$ which is the only minimal tetracyclic subgraph of Ω, by adding some trees to some vertices of $B(\Omega)$.

The lemma highlights the crucial role played by the rule governing the base of an extremal graph that attains the maximum *SLEE*.

Lemma 3.1. *Let an extremal graph Ω that possesses the maximum SLEE among all graphs in $\mathcal{F}_n$. In this graph, each vertex belongs to either $B(\Omega)$ or is a pendant vertex.*

Proof. Consider a subgraph T of Ω that shares exactly one vertex, denoted as x, with $B(\Omega)$. If T is not a star with x as the center vertex, then there exists a neighbor y of x in T such that the degree of y in Ω is greater than 1. Let Ω' be the graph obtained by transferring all vertices in $N(y)\setminus\{x\}$ from the neighborhood of y to the neighborhood of x, and let Γ be the corresponding transfer route graph. According to Lemma 2.4, it implies that $(\Gamma; y) \prec_s (\Gamma; x)$, leading to $SLEE(\Omega) < SLEE(\Omega')$, by Lemma 2.3, which results in a contradiction. Therefore, every subgraph of Ω sharing a single vertex x with $B(\Omega)$ is indeed a star with x as the center vertex, and this establishes the desired result. □

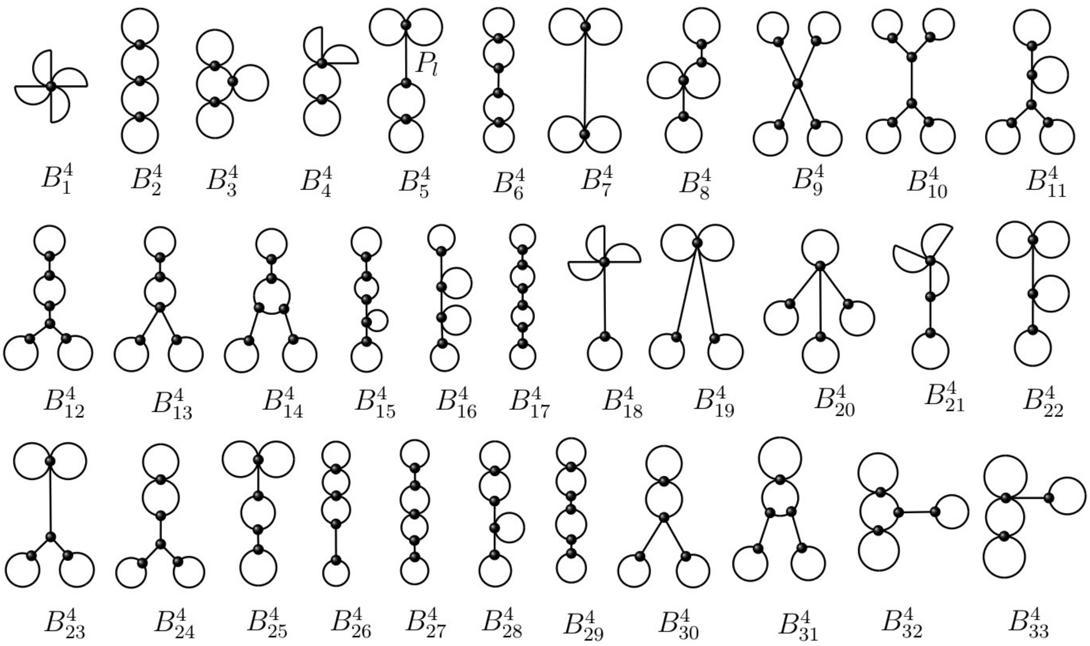

Figure 1: The graphs $B_i^4 : i \in \{1, 2, 3, ...33\}$

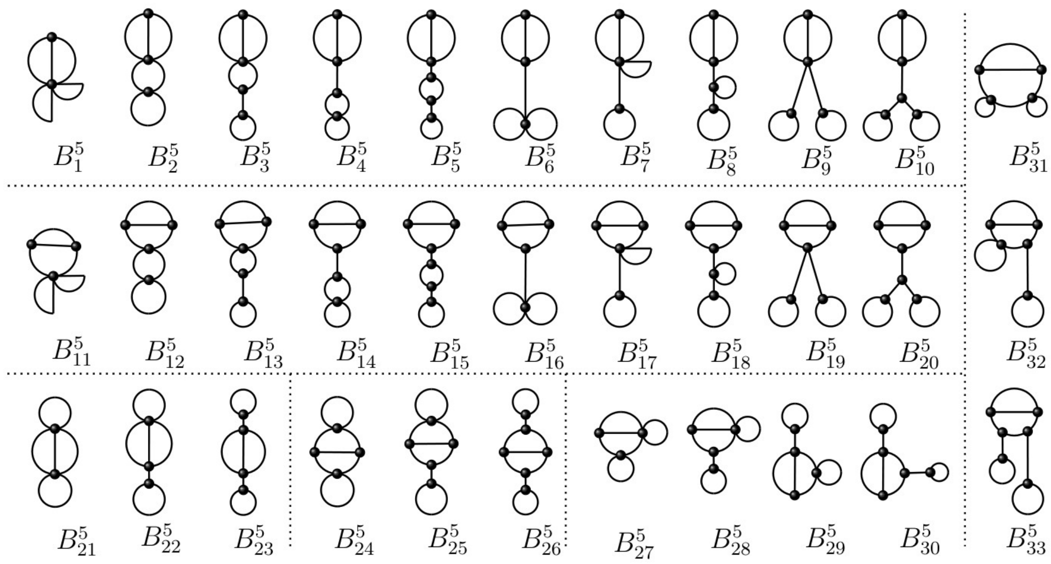

Figure 2: The graphs $B_i^5 : i \in \{1, 2, 3, ...33\}$

