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# On weak law of large numbers and $L^p$-convergence for weighted random variables

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**Abstract.** In the present paper, we consider a sequence of the weighted random variables, which include the sequences of martingale differences, and establish the weak law of large numbers and the convergence in $L^p$ under some weaker conditions. Based on a general normalizing function that satisfies some specific conditions, we extend the weak law of large numbers for general random variables.

## 1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with partial sums $S_n = X_1 + \cdots + X_n$ for every $n \geq 1$. For the independent identically distributed random variables, the following Kolmogorov-Feller theorem provides a necessary and sufficient condition for the weak law of large numbers to hold.

**Theorem 1.1.** ([6, P. 250]) *Assume that $\{X, X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables. Then*

$$\frac{S_n - n\mathbb{E}(X\mathbb{I}\{|X| \leq n\})}{n} \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty$$

*if and only if*

$$x\mathbb{P}(|X| > x) \to 0 \quad as \quad x \to \infty.$$

Klass and Teicher [13] extended the Kolmogorov-Feller weak law of large numbers for asymmetric random variables barely with or without finite mean, using the sequence $\{b_n, n \geq 1\}$ which is a restricted sequence of constants. Gut [8] also proved the statement by $b_n/n$ is slowly varying. Later on, the statement of Klass and Teicher [13] has been generalized to maxima of partial sums of negatively associated and identically distributed random variables by Kruglov [14]. He obtained the independent identically distributed case under the following conditions.

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**Theorem 1.2.** ([14, Theorem 2]) *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, and let $\{b_n, n \geq 1\}$ be a non-decreasing sequence of positive constants, such that*

$$\sum_{k=1}^{n} \frac{b_k^2}{k^2} = O\left(\frac{b_n^2}{n}\right). \tag{1.1}$$

*Then*

$$\frac{1}{b_n} \max_{1\leq k\leq n} \left| S_k - k\mathbb{E}\Big(X\mathbb{I}\{|X| \leq b_n\}\Big) \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty \tag{1.2}$$

*if and only if*

$$n\mathbb{P}(|X| > b_n) \to 0 \quad as \quad n \to \infty. \tag{1.3}$$

Gao and Miao [7] considered a sequence of the general random variables, which included the sequences of martingale differences, and established the weak law of large numbers and the convergence in $L^p$ under some weaker conditions. Weighted versions of the law of large numbers has been considered by many authors. For example, Adler and Rosalsky [1] established the weak law of large numbers for normed weighted sums of independent identically distributed random variables which extend the classic Kolmogorov-Feller weak law of large numbers. Sung [20] obtained the weak law for weighted pairwise independent random variables with an array of constants. In the present work, we consider a large class of summability methods which are defined by Jajte [11] as follows.

**Theorem 1.3.** ([11, Theorem]) *Assume that $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, and let $g(x)$ be a positive increasing function and $h(x)$ a positive function such that $\phi(x) = g(x)h(x)$ satisfies the following conditions: (i) $\phi(x)$ is strictly increasing and $\phi([d, +\infty)) = [0, +\infty)$ for some $d \geq 0$; (ii) there exist $C > 0$ and $k_0 \geq 1$ such that $\phi(x + 1)/\phi(x) \leq C$, for $x \geq k_0$; (iii) there exist constants $a$ and $b$ such that $\phi^2(s) \int_s^\infty \frac{1}{\phi^2(x)} dx \leq as + b$, for $s > d$.Then*

$$\frac{1}{g(n)} \sum_{i=1}^{n} \frac{X_i - \mathbb{E}\Big(X\mathbb{I}\{|X| \leq \phi(i)\}\Big)}{h(i)} \xrightarrow{a.s.} 0 \quad as \quad n \to \infty$$

*if and only if*

$$\mathbb{E}\left(\phi^{-1}(|X|)\right) < \infty, \tag{1.4}$$

*where $\phi^{-1}$ is the inverse of $\phi$.*

Jing and Liang [12] extended the result of Jajte [11] to the negatively associated random variables with identical distribution which contains the case of independent identically distributed random variables. Sung [21] established the sufficient conditions for weighted strong laws of large numbers for identically distributed random variables by introducing three series. Miao et al. [15] established the strong law of large numbers for identically distributed martingale sequence, and further studied the case which is weaker by assuming the random variables are uniformly dominated random variables.

On the other hand, the summability methods of Jajte [11] were also considered to the weak law of large numbers. Balan and Stoica [2] proved the weak law of large numbers for the sequences of free identically distributed random variables which are obtained under certain regularity conditions. Recently, Naderi et al. [17] established the Kolmogorov-Feller weak law of large numbers for maximal weighted sums of independent identically distributed random variables which are slightly weaker than the condition (1.4). The statement as follows.

**Theorem 1.4.** ([17, Theorem 1]) *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables. Let $g(x)$ and $h(x)$ be two nonnegative functions defined on $[0, +\infty)$, $\phi(x) = g(x)h(x)$, satisfying that $h$ is nondecreasing and $\phi$ is strictly increasing with $\phi\Big([0, +\infty)\Big) = [0, +\infty)$, and*

$$\sum_{k=1}^{n} \frac{1}{h^2(k)} = O\Big(\frac{n}{h^2(n)}\Big). \tag{1.5}$$

*Assume that $x \mapsto \mathbb{P}\Big(|X| > x\Big)$ is regularly varying at infinity with index $\rho$ for some $\rho \geq -2$. Then*

$$\frac{1}{g(n)} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}\Big(X\mathbb{I}\{|X| \leq \phi(n)\}\Big)}{h(i)} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty$$

*if and only if*

$$n\mathbb{P}\Big(|X| > \phi(n)\Big) \to 0 \quad as \quad n \to \infty.$$

Naderi et al. [18] further studyed the Kolmogorov-Feller weak law of large numbers for maximal weighted sums of negatively supersdditive dependent random variables, and simulated the asymptotic behavior in the sense of convergence in probability. Boukhari and Boudjemaa [4] obtained the weighted weak law of large numbers for general random variables which are stochastically dominated by a random variable $\xi$. More recently, Boukhari [3] proved the sufficiency part in Theorem 1.4 is valid for a large class of functions $\phi(x)$ without the regularly varying restriction.

