Filomat 39:5 (2025), 1457–1464
https://doi.org/10.2298/FIL2505457O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

General Tauberian theorems for the statistically $(C,1)$ summable integrals

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Abstract. Let f be a real- or complex-valued function that is measurable in Lebesgue's sense on some interval (x_0,∞), where $x_0 \geq 0$. It is known that the existence of ordinary limit of function f implies the statistical limit of f. However, the converse implication is not always true. In this study we introduce some Tauberian conditions in terms of the general control modulo of integer order $r \geq 1$. Also we consider the Tauberian conditions of slow decrease and slow oscillation. Under these Tauberian conditions, we obtain the ordinary limit of a function from its statistical limit. The main results generalize some classical type Tauberian theorems given for statistical convergence.

1. Summability $(C,1)$ of integrals over $\mathbb{R}_+$ and Tauberian theorems

Let f be a real- or complex-valued function on $(0,\infty)$ which is locally integrable in Lebesgue's sense on $(0,\infty)$, in symbols: $f \in L^1_{loc}(\mathbb{R}_+)$ and

$$s(x) = \int_0^x f(t)dt \tag{1}$$

for $x > 0$. The $(C,1)$ mean of (1) is defined by

$$\sigma(x) = \frac{1}{x}\int_0^x s(u)du \tag{2}$$

for $x > 0$. The k-fold application of $(C,1)$ summability method gives $\sigma_{(k)}(x)$ as follows

$$\sigma_{(k)}(x) = \frac{1}{x}\int_0^x \sigma_{(k-1)}(u)du \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 40E05; Secondary 40G05, 40C10, 40G15.
Keywords. Tauberian theorem, statistically convergence, general control modulo, slow decrease.
Received: 12 August 2024; Revised: 07 October 2024; Accepted: 12 October 2024
Communicated by Eberhard Malkowsky
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for $x > 0$ and $k \geq 1$, where $\sigma_{(0)}(x) = s(x)$ and $\sigma_{(1)}(x) = \sigma(x)$. The integral $\int_0^\infty f(t)dt$ is said to be summable $(C, 1)$ to a finite number α, symbolically $\lim_{x\to\infty} s(x) = \alpha(C, 1)$ if the limit of (2)

$$\lim_{x\to\infty} \sigma(x) = \alpha \tag{4}$$

exists. Naturally the integral $\frac{1}{x}\int_0^x \sigma_{(k-1)}(u)du$ is said to be summable $(C, 1)$ to a finite number α, symbolically $\lim_{x\to\infty} \sigma_{(k-1)}(x) = \alpha(C, 1)$ if the limit of (3)

$$\lim_{x\to\infty} \sigma_{(k)}(x) = \alpha \tag{5}$$

exists. It is clear that if the limit of (1)

$$\lim_{x\to\infty} s(x) = \alpha \tag{6}$$

exists, then limit (4) and (5) also exists. However the existence of the limit (4) or (5) does not necessitate the existence of the limit (6) in general. We need some additional conditions to make this statement true. Such conditions are called 'Tauberian conditions' and the resulting theorems are called 'Tauberian theorems' after A. Tauber [15], who was the first to prove such theorems.

For a function, we have the following identiy:

$$s(x) - \sigma(x) = \tau(x), \tag{7}$$

where $\tau(x) = \dfrac{1}{x}\displaystyle\int_0^x tf(t)dt$. The k-fold application of $(C, 1)$ summability method to $\tau(x)$ gives $\tau_{(k)}(x)$ as follows

$$\tau_{(k)}(x) = \frac{1}{x}\int_0^x \tau_{(k-1)}(u)du \tag{8}$$

for $x > 0$ and $k \geq 1$, where $\tau_{(0)}(x) = \tau(x)$. The identity (7) is known as the Kronecker identity for the functions in literature. The classical control modulo of $s(x)$ is denoted by

$$\omega_{(0)}(x) = xf(x) \tag{9}$$

and general control modulo of integer order $r \geq 1$ of $s(x)$ is defined by

$$\omega_{(r)}(x) = \omega_{(r-1)}(x) - \sigma(\omega_{(r-1)}(x)). \tag{10}$$

The concepts of the classical and general control modulo were introduced by Dik [2] for sequences.