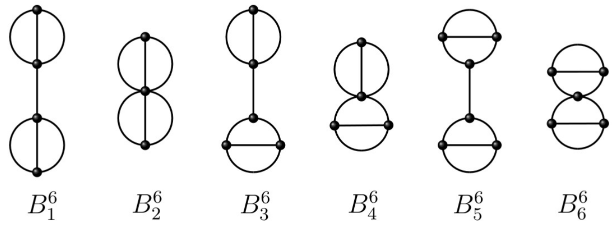

Figure 3: The graphs $B_i^6 : i \in \{1, 2, 3, ...6\}$

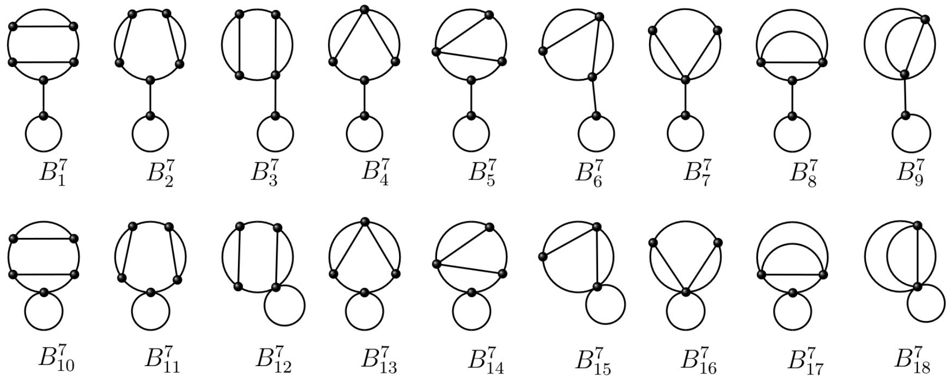

Figure 4: The graphs $B_i^7 : i \in \{1, 2, 3, ...18\}$

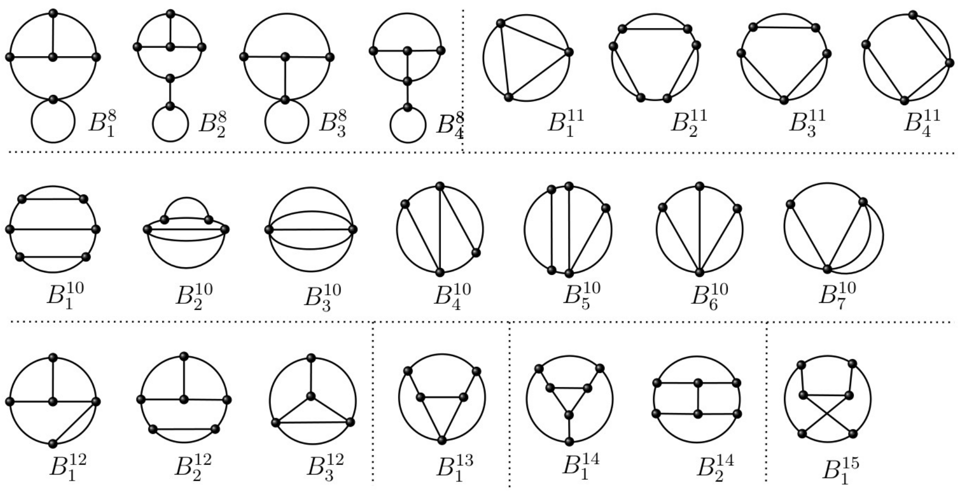

Figure 5: The graphs $B_i^8 : i \in \{1, 2, 3, 4\}$, $B_i^{10} : i \in \{1, 2, 3, ..., 7\}$, $B_i^{11} : i \in \{1, 2, 3, 4\}$, $B_i^{12} : i \in \{1, 2, 3\}$, $B_i^{13} : i \in \{1\}$, $B_i^{14} : i \in \{1, 2\}$, $B_i^{15} : i \in \{1\}$

According to [3], the following types of bases are presented in tetracyclic graphs, as depicted in Figure. 1–5. Let

$$\mathcal{F}_n^4 = \left\{\Omega | B(\Omega) \cong B_i^4, i \in \{1, 2, 3, ...33\}\right\};$$
$$\mathcal{F}_n^5 = \left\{\Omega | B(\Omega) \cong B_i^5, i \in \{1, 2, 3, ...33\}\right\};$$
$$\mathcal{F}_n^6 = \left\{\Omega | B(\Omega) \cong B_i^6, i \in \{1, 2, 3, ...6\}\right\};$$
$$\mathcal{F}_n^7 = \left\{\Omega | B(\Omega) \cong B_i^7, i \in \{1, 2, 3, ...18\}\right\};$$
$$\mathcal{F}_n^8 = \left\{\Omega | B(\Omega) \cong B_i^8, i \in \{1, 2, 3, 4\}\right\};$$
$$\mathcal{F}_n^{10} = \left\{\Omega | B(\Omega) \cong B_i^{10}, i \in \{1, 2, 3, ...7\}\right\};$$
$$\mathcal{F}_n^{11} = \left\{\Omega | B(\Omega) \cong B_i^{11}, i \in \{1, 2, 3, 4\}\right\};$$
$$\mathcal{F}_n^{12} = \left\{\Omega | B(\Omega) \cong B_i^{12}, i \in \{1, 2, 3\}\right\};$$
$$\mathcal{F}_n^{13} = \left\{\Omega | B(\Omega) \cong B_i^{13}, i \in \{1\}\right\};$$