**Theorem 1.5.** ([3, Theorem 3.1]) *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, and let $g(x)$ and $h(x)$ be two positive functions defined on $[0, +\infty)$, $\phi(x) = g(x)h(x)$ satisfying either*

($H_1$) *$\phi$ is increasing with $\phi([0, +\infty)) = [0, +\infty)$;*

($H_2$) *The function $h$ is nondecreasing and $\sum_{k=1}^{n} \frac{1}{h^2(k)} = O\Big(\frac{n}{h^2(n)}\Big)$;*

($H_3$) $\sum_{k=1}^{n} \frac{\phi^2(k)}{k^2} = O\Big(\phi^2(n)/n\Big)$,

*or ($H_1$), ($H_2$) and*

($H_4$) $\lim_{n\to\infty} \phi(n)/n = +\infty$.

*If $n\mathbb{P}\Big(|X| > \phi(n)\Big) \to 0$ as $n \to \infty$, then we have*

$$\frac{1}{g(n)} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}\Big(X\mathbb{I}\{|X| \leq \phi(n)\}\Big)}{h(i)} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty.$$

Boukhari [3] remarked that when $h(x) = 1$ then ($H_2$) is fulfilled and $\phi(x) = g(x)$, and if $\phi(x)$ satisfied ($H_1$) and either ($H_3$) or ($H_4$), then the conclusion strengthened the result of Klass and Teicher [13]. Motivated by the above results, the aim of the present paper is further to study the weighted weak law of large numbers under the condition ($H_3$) and weaker dependence restrictions. In addition, we will study the $L^p$-convergence for these random sequences. Throughout the paper, let $C$ denote a positive constant not depending on $n$, which may be different in various places.

## 2. Main results

Firstly, we recall the usual concept of stochastic domination. A sequence of $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable $X$, if

$$\mathbb{P}(|X_n| > x) \leq \mathbb{P}(|X| > x) \tag{2.1}$$

for every $x \geq 0$ and $n \geq 1$. Many authors use an apparently weaker definition of $\{X_n, n \geq 1\}$ being stochastically dominated by a nonnegative random variable $Y$, namely that for every $x \geq 0$ and $n \geq 1$,

$$\mathbb{P}(|X_n| > x) \leq C_1\mathbb{P}(C_2|Y| > x), \tag{2.2}$$

for some $C_1, C_2 \in (0, \infty)$. Rosalsky and Thanh [19, Theorem 2.4] showed that (2.1) and (2.2) are indeed equivalent.

**Theorem 2.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, let $g(x)$ and $h(x)$ be two positive functions defined on $[0, +\infty)$ and $\phi(x) = g(x)h(x)$. Assume that $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable $\xi$, and satisfying the following conditions:*

($B_1$) *$\phi$ is increasing with $\phi([0, +\infty)) = [0, +\infty)$;*

($B_2$) *The function $h$ is nondecreasing and $\sum_{k=1}^{n} \frac{1}{h^2(k)} = O\Big(\frac{n}{h^2(n)}\Big)$;*

($B_3$) $\sum_{k=1}^{n} \frac{\phi^2(k)}{k^2} = O\Big(\phi^2(n)/n\Big)$;

($B_4$) *$n\mathbb{P}\Big(|\xi| > \phi(n)\Big) \to 0$ as $n \to \infty$.*

*Then we have*

$$\frac{1}{g(n)} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}\Big(X_i\mathbb{I}(|X_i| \leq \phi(n))|\mathcal{F}_{i-1}\Big)}{h(i)} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty, \tag{2.3}$$

*where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ for every $n \geq 1$.*

*Furthermore, if the following condition holds*

($B_5$) $\sum_{k=n}^{\infty} \frac{\phi(k+1) - \phi(k)}{k} = O\Big(\phi(n)/n\Big)$,

*then we have*

$$\frac{1}{g(n)} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{h(i)} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty. \tag{2.4}$$

**Remark 2.1.** *Let $h(x) = 1$, then we have $\phi(x) = g(x)$, and then the above weighted types of weak law of large numbers transforms into the general types of weak law of large numbers.*

**Remark 2.2.** *Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of martingale differences. Hall and Heyde [10, Theorem 2.13] established a general weak law of large numbers as follows. If*

($C_1$) $\sum_{i=1}^{n} \mathbb{P}(|X_i| > b_n) \to 0$, (C$_2$) $\frac{1}{b_n}\sum_{i=1}^{n}\mathbb{E}\Big(X_i\mathbb{I}(|X_i|\le b_n)|\mathcal{F}_{i-1}\Big)\xrightarrow{\mathbb{P}}0,$

(C$_3$) $\frac{1}{b_n^2}\sum_{i=1}^{n}\Big(\mathbb{E}\Big[X_i^2\mathbb{I}(|X_i|\le b_n)\Big]-\mathbb{E}\Big[\mathbb{E}\Big(X_i\mathbb{I}(|X_i|\le b_n)|\mathcal{F}_{i-1}\Big)\Big]^2\Big)\to 0,$

*as $n\to\infty$, then we have*

$$\frac{1}{b_n}\sum_{i=1}^{n}X_i\xrightarrow{\mathbb{P}}0 \quad as \quad n\to\infty.$$

*Hence Theorem 2.1 is a strong version which gives the sufficient conditions of the weak law of large numbers of the maximum of martingale.*

**Remark 2.3.** *Naderi et al. [16, Theorem 1] established the weak law of large numbers for weighted negatively superadditive dependent random variables. Let $g(x)$, $h(x)$ and $\phi(x)$ satisfy the conditions $(B_1)$-$(B_4)$, then*

$$\frac{1}{g(n)}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_i-\mathbb{E}\Big(X_i\mathbb{I}\left(|X_i|\le\phi(n)\right)\Big)}{h(i)}\right|\xrightarrow{\mathbb{P}}0 \quad as \quad n\to\infty.$$

*Hence Theorem 2.1 is an interesting supplement for their works which gives the sufficient conditions of the weighted weak law of large numbers of the martingale difference sequences.*

**Remark 2.4.** *Chang and Miao [5, Theorem 2.1] obtained the weak law of large numbers for the sequence of identically distributed random variables. Let $\{b_n, n\ge 1\}$ be an increasing sequence of positive real numbers. Assume that there exists a nondecreasing sequence of positive real numbers $\{a_n, n\ge 1\}$ such that*

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni}-\mathbb{E}X_{ni})\right|\right)^2\le a_n^2\sum_{i=1}^{n}\mathbb{E}\left(X_{ni}-\mathbb{E}X_{ni}\right)^2,$$

*where $X_{ni}=-\frac{b_n}{a_n}\mathbb{I}\{X_i\le-\frac{b_n}{a_n}\}+X_i\mathbb{I}\{|X_i|\le\frac{b_n}{a_n}\}+\frac{b_n}{a_n}\mathbb{I}\{X_i>\frac{b_n}{a_n}\}, 1\le i\le n$. Suppose that $\{b_n/a_n, n\ge 1\}$ be an increasing sequence, and for any $n\ge 1$, suppose that $h(n)=1$ and $g(n)=b_n/a_n$ satisfy the conditions $(B_1)$-$(B_4)$, then*

$$\frac{1}{b_n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(X_i-\mathbb{E}\left(X_i\mathbb{I}\left(|X_i|\le\frac{b_n}{a_n}\right)\right)\right)\right|\xrightarrow{\mathbb{P}}0 \quad as \quad n\to\infty.$$

*Hence Theorem 2.1 further study the weak law of large numbers which discard the above inequality of $a_n$ and introduce the martingale difference sequences.*

**Theorem 2.2.** *Under the conditions in Theorem 2.1, we have*

$$\frac{1}{g(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_i-\mathbb{E}(X_i|\mathcal{F}_{i-1})}{h(i)}\right|\right)\to 0 \quad as \quad n\to\infty, \tag{2.5}$$

*where $\mathcal{F}_0=\{\emptyset,\Omega\}$ and $\mathcal{F}_n=\sigma\{X_k, 1\le k\le n\}$ for every $n\ge 1$.*

**Remark 2.5.** *For the case $h(n)=1$ and $\phi(n)=g(n)$, the conditions $(B_1)$ and $(B_2)$ hold, then the conditions $(B_3)$ and $(B_4)$ implies*

$$\frac{1}{g(n)}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\Big(X_i-\mathbb{E}\Big(X_i\mathbb{I}(|X_i|\le g(n))|\mathcal{F}_{i-1}\Big)\Big)\right|\xrightarrow{\mathbb{P}}0 \quad as \quad n\to\infty. \tag{2.6}$$

*When $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables, Kruglov [14] (see Theorem 1.2) proved that (2.6) is equivalent to the condition $(B_4)$ under the property $(B_3)$ by using the symmetrization method. In particular, when $\phi(n) = g(n) = n^{1/p}$ for some $1 < p < 2$, then the conditions $(B_3)$ and $(B_5)$ hold. In fact, for the condition $(B_5)$, we have*

$$\sum_{k=n}^{\infty} \frac{\phi(k+1) - \phi(k)}{k} = \sum_{k=n}^{\infty} \frac{(k+1)^{1/p} - k^{1/p}}{k}$$
$$= \sum_{k=n}^{\infty} \frac{k^{1/p}}{k} \left[ \left(1 + \frac{1}{k}\right)^{1/p} - 1 \right] \leq C \sum_{k=n}^{\infty} \frac{k^{1/p}}{k^2} \leq \frac{C}{n^{1-1/p}}.$$

*Hence from Theorem 2.1 and Theorem 2.2, the condition $(B_4)$ implies that*

$$\frac{1}{n^{1/p}} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left( X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1}) \right) \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty$$

*and*

$$\frac{1}{n^{1/p}} \mathbb{E}\left( \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left( X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1}) \right) \right| \right) \to 0 \quad as \quad n \to \infty.$$

*The key technology to prove the sufficiency and necessity of Theorem 1.2 is the symmetrization method (or Levy's inequality) for the independent random variables. The Levy type inequality does not hold for general dependent random variables, so we can not give the necessary conditions for Theorem 2.1 and Theorem 2.2.*

**Corollary 2.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $\xi$. Let $\alpha > 0, \beta \geq 0, 0 \leq \tau < 2^{-1}, 0 < \alpha + \beta + \tau < 1, \alpha + \tau > 2^{-1}$, and assume that*

$$n\mathbb{P}\left(|\xi| > n^{\alpha+\tau} \log^{\beta} n\right) \to 0 \quad as \quad n \to \infty,$$

*then we have*

$$\frac{1}{n^{\alpha}} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^{\tau} \log^{\beta} i} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty \tag{2.7}$$

*and*

$$\frac{1}{n^{\alpha}} \mathbb{E}\left( \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^{\tau} \log^{\beta} i} \right| \right) \to 0 \quad as \quad n \to \infty, \tag{2.8}$$

*where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ for every $n \geq 1$.*

**Corollary 2.2.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $\xi$. Let $\alpha > 0, \beta \geq 0, 0 \leq \tau < 2^{-1}, 0 < \alpha + \beta + \tau < 1, \alpha + \tau > 2^{-1}$, and assume that*

$$n\mathbb{P}\left(|\xi| > n^{\alpha+\tau} \log^{\beta} n\right) \to 0 \quad as \quad n \to \infty,$$

*then we have*

$$\frac{1}{n^{\alpha} \log^{\beta} n} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^{\tau}} \right| \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty$$

*and*

$$\frac{1}{n^{\alpha} \log^{\beta} n} \mathbb{E}\left( \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^{\tau}} \right| \right) \to 0 \quad as \quad n \to \infty,$$

*where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ for every $n \geq 1$.*

**Remark 2.6.** *By comparing the Corollary 2.1 and the Corollary 2.2, it is necessary to consider the following cases. When $\alpha = 0$, neither of these corollaries holds true. When $\beta = 0$ or $\tau = 0$, these two corollaries are true.*

**Theorem 2.3.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $\xi$, let $g(x)$ and $h(x)$ be two positive functions defined on $[0, +\infty)$, and $\phi(x) = g(x)h(x)$. Assume that the conditions $(B_1)$ holds. In addition, suppose that for some $1 < p < 2$,*

$(B_2^{'})$ *The function h is nondecreasing and* $\sum_{k=1}^{n} \frac{1}{h^p(k)} = O\Big(\frac{n}{h^p(n)}\Big)$,

$(B_3^{'})$ $\sum_{k=1}^{n} \frac{\phi^p(k)}{k^2} = O\left(\phi^p(n)/n\right)$,

$(B_4^{'})$ $n\mathbb{P}\Big(|\xi|^p > \phi(n)\Big) \to 0$ *as* $n \to \infty$,

*and*

$(B_5^{'})$ $\sum_{k=n}^{\infty} \frac{\phi(k+1) - \phi(k)}{k} = O\left(\phi^p(n)/n\right)$,

*then we have*

$$\frac{1}{g^p(n)}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{h(i)}\right|^p\right) \to 0 \quad as \quad n \to \infty, \tag{2.9}$$