$$\mathcal{F}_n^{14} = \left\{\Omega | B(\Omega) \cong B_i^{14}, i \in \{1, 2\}\right\};$$
$$\mathcal{F}_n^{15} = \left\{\Omega | B(\Omega) \cong B_i^{15}, i \in \{1\}\right\}.$$

By employing this classification, we can confidently assert that $\mathcal{F}_n = \mathcal{F}_n^4 \cup \mathcal{F}_n^5 ... \cup \mathcal{F}_n^8 \cup \mathcal{F}_n^{10} \cup \mathcal{F}_n^{11} ... \cup \mathcal{F}_n^{15}$. Within the vast set of bases for tetracyclic graphs, only eight bases exhibit the distinct property that every edge within them is part of atleast one triangle. These bases are denoted as A_i^j (as illustrated in figure 6) and are defined as follows:
$B(\Omega) \cong A_i^j$, for some $j \in \{4, 10\}$, $i \in \{1, 2, 3, 4\}$,
$B(\Omega) \cong A_i^5$, for some $i \in \{1, 2, 3, ..., 7\}$,
$B(\Omega) \cong A_i^6$, for some $i \in \{1, 2, 3\}$,
$B(\Omega) \cong A_i^7$, for some $i \in \{1, 2, 3, 4, 5\}$,
$B(\Omega) \cong A_1^j$, for some $j \in \{8, 11, 12, 13\}$.
Based on the above notations, the objective of this section is to demonstrate that Γ_j^n is the unique extremal graph with the maximum *SLEE* among the members of $\mathcal{F}_n^j$, where $j \in \{4, 5, 6, 7, 8, 10, 11, 12, 13\}$. To achieve this objective, we require a suitable tool for comparing the *SLEE* values of graphs with the same type of bases. The following lemma provides such a tool.