*where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ for every $n \geq 1$.*

**Remark 2.7.** *Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of martingale differences. Hall and Heyde [10, Theorem 2.22] established the convergence in $L^p$ as follows. If $1 \leq p < 2$ and $\{|X_n|^p, n \geq 1\}$ is uniformly integrable, then*

$$\frac{1}{n}\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \to 0 \quad as \quad n \to \infty.$$

*Hence Theorem 2.2 and Theorem 2.3 extend and strengthen the above results of Hall and Heyde [10].*

**Corollary 2.3.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $\xi$. Let $\alpha > 0$, $\beta \geq 0$, $\tau \geq 0$, $1 < p < 2$, $0 < \alpha + \beta + \tau < 1$, $p(\alpha + \tau) > 1$, $p\tau < 1$, and assume that*

$$n\mathbb{P}\Big(|\xi|^p > n^{\alpha+\tau}\log^\beta n\Big) \to 0 \;\; as \;\; n \to \infty,$$

*then we have*

$$\frac{1}{n^{p\alpha}}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} \frac{X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^\tau \log^\beta i}\right|^p\right) \to 0 \quad as \quad n \to \infty, \tag{2.10}$$

*where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ for every $n \geq 1$.*

**Corollary 2.4.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $\xi$. Let $\alpha > 0$, $\beta \geq 0$, $\tau \geq 0$, $1 < p < 2$, $0 < \alpha + \beta + \tau < 1$, $p(\alpha + \tau) > 1$, $p\tau < 1$, and assume that*

$$n\mathbb{P}\Big(|\xi|^p > n^{\alpha+\tau}\log^\beta n\Big) \to 0 \;\; as \;\; n \to \infty,$$

*then we have*

$$\frac{1}{n^{p\alpha}\log^{p\beta} n}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_i-\mathbb{E}(X_i|\mathcal{F}_{i-1})}{i^{\tau}}\right|^{p}\right)\to 0 \quad as \quad n\to\infty,$$

*where* $\mathcal{F}_0=\{\emptyset,\Omega\}$ *and* $\mathcal{F}_n=\sigma\{X_k, 1\le k\le n\}$ *for every* $n\ge 1$.

**Remark 2.8.** *By comparing the Corollary 2.3 and the Corollary 2.4, it is necessary to consider the following cases. When* $\alpha=0$*, neither of these corollaries holds true. When* $\beta=0$ *or* $\tau=0$*, these two corollaries are true.*

## 3. Proofs of main results

*Proof.* **[Proof of Theorem 2.1]** For $1\le k\le n$, let

$$X_{nk}=X_k\mathbb{I}(|X_k|\le\phi(n)).$$

For any $r>0$, by using Doob maximal inequality for martingale and the Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_i-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|>rg(n)\right)\\
\le&\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|>rg(n)\right)+\sum_{k=1}^{n}\mathbb{P}\Big(|X_k|>\phi(n)\Big)\\
\le&\frac{1}{r^2g^2(n)}\mathbb{E}\left(\sum_{i=1}^{n}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right)^2+Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\\
\le&\frac{1}{r^2g^2(n)}\sum_{i=1}^{n}\frac{\mathbb{E}\Big(X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})\Big)^2}{h^2(i)}+Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\\
\le&\frac{1}{r^2g^2(n)}\sum_{i=1}^{n}\frac{\mathbb{E}\Big(X_i^2\mathbb{I}(|X_i|\le\phi(n))\Big)}{h^2(i)}+Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\\
\le&\frac{1}{r^2g^2(n)}\sum_{i=1}^{n}\frac{1}{h^2(i)}\Big(\mathbb{E}\Big(\xi^2\mathbb{I}(|\xi|\le\phi(n))+\phi^2(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)\Big)\Big)+Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\\
\le&\frac{Cn}{\phi^2(n)}\mathbb{E}\left[\xi^2\mathbb{I}(|\xi|\le\phi(n))\right]+Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big).
\end{aligned}$$

In order to prove the claim (2.3), from the condition $n\mathbb{P}\Big(|\xi|>\phi(n)\Big)\to 0$, it is enough to show

$$\frac{n}{\phi^2(n)}\mathbb{E}\left[\xi^2\mathbb{I}(|\xi|\le\phi(n))\right]\to 0. \tag{3.1}$$

From the condition $(B_3)$, we know that there exists a positive constant $C$, such that for all $n\ge 1$,

$$\sum_{k=1}^{n}\frac{\phi^2(k)}{k^2}\le C\frac{\phi^2(n)}{n},$$

which implies

$$\phi^2(1)\le C\frac{\phi^2(k)}{k} \quad \text{for any} \quad k\ge 1$$

and

$$C^2\frac{\phi^2(n)}{n}\geq C\sum_{k=1}^{n}\frac{\phi^2(k)}{k^2}\geq\phi^2(1)\sum_{k=1}^{n}\frac{1}{k}\to\infty.$$

Hence from the above fact $n/\phi^2(n)\to 0$ as $n\to\infty$, for every fixed $1<R<n$, we have

$$\frac{n}{\phi^2(n)}\mathbb{E}\left[\xi^2\mathbb{I}(|\xi|\leq\phi(R))\right]\to 0. \tag{3.2}$$

Furthermore, we have

$$\begin{aligned}
&\frac{n}{\phi^2(n)}\mathbb{E}\left[\xi^2\mathbb{I}(\phi(R)<|\xi|\leq\phi(n))\right]\\
=&\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n}\mathbb{E}\left[\xi^2\mathbb{I}(\phi(k-1)<|\xi|\leq\phi(k))\right]\\
\leq&\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n}\phi^2(k)\mathbb{P}\Big(\phi(k-1)<|\xi|\leq\phi(k)\Big)\\
=&\frac{n}{\phi^2(n)}\left(\sum_{k=R}^{n-1}\phi^2(k+1)\mathbb{P}\Big(|\xi|>\phi(k)\Big)-\sum_{k=R+1}^{n}\phi^2(k)\mathbb{P}\Big(|\xi|>\phi(k)\Big)\right)\\
\leq&\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n-1}\Big(\phi^2(k+1)-\phi^2(k)\Big)\mathbb{P}\Big(|\xi|>\phi(k)\Big)\\
&\quad+\frac{n\phi^2(R+1)}{\phi^2(n)}\mathbb{P}\Big(|\xi|>\phi(R)\Big)-n\mathbb{P}\Big(|\xi|>\phi(n)\Big).
\end{aligned}$$

For any $\varepsilon>0$ and all $R$ large enough, we have

$$\begin{aligned}
&\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n-1}\Big(\phi^2(k+1)-\phi^2(k)\Big)\mathbb{P}\Big(|\xi|>\phi(k)\Big)\\
=&\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n-1}\frac{1}{k}\Big(\phi^2(k+1)-\phi^2(k)\Big)k\mathbb{P}\Big(|\xi|>\phi(k)\Big)\\
\leq&\varepsilon\frac{n}{\phi^2(n)}\sum_{k=R+1}^{n-1}\frac{1}{k}\Big(\phi^2(k+1)-\phi^2(k)\Big)\\
=&\varepsilon\frac{n}{\phi^2(n)}\left(\sum_{k=R+2}^{n-1}\left(\frac{1}{k-1}-\frac{1}{k}\right)\phi^2(k)+\frac{\phi^2(n)}{n-1}-\frac{\phi^2(R+1)}{R+1}\right)\\
\leq&C\varepsilon\frac{n}{\phi^2(n)}\left(\sum_{k=R+2}^{n-1}\frac{\phi^2(k)}{k^2}+\frac{\phi^2(n)}{n-1}-\frac{\phi^2(R+1)}{R+1}\right)\leq C\varepsilon.
\end{aligned}$$

From the conditions $n\mathbb{P}\Big(|\xi|>\phi(n)\Big)\to 0$ and $n/\phi^2(n)\to 0$ as $n\to\infty$, and together with (3.2), the claim (3.1) holds.

Next we shall prove the claim (2.4). In order to prove the claim (2.4), from the claim (2.3), it is enough to show that for any $r>0$,

$$\mathbb{P}\left(\frac{1}{g(n)}\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\frac{\mathbb{E}\Big(X_i\mathbb{I}(|X_i|>\phi(n))|\mathcal{F}_{i-1}\Big)}{h(i)}\right|>r\right)\to 0.$$

Here we first give the following fact that

$$
\begin{aligned}
&\mathbb{E}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))\Big)=\int_0^\infty \mathbb{P}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))>t\Big)dt\\
=&\int_0^{\phi(n)} \mathbb{P}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))>t\Big)dt+\int_{\phi(n)}^\infty \mathbb{P}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))>t\Big)dt\\
=&\phi(n)\mathbb{P}\Big(|X_i|>\phi(n)\Big)+\int_{\phi(n)}^\infty \mathbb{P}\Big(|X_i|>t\Big)dt\\
\le& C\phi(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)+\int_{\phi(n)}^\infty C\mathbb{P}\Big(|\xi|>t\Big)dt\\
=&C\phi(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)+C\sum_{k=n}^\infty\int_{\phi(k)}^{\phi(k+1)} \mathbb{P}\Big(|\xi|>t\Big)dt\\
\le& C\phi(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)+C\sum_{k=n}^\infty\frac{\phi(k+1)-\phi(k)}{k}k\mathbb{P}\Big(|\xi|>\phi(k)\Big).
\end{aligned}
$$

Hence for any $r>0$, from the conditions $(B_2)$, $(B_4)$ and $(B_5)$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\frac{1}{g(n)}\max_{1\le k\le n}\left|\sum_{i=1}^k\frac{\mathbb{E}\Big(X_i\mathbb{I}(|X_i|>\phi(n))|\mathcal{F}_{i-1}\Big)}{h(i)}\right|>r\right)\\
\le&\mathbb{P}\left(\frac{1}{g(n)}\sum_{i=1}^n\frac{\mathbb{E}\Big(|X_i|\,\mathbb{I}(|X_i|>\phi(n))|\mathcal{F}_{i-1}\Big)}{h(i)}>r\right)\\
\le&\frac{C}{g(n)}\sum_{i=1}^n\frac{\mathbb{E}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))\Big)}{h(i)}\\
\le&\frac{Ch(n)}{g(n)}\sum_{i=1}^n\frac{\mathbb{E}\Big(|X_i|\mathbb{I}(|X_i|>\phi(n))\Big)}{h^2(i)}\\
\le&\frac{Cn}{\phi(n)}\left(C\phi(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)+C\sum_{k=n}^\infty\frac{\phi(k+1)-\phi(k)}{k}k\mathbb{P}\Big(|\xi|>\phi(k)\Big)\right)\\
=&Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)+C\frac{n}{\phi(n)}\sum_{k=n}^\infty\frac{\phi(k+1)-\phi(k)}{k}k\mathbb{P}\Big(|\xi|>\phi(k)\Big)\\
\le&Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\to 0,
\end{aligned}
$$

which yields the desired results. □

*Proof.* **[Proof of Theorem 2.2]** For $1\le k\le n$, let

$$X_{nk}=X_k\mathbb{I}(|X_k|\le\phi(n)) \ \text{ and } \ Y_{nk}=X_k\mathbb{I}(|X_k|>\phi(n)).$$

In order to prove the claim (2.5), it is enough to check

$$\frac{1}{g(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^k\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|\right)\to 0$$

and

$$\frac{1}{g(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|\right)\to 0.$$

By using Jensen's inequality and Burkholder's inequality (see Gut [9, Theorem 9.5]), we have

$$\begin{aligned}
&\frac{1}{g(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|\right)\\
\le&\frac{1}{g(n)}\left(\mathbb{E}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|^2\right)^{1/2}\\
\le&\frac{C}{g(n)}\left(\mathbb{E}\left(\sum_{i=1}^{n}\left(\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right)^2\right)\right)^{1/2}\\
\le&\frac{C}{g(n)}\left(\sum_{i=1}^{n}\frac{\mathbb{E}X_{ni}^2}{h^2(i)}\right)^{1/2}\\
=&\frac{C}{g(n)}\left(\sum_{i=1}^{n}\frac{\mathbb{E}X_i^2\mathbb{I}(|X_i|\le\phi(n))}{h^2(i)}\right)^{1/2}\\
\le&\frac{C}{g(n)}\left(\sum_{i=1}^{n}\frac{1}{h^2(i)}\Big(\mathbb{E}\big(\xi^2\mathbb{I}(|\xi|\le\phi(n))\big)+\phi^2(n)\mathbb{P}\big(|\xi|>\phi(n)\big)\Big)\right)^{1/2}\\
\le&C\left(\frac{n}{\phi^2(n)}\mathbb{E}\big(\xi^2\mathbb{I}(|\xi|\le\phi(n))\big)+n\mathbb{P}\big(|\xi|>\phi(n)\big)\right)^{1/2}\\
\to&0,
\end{aligned}$$

where, in the above last step, we use the similar method as Theorem 2.1.