Lemma 3.2. *Let Ω be a tetracyclic graph with $x, y \in V(\Omega)$, where $e = xy \in E(\Omega)$ and $N(x) \cap N(y) = \emptyset$. If $\Omega \in \mathcal{F}_n{}^j$, for some $j \in \{4, 5, 6, 7, 8, 10, ..., 13\}$, then there exists another graph $\Omega' \in \mathcal{F}_n^j$ such that $SLEE(\Omega) < SLEE(\Omega')$.*

Proof. By Ω' represent the graph obtained from Ω by transferring all of the vertices in $N(y) \setminus \{x\}$ from $N(y)$ to $N(x)$, and Γ represent a transfer route graph. According to Lemma 2.4, $SLEE(\Omega) < SLEE(\Omega')$ is implied, by Lemma 2.3, which states that $(\Gamma; y) \prec_s (\Gamma; x)$. However, the aforementioned transfer has no effect on either the number of simple cycles or edges, therefore, we deduce that $\Omega' \in \mathcal{F}_n^j$. □

Note: According to Lemma 3.2, if the base of the tetracyclic graph Ω contains a path that is not part of a simple cycle. For example, Figure 1.(B_5^4) shows that Ω is not a maximal *SLEE* if it has a path P_l with no simple cycle. Additionally, if two consecutive vertices y_1 and y_2 of a simple cycle $C_q = y_1 y_2 ... y_q y_1$ of Ω do not share any neighbors (i.e.$N(y_1) \cap N(y_2) = \emptyset$). Then Ω is not an *SLEE* maximal graph in $\mathcal{F}_n^j$ for $j \in \{4, 5, 6, 7, 8, 10, 11, 12, 13\}$. As a result, if $B(\Omega)$ is the base of an extremal tetracyclic graph in $\mathcal{F}_n^j$ with maximum *SLEE*, then $B(\Omega) \cong A_i^j$, $j \in \{4, 5, 6, 7, 8, 10, 11, 12, 13\}$.

Lemma 3.3. *If Ω is an SLEE maximal graph in $\mathcal{F}_n^j$, for $j = 4, 5, 6, 7, 8, 10, 12, 13$, then $B(\Omega) \cong A_1^j$.*

Proof. Let $B(\Omega) \cong A_1^4$. Assume Ω is an *SLEE* maximal graph in $\mathcal{F}_n^4$ such that $B(\Omega) \cong A_2^4$. Assume that Ω' is a graph that results from Ω by transferring all vertices $N(x) \setminus \{z, y\}$ from $N(x)$ to $N(z)$, and that Γ is the transfer route graph. It is noteworthy that $B(\Omega') = A_1^4$. Since $N_\Gamma(x) \subseteq N_\Gamma(z) \cup \{z\}$, it implied by Lemma 2.4 that $(\Gamma; x) \prec_s (\Gamma; z)$. This led to the contradiction $SLEE\,(\Omega) < SLEE\,(\Omega')$ in Lemma 2.3. Therefore if Ω is an *SLEE* maximal graph in $\mathcal{F}_n^4$, then $B(\Omega) = A_1^4$. Similar verification was made for the proof for the case $B(\Omega) \cong A_1^j$ for $j = \{5, 6, 7, 8, 10, 12, 13\}$. □