Furthermore, as the similar proof of Theorem 2.1, we can get

$$\begin{aligned}
&\frac{1}{g(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|\right)\\
\le&\frac{1}{g(n)}\mathbb{E}\left(\sum_{i=1}^{n}\frac{\big|Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})\big|}{h(i)}\right)\\
\le&\frac{1}{g(n)}\sum_{i=1}^{n}\frac{1}{h(i)}\mathbb{E}\big|Y_{ni}\big|\\
\le&\frac{Ch(n)}{g(n)}\sum_{i=1}^{n}\frac{1}{h^2(i)}\mathbb{E}\big(|X_i|\mathbb{I}(|X_i|>\phi(n))\big)\to 0.
\end{aligned}$$

□

*Proof.* **[Proof of Corollary 2.1]** Let $g(n)=n^{\alpha}$ and $h(n)=n^{\tau}\log^{\beta}n$, then from Theorem 2.1, it is enough to check the conditions $(B_2)$, $(B_3)$ and $(B_5)$. The condition $(B_2)$ holds by showing

$$\sum_{k=1}^{n}\frac{1}{h^2(k)}=\sum_{k=1}^{n}\frac{1}{k^{2\tau}\log^{2\beta}k}\le C\frac{1}{n^{2\tau-1}\log^{2\beta}n}=O\left(\frac{n}{h^2(n)}\right).$$

The condition $(B_3)$ holds by showing

$$\sum_{k=1}^{n}\frac{\phi^2(k)}{k^2}=\sum_{k=1}^{n}\frac{k^{2(\alpha+\tau)}\log^{2\beta}k}{k^2}=\sum_{k=1}^{n}\frac{\log^{2\beta}k}{k^{2-2(\alpha+\tau)}}\leq C\frac{\log^{2\beta}n}{n^{1-2(\alpha+\tau)}}=O\left(\frac{\phi^2(n)}{n}\right).$$

Since $x^{-1}\log x$ is a monotonically decreasing function for $x>e$, we have

$$\frac{\log(x+1)}{\log x}\leq\frac{x+1}{x}.$$

Furthermore, it is easy to see that for any $0<t<1$ and $x\geq -1$,

$$(1+x)^t<1+tx.$$

Hence we have

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{\phi(k+1)-\phi(k)}{k}&=\sum_{k=n}^{\infty}\frac{(k+1)^{\alpha+\tau}\log^{\beta}(k+1)-k^{\alpha+\tau}\log^{\beta}k}{k}\\&=\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k}\left(\left(1+\frac{1}{k}\right)^{\alpha+\tau}\left(\frac{\log(k+1)}{\log k}\right)^{\beta}-1\right)\\&\leq\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k}\left(\left(1+\frac{1}{k}\right)^{\alpha+\beta+\tau}-1\right)\\&\leq(\alpha+\beta+\tau)\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k^2}\\&\leq C\frac{\log^{\beta}n}{n^{1-(\alpha+\tau)}}=O\left(\frac{\phi(n)}{n}\right),\end{aligned}$$

which yields the condition $(B_5)$. □

*Proof.* **[Proof of Corollary 2.2]** Using the similar proof of the claims (2.7) and (2.8), we can get the desire results. □

*Proof.* **[Proof of Theorem 2.3]** For $1\leq k\leq n$, let

$$X_{nk}=X_k\mathbb{I}(|X_k|\leq\phi(n))\ \text{ and }\ Y_{nk}=X_k\mathbb{I}(|X_k|>\phi(n)).$$

In order to prove the claim (2.9), it is enough to check

$$\frac{1}{g^p(n)}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|^p\right)\to 0$$

and

$$\frac{1}{g^p(n)}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|^p\right)\to 0.$$

By using Burkholder's inequality (see Gut [9, Theorem 9.5]), Cr inequality and noting $0 < p/2 < 1$, we have

$$\begin{aligned}
&\frac{1}{g^p(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|^p\right)\\
\le &C\mathbb{E}\left(\left|\sum_{i=1}^{n}\left(\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{g(n)h(i)}\right)^2\right|^{p/2}\right)\\
\le &C\sum_{i=1}^{n}\mathbb{E}\left(\left|\frac{X_{ni}-\mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{g(n)h(i)}\right|^p\right)\\
\le &C\sum_{i=1}^{n}\frac{\mathbb{E}\big(|X_i|^p\mathbb{I}(|X_i|\le\phi(n))\big)}{g^p(n)h^p(i)}\\
\le &C\frac{n}{\phi^p(n)}\mathbb{E}\big(|\xi|^p\mathbb{I}(|\xi|\le\phi(n))\big)+Cn\mathbb{P}\big(|\xi|>\phi(n)\big).
\end{aligned}$$

Since the condition $(B_4^{'})$ holds, we get

$$0 \le n\mathbb{P}\big(|\xi|>\phi(n)\big) \le n\mathbb{P}\big(|\xi|^p>\phi(n)\big) \to 0 \ \text{ as } \ n\to\infty,$$

that is to say the condition $(B_4)$ holds.

As similar as the proof of Theorem 2.1, we have the fact $n/\phi^p(n) \to 0$ as $n \to \infty$. Hence for every fixed $1 < R < n$, we have

$$\frac{n}{\phi^p(n)}\mathbb{E}\left[|\xi|^p\mathbb{I}(|\xi|\le\phi(R))\right]\to 0. \tag{3.3}$$

Furthermore, we have

$$\begin{aligned}
&\frac{n}{\phi^p(n)}\mathbb{E}\left[|\xi|^p\mathbb{I}(\phi(R)<|\xi|\le\phi(n))\right]\\
=&\frac{n}{\phi^p(n)}\sum_{k=R+1}^{n}\mathbb{E}\left[|\xi|^p\mathbb{I}(\phi(k-1)<|\xi|\le\phi(k))\right]\\
\le&\frac{n}{\phi^p(n)}\sum_{k=R+1}^{n}\phi^p(k)\mathbb{P}\big(\phi(k-1)<|\xi|\le\phi(k)\big)\\
=&\frac{n}{\phi^p(n)}\sum_{k=R+1}^{n}\phi^p(k)\Big(\mathbb{P}\big(|\xi|>\phi(k-1)\big)-\mathbb{P}\big(|\xi|>\phi(k)\big)\Big)\\
=&\frac{n}{\phi^p(n)}\left(\sum_{k=R}^{n-1}\phi^p(k+1)\mathbb{P}\big(|\xi|>\phi(k)\big)-\sum_{k=R+1}^{n}\phi^p(k)\mathbb{P}\big(|\xi|>\phi(k)\big)\right)\\
\le&\frac{n}{\phi^p(n)}\sum_{k=R+1}^{n-1}\big(\phi^p(k+1)-\phi^p(k)\big)\mathbb{P}\big(|\xi|>\phi(k)\big)\\
&\quad+\frac{n\phi^p(R+1)}{\phi^p(n)}\mathbb{P}\big(|\xi|>\phi(R)\big)-n\mathbb{P}\big(|\xi|>\phi(n)\big).
\end{aligned}$$

From the conditions $n\mathbb{P}\Big(|\xi| > \phi(n)\Big) \to 0$ and $n/\phi^p(n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{n}{\phi^p(n)} \mathbb{E}\left[|\xi|^p \mathbb{I}(\phi(R) < |\xi| \le \phi(n))\right] \\ &\quad \le \lim_{n\to\infty} \frac{n}{\phi^p(n)} \sum_{k=R+1}^{n-1} \Big(\phi^p(k+1) - \phi^p(k)\Big)\mathbb{P}\Big(|\xi| > \phi(k)\Big). \end{aligned} \tag{3.4}$$

For any $\varepsilon > 0$ and all $R$ large enough, from $(B_3^{'})$ and $(B_4)$, we have

$$\begin{aligned} &\frac{n}{\phi^p(n)} \sum_{k=R+1}^{n-1} \Big(\phi^p(k+1) - \phi^p(k)\Big)\mathbb{P}\Big(|\xi| > \phi(k)\Big) \\ &= \frac{n}{\phi^p(n)} \sum_{k=R+1}^{n-1} \frac{1}{k}\Big(\phi^p(k+1) - \phi^p(k)\Big)k\mathbb{P}\Big(|\xi| > \phi(k)\Big) \\ &\le \varepsilon \frac{n}{\phi^p(n)} \sum_{k=R+1}^{n-1} \frac{1}{k}\Big(\phi^p(k+1) - \phi^p(k)\Big) \\ &= \varepsilon \frac{n}{\phi^p(n)} \left(\sum_{k=R+2}^{n-1} \Big(\frac{1}{k-1} - \frac{1}{k}\Big)\phi^p(k) + \frac{\phi^p(n)}{n-1} - \frac{\phi^p(R+1)}{R+1}\right) \\ &\le C\varepsilon \frac{n}{\phi^p(n)} \left(\sum_{k=R+2}^{n-1} \frac{\phi^p(k)}{k^2} + \frac{\phi^p(n)}{n-1} - \frac{\phi^p(R+1)}{R+1}\right) \le C\varepsilon, \end{aligned}$$

which, together with (3.4), implies

$$\lim_{n\to\infty} \frac{n}{\phi^p(n)} \mathbb{E}\left[|\xi|^p \mathbb{I}(\phi(R) < |\xi| \le \phi(n))\right] = 0. \tag{3.5}$$

From above discussion, we have

$$\frac{1}{g^p(n)} \mathbb{E}\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} \frac{X_{ni} - \mathbb{E}(X_{ni}|\mathcal{F}_{i-1})}{h(i)}\right|^p\right) \to 0.$$

Furthermore, using the similar method as Theorem 2.1, we can get the fact that

$$\begin{aligned} &\mathbb{E}\Big(|X_i|^p\, \mathbb{I}(|X_i| > \phi(n))\Big) = \int_0^\infty \mathbb{P}\Big(|X_i|^p\, \mathbb{I}(|X_i| > \phi(n)) > t\Big)dt \\ &= \int_0^{\phi^p(n)} \mathbb{P}\Big(|X_i|^p\, \mathbb{I}(|X_i| > \phi(n)) > t\Big)dt + \int_{\phi^p(n)}^\infty \mathbb{P}\Big(|X_i|^p\, \mathbb{I}(|X_i| > \phi(n)) > t\Big)dt \\ &= \phi^p(n)\mathbb{P}\Big(|X_i| > \phi(n)\Big) + \int_{\phi^p(n)}^\infty \mathbb{P}\Big(|X_i|^p > t\Big)dt \\ &\le C\phi^p(n)\mathbb{P}\Big(|\xi| > \phi(n)\Big) + C\int_{\phi^p(n)}^\infty \mathbb{P}\Big(|\xi|^p > t\Big)dt \\ &\le C\phi^p(n)\mathbb{P}\Big(|\xi| > \phi(n)\Big) + C\int_{\phi(n)}^\infty \mathbb{P}\Big(|\xi|^p > t\Big)dt \\ &= C\phi^p(n)\mathbb{P}\Big(|\xi| > \phi(n)\Big) + C\sum_{k=n}^\infty \int_{\phi(k)}^{\phi(k+1)} \mathbb{P}\Big(|\xi|^p > t\Big)dt \\ &\le C\phi^p(n)\mathbb{P}\Big(|\xi| > \phi(n)\Big) + C\sum_{k=n}^\infty \frac{\phi(k+1) - \phi(k)}{k} k\mathbb{P}\Big(|\xi|^p > \phi(k)\Big). \end{aligned}$$

So we have

$$
\begin{aligned}
&\frac{1}{g^p(n)}\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{h(i)})\right|^p\right)\\
\le& C\mathbb{E}\left(\left|\sum_{i=1}^{n}\left(\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{g(n)h(i)}\right)^2\right|^{p/2}\right)\\
\le& C\sum_{i=1}^{n}\mathbb{E}\left(\left|\frac{Y_{ni}-\mathbb{E}(Y_{ni}|\mathcal{F}_{i-1})}{g(n)h(i)}\right|^p\right)\\
\le& C\sum_{i=1}^{n}\frac{\mathbb{E}\Big(|X_i|^p\,\mathbb{I}(|X_i|>\phi(n))\Big)}{g^p(n)h^p(i)}\\
\le& \frac{Cn}{g^p(n)h^p(n)}\left(C\phi^p(n)\mathbb{P}\Big(|\xi|>\phi(n)\Big)+C\sum_{k=n}^{\infty}\frac{\phi(k+1)-\phi(k)}{k}k\mathbb{P}\Big(|\xi|^p>\phi(k)\Big)\right)\\
\le& Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)+\frac{Cn}{\phi^p(n)}\sum_{k=n}^{\infty}\frac{\phi(k+1)-\phi(k)}{k}k\mathbb{P}\Big(|\xi|^p>\phi(k)\Big)\\
\le& Cn\mathbb{P}\Big(|\xi|>\phi(n)\Big)\to 0.
\end{aligned}
$$