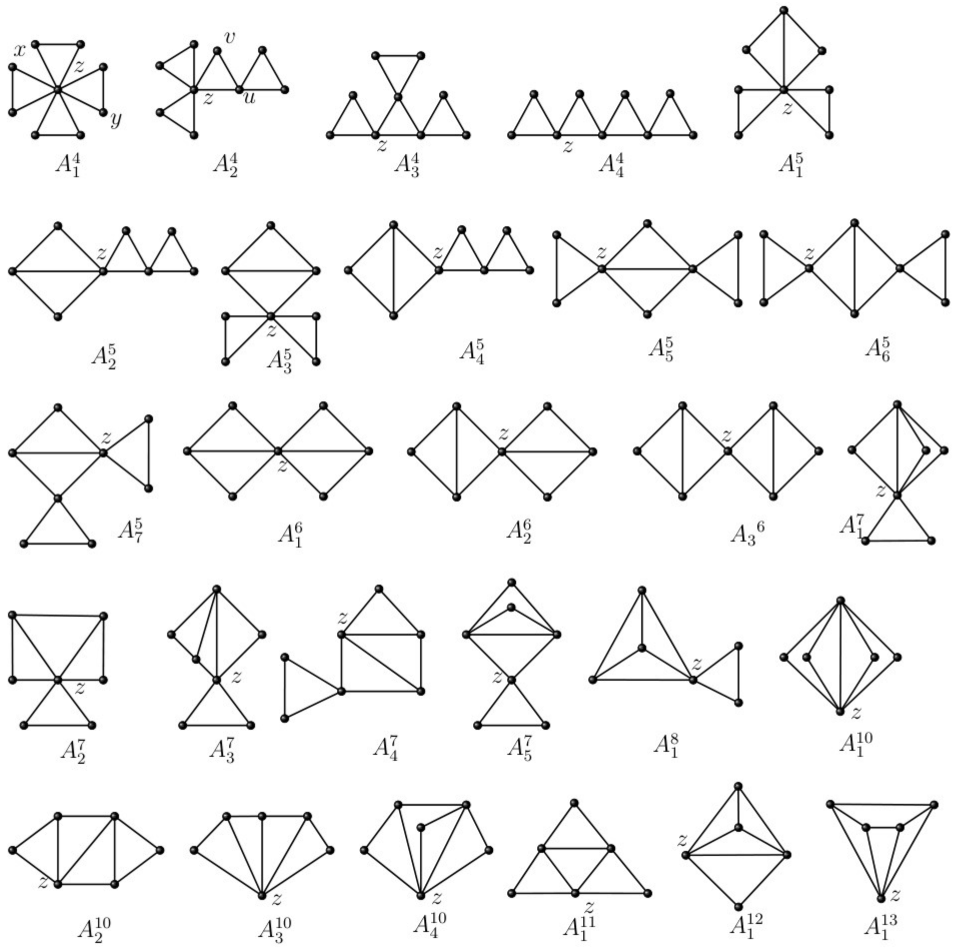

Figure 6: The bases of tetracyclic graphs

4. Main results

Theorem 4.1. *Let Ω be the n-vertex tetracyclic graphs and exactly j simple cycles (i.e., a cycle with no repeated vertices) where $j \in \{4, 5, ..., 8, 10, 12, 13\}$. If Ω is an extremal graph with the maximum SLEE, then $\Omega \cong \Gamma_j^n$, where Γ_j^n is as shown in figure 7, for $j = 4, 5, 6, 7, 8, 10, 12, 13$.*

Proof. Assume that Ω is a *SLEE* maximal graph over $\mathcal{F}_n^j$, where $j \in \{4, 5, ...8, 10, 12, 13\}$. Ω is obtained by attaching a few pendant vertices of A_1^j. Assume that u is a vertex of A_1^j, where $u \neq z$ and it has a few pendant

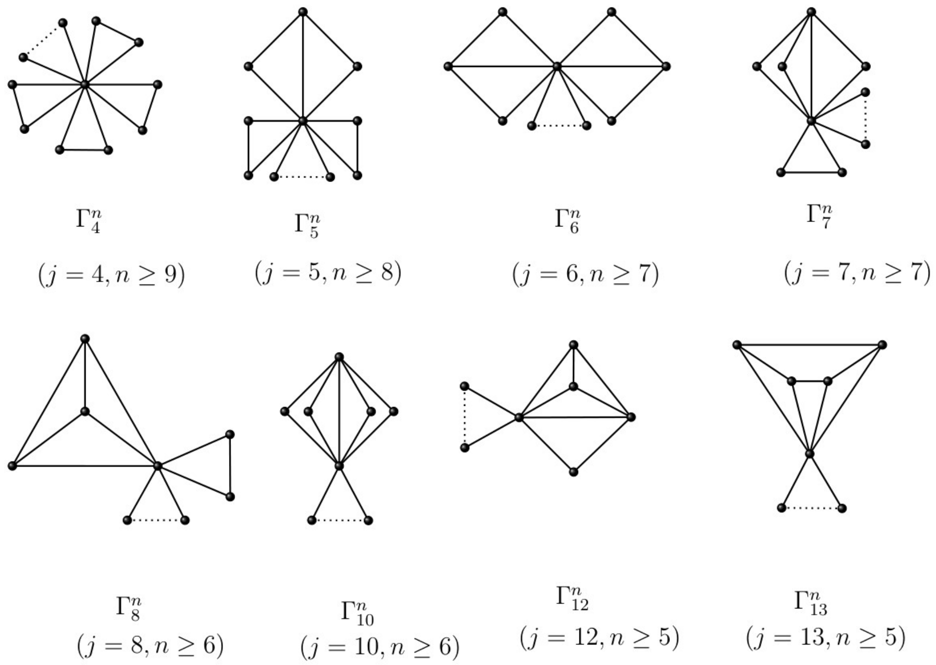

Figure 7: Extremal graphs with the maximum *SLEE*

neighbors (fig.6 shows the vertex z). Let $N^{np}(u)$ is the collection of all non- pendant neighbors of u. Since $N^{np}(u) \subseteq N(z) \cup \{z\}$, by transferring pendant neighbors of u from $N(u)$ to $N(z)$ and by Lemma 2.4 is used to obtain the graph Ω' which results in the contradiction $SLEE\,(\Omega) < SLEE\,(\Omega')$. Therefore all of the $n - \left|V(A_1^j)\right|$ pendant vertices of Ω are attached to z. Hence $\Omega \cong \Gamma_j^n$. □

According to the aforementioned Theorem 4.1. The extremal graph Γ_j^n with the maximum *SLEE* among the members of A_i^j (as shown in fig 7.) For $j = 4, 5...8, 10, 12, 13$, $\phi(\Gamma_j^n, x)$, be the characteristic polynomial of Γ_j^n are as follows.

$$\begin{aligned}
\phi(\Gamma_4^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+54)x^4-(54n+124)x^3+(108n+225)x^2 \\
&\quad -(81n+432)x+432] = (-1)^n(x-1)^6 f_4(x) \\
\phi(\Gamma_5^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+53)x^4-(53n+118)x^3+(102n+208)x^2 \\
&\quad -(72n+376)x+336] = (-1)^n(x-1)^6 f_5(x) \\
\phi(\Gamma_6^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+52)x^4-(52n+112)x^3+(96n+192)x^2 \\
&\quad -(64n+320)x+256] = (-1)^n(x-1)^6 f_6(x) \\
\phi(\Gamma_7^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+51)x^4-(51n+108)x^3+(92n+180)x^2 \\
&\quad -(60n+288)x+224] = (-1)^n(x-1)^6 f_7(x) \\
\phi(\Gamma_8^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+51)x^4-(51n+108)x^3+(92n+180)x^2 \\
&\quad -(60n+288)x+224] = (-1)^n(x-1)^6 f_8(x) \\
\phi(\Gamma_{10}^n) &= (-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+48)x^4-(48n+96)x^3+(80n+144)x^2 \\
&\quad -(48n+192)x+128] = (-1)^n(x-1)^6 f_{10}(x)
\end{aligned}$$

n	5	6	7	8	9	10	11	12
Γ_{10}^n	-	1822.75	3103.12	6087.59	13361.49	31654.47	78663.07	201279.65
Γ_{12}^n	974.53	1514.69	2725.89	5596.12	12673.44	30614.29	76969.21	198326.21
Γ_{13}^n	761.71	1262.34	2410.23	5174.69	12068.03	29676.41	75407.79	195560.12
Γ_{7}^n	-	-	2285.33	4982.49	11754.15	29134.39	74424.25	193696.10
Γ_{8}^n	-	1176.12	2285.33	4982.49	11754.15	29134.39	74424.25	193696.10
Γ_{6}^n	-	-	2061.68	4671.57	11288.58	28384.29	73133.19	191344.67
Γ_{5}^n	-	-	-	4502.34	11015.56	27913.44	72274.09	189703.10
Γ_{4}^n	-	-	-	-	10743.85	27444.88	71418.6	188066.1

Table 1: $SLEE(\Gamma_j^n)$

$$\begin{aligned}\phi(\Gamma_{12}^n) = &(-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+49)x^4-(49n+98)x^3+(82n+152)x^2\\ &-(48n+216)x+144] = (-1)^n(x-1)^6 f_{12}(x)\\ \phi(\Gamma_{13}^n) = &(-1)^n(x-1)^6[x^6-(n+12)x^5+(12n+50)x^4-(50n+100)x^3+(84n+157)x^2\\ &-(45n+240)x+144] = (-1)^n(x-1)^6 f_{13}(x)\end{aligned}$$

For $j = 4,5,6,7,8,10,12,13$, the signless Laplacian Estrada index $SLEE(\Gamma_j^n)$ are computed in Tabel 1.