Hence the desired results can be obtained. □

*Proof.* **[Proof of Corollary 2.3]** Let $g(n) = n^{\alpha}$ and $h(n) = n^{\tau}\log^{\beta} n$, then from Theorem 2.3, it is enough to check the conditions $(B_2^{'})$, $(B_3^{'})$ and $(B_5^{'})$. The condition $(B_2^{'})$ holds by showing

$$\sum_{k=1}^{n}\frac{1}{h^p(k)}=\sum_{k=1}^{n}\frac{1}{k^{p\tau}\log^{p\beta}k}\le C\frac{1}{n^{p\tau-1}\log^{p\beta}n}=O\left(\frac{n}{h^p(n)}\right).$$

The condition $(B_3^{'})$ holds by showing

$$\sum_{k=1}^{n}\frac{\phi^p(k)}{k^2}=\sum_{k=1}^{n}\frac{k^{p(\alpha+\tau)}\log^{p\beta}k}{k^2}=\sum_{k=1}^{n}\frac{\log^{p\beta}k}{k^{2-p(\alpha+\tau)}}\le C\frac{\log^{p\beta}n}{n^{1-p(\alpha+\tau)}}=O\left(\frac{\phi^p(n)}{n}\right).$$

Furthermore, we have

$$
\begin{aligned}
\sum_{k=n}^{\infty}\frac{\phi(k+1)-\phi(k)}{k}=&\sum_{k=n}^{\infty}\frac{(k+1)^{\alpha+\tau}\log^{\beta}(k+1)-k^{\alpha+\tau}\log^{\beta}k}{k}\\
=&\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k}\left(\left(1+\frac{1}{k}\right)^{\alpha+\tau}\left(\frac{\log(k+1)}{\log k}\right)^{\beta}-1\right)\\
\le&\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k}\left(\left(1+\frac{1}{k}\right)^{\alpha+\beta+\tau}-1\right)\\
\le&(\alpha+\beta+\tau)\sum_{k=n}^{\infty}\frac{k^{\alpha+\tau}\log^{\beta}k}{k^2}\\
\le& C\frac{\log^{\beta}n}{n^{1-(\alpha+\tau)}}=O\left(\frac{\phi(n)}{n}\right)=O\left(\frac{\phi^p(n)}{n}\right),
\end{aligned}
$$

which yields the condition $(B_5^{'})$. □

*Proof.* **[Proof of Corollary 2.4]** Using the similar proof of the claim (2.10), we can get the desire results. □

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## References

[1] A. Adler and A. Rosalsky, *On the weak law of large numbers for normed weighted sums of i.i.d. random variables*, Internat. J. Math. Math. Sci. **14** (1991), no. 1, 191–202.
[2] R. Balan and G. Stoica, *A note on the weak law of large numbers for free random variables*, Ann. Sci. Math. Quebec **31** (2007), no. 1, 23–30.
[3] F. Boukhari, *Weak laws of large numbers for maximal weighted sums of random variables*, Comm. Statist. Theory Methods **50** (2021), no. 1, 105–115.
[4] F. Boukhari and M. R. M. Boudjemaa, *A weighted weak law of large numbers for general random variables*, Preprint, 2021.
[5] M. M. Chang and Y. Miao, *Weak law of large numbers and complete convergence for general dependent sequences*, Acta Math. Hungar. **169** (2023), no. 2, 469–488.
[6] W. Feller, *An introduction to probability theory and its applications. Vol. II*, (Second edition), John Wiley & Sons, Inc., New York-London-Sydney, 1971.
[7] Q. Gao and Y. Miao, *On a weak law of large numbers and $L_p$-convergence for general random variables.* Quaest. Math. Doi: 10.2989/16073606.2024.2403745.
[8] A. Gut, *An extension of the Kolmogorov-Feller weak law of large numbers with an application to the St. Petersburg game*, J. Theoret. Probab. **17** (2004), no. 3, 769–779.
[9] A. Gut, *Probability: a graduate course*, (Second edition). Springer, New York, 2013.
[10] P. Hall and C. C. Heyde, *Martingale limit theory and its application*, Academic Press, Inc., New York-London, 1980.
[11] R. Jajte, *On the strong law of large numbers*, Ann. Probab. **31** (2003), no. 1, 409–412.
[12] B. Y. Jing and H. Y. Liang, *Strong limit theorems for weighted sums of negatively associated random variables*, J. Theoret. Probab. **21** (2008), no. 4, 890–909.
[13] M. Klass and H. Teicher, *Iterated logarithm laws for asymmetric random variables barely with or without finite mean*, Ann. Probab. **5** (1977), no. 6, 861–874.
[14] V. M. Kruglov, *A generalization of weak law of large numbers*, Stoch. Anal. Appl. **29** (2011), no. 4, 674–683.
[15] Y. Miao, J. Y. Mu and S. L. Zhang, *Limit theorems for identically distributed martingale difference*, Comm. Statist. Theory Methods **49** (2020), no. 6, 1435–1445.
[16] H. Naderi, F. Boukhari and P. Matula, *A note on the weak law of large numbers for weighted negatively superadditive dependent random variables*, Comm. Statist. Theory Methods **51** (2022), no. 21, 7465–7475.
[17] H. Naderi, P. Matula, M. Amini and H. Ahmadzade, *A version of the Kolmogrov-Feller weak law of large numbers for maximal weighted sums of random variables*, Comm. Statist. Theory Methods **48** (2019), no. 21, 5414–5418.
[18] H. Naderi, P. Matula, M. Salehi and M. Amini, *On weak law of large numbers for sums of negatively superadditive dependent random variables*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 1, 13–21.
[19] A. Rosalsky and L. V. Thanh, *A note on the stochastic domination condition and uniform integrability with applications to the strong law of large numbers*, Statist. Probab. Lett. **178** (2021), Paper No. 109181, 10 pp.
[20] S. H. Sung, *Weak laws for weighted sums of random variables*, Bull. Korean Math. Soc. **41** (2004), no. 2, 275–282.
[21] S. H. Sung, *On the strong law of large numbers for weighted sums of random variables*, Comput. Math. Appl. **62** (2011), no. 11, 4277–4287.