Lemma 4.2. *Let Ω be an extremal graph with the maximum SLEE in $\mathcal{F}_n$. Then $B(\Omega) \cong A_1^{10}$.*

Proof. Consider Ω be an extremal graph with the maximum *SLEE* in $\mathcal{F}_n$. According to Lemma 3.3, proving that $B(\Omega)$ is not isomorphic to either A_1^4 or A_1^5 is sufficient. Let us assume $B(\Omega) \cong A_1^4$ or A_1^5. Now, consider Ω' the graph obtained by relocating all vertices in $N(v)\backslash z$ from $N(v)$ to $N(u)$, and let H be the transfer route graph. Since $N_H(v) \subset N_H(u)$, it follows that $(H; v) \prec_s (H; u)$ and $(H; w, v) \prec_s (H; w, u)$, by Lemma 2.4, for each $w \in N(v)\backslash z$. Consequently, by Lemma 2.3, we observe that $SLEE(\Omega) < SLEE(\Omega')$, leading to a contradiction. Therefore $B(\Omega) \cong A_1^{10}$. □

Theorem 4.3. *If Ω is an n-vertex tetracyclic graph ($n \geq 6$), then $SLEE(\Omega) \leq SLEE\left(\Gamma_{10}^n\right)$ with equality if and only if $\Omega \cong \Gamma_{10}^n$.*

Proof. Let Ω be a graph in Γ_j^n. The characteristic polynomial of Γ_j^n is $\phi\left(\Gamma_j^n\right) = (-1)^n(x-1)^6 f_j(x)$, for $j = 4,5,6,7,8,10,12,13$.
Consider

$$f_{10}(x) = x^6-(n+12)x^5+(12n+48)x^4-(48n+96)x^3+(80n+144)x^2-(48n+192)x+128.$$

Let $x = n - 3$, then

$$f_4(n-3) = -3n^5 + 81n^4 - 880n^3 + 4824n^2 - 13392n + 15120$$
$$f_5(n-3) = -3n^5 + 81n^4 - 877n^3 + 4771n^2 - 13072n + 14460$$
$$f_6(n-3) = -3n^5 + 81n^4 - 874n^3 + 4718n^2 - 12755n + 13825$$
$$f_7(n-3) = -3n^5 + 81n^4 - 871n^3 + 4671n^2 - 12510n + 13400$$
$$f_8(n-3) = -3n^5 + 81n^4 - 871n^3 + 4671n^2 - 12510n + 13400$$
$$f_{10}(n-3) = -3n^5 + 81n^4 - 862n^3 + 4530n^2 - 11775n + 12125$$
$$f_{12}(n-3) = -3n^5 + 81n^4 - 865n^3 + 4571n^2 - 11964n + 12420$$
$$f_{13}(n-3) = -3n^5 + 81n^4 - 868n^3 + 4612n^2 - 12144n + 12672$$

For $n \geq 6$, we get $f_{10}(n-3) - f_4(n-3) > 0 \Rightarrow SLEE\left(\Gamma_{10}^n\right) > SLEE\left(\Gamma_4^n\right)$. Therefore, we deduce that, for $n \geq 6$, $f_{10}(n-3) - f_j(n-3) > 0$ where $j = 4, 5, 6, 7, 8, 12, 13$.
This suggests that $SLEE\left(\Gamma_{10}^n\right) > SLEE\left(\Gamma_j^n\right)$.
As a result, let Γ_{10}^n be the only extremal graph with the maximum $SLEE$. □

5. Conclusion

The study has determined the maximum value of the signless Laplacian Estrada index among tetracyclic graphs. This value represents the highest possible spectral characteristic within this class of graphs. The tetracyclic graphs attaining the largest signless Laplacian Estrada index offer valuable insights into their structural properties. These graphs exhibit unique connectivity patterns and cycle structures that contribute to their maximized index value.

Acknowledgement: The authors are grateful to the referees for carefully reading the paper and making valuable suggestions and comments.

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