We recall that a real-valued function $s(x)$ is said to be slowly decreasing if

$$\lim_{\lambda\to 1^+} \liminf_{x\to\infty} \min_{x\leq t\leq \lambda x} (s(t) - s(x)) \geq 0 \tag{11}$$

Furthermore, if a function $s(x)$ is slowly decreasing, then its $(C, 1)$ mean is also slowly decreasing [8] and we denote the set of all slowly decreasing functions by $\mathcal{SD}$.

It is easy to check that (11) is satisfied if and only if for every $\epsilon > 0$ there exist $x_0 = x_0(\epsilon) > 0$ and $\lambda_0 = \lambda_0(\epsilon) > 1$, such that

$$s(t) - s(x) \geq -\epsilon$$

whenever $x_0 \leq x \leq t \leq \lambda_0 x_0$. Historically, the term of 'slowly decreasing' for a sequence of real numbers was introduced by Schmidt [12].

We recall that a complex-valued function $s(x)$ is said to be slowly oscillating if

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \max_{x\leq t\leq \lambda x} |s(t) - s(x)| = 0. \tag{12}$$

Furthermore, if a function $s(x)$ is slowly oscillating, then its $(C,1)$ mean is also slowly oscillating [8] and we denote the set of all slowly oscillating functions by $\mathcal{SO}$.

It is easy to check that (12) is satisfied if and only if for every $\epsilon > 0$ there exist $x_0 = x_0(\epsilon) > 0$ and $\lambda_0 = \lambda_0(\epsilon) > 1$, such that

$$|s(t) - s(x)| \leq \epsilon$$

whenever $x_0 \leq x \leq t \leq \lambda_0 x_0$. Historically, the term of 'slowly oscillating' for a sequence of complex numbers was introduced by Hardy [6].

Throughout this paper, we accept the following notational conventions:

(i) $f(x) = O(1)$ denotes that $|f(x)| \leq H$ for $H > 0$ and sufficiently large x.

(ii) $f(x) = O_L(1)$ means that $f(x) \geq -K$ for $K > 0$ and sufficiently large x.

(iii) $f(x) \in \mathcal{SO}$ means that $f(x)$ is slowly oscillating function.

(iv) $f(x) \in \mathcal{SD}$ means that $f(x)$ is slowly decreasing function.

In [10], it was investigated that slow decrease of $s(x)$ and $x\frac{d}{dx}s(x) = O_L(1)$ are Tauberian conditions for $(C,1)$ summable integrals of real-valued functions.

Theorem 1.1. *If a real-valued function $f \in L^1_{loc}(\mathbb{R}_+)$ is such that integral (1) is summable $(C,1)$ to a finite number α and slowly decreasing, then $s(x)$ is convergent to α.*

Theorem 1.2. *If a real-valued function $f \in L^1_{loc}(\mathbb{R}_+)$ is such that integral (1) is summable $(C,1)$ to α and $x\frac{d}{dx}s(x) = O_L(1)$, then $s(x)$ is convergent to α.*

It is easy to check that the condition of slow decrease is satisfied if the condition $x\frac{d}{dx}s(x) = O_L(1)$ holds. The following theorems are used in important steps of proofs of the our main theorems.

Analogously, it was investigated that slow oscillation of $s(x)$ and $x\frac{d}{dx}s(x) = O(1)$ are Tauberian conditions for $(C,1)$ summable integrals of complex-valued functions in [10].

Theorem 1.3. *If a complex-valued function $f \in L^1_{loc}(\mathbb{R}_+)$ is such that integral (1) is summable $(C,1)$ to α and slowly oscillating, then $s(x)$ is convergent to α.*

Theorem 1.4. *If a complex-valued function $f \in L^1_{loc}(\mathbb{R}_+)$ is such that integral (1) is summable $(C,1)$ to α and $x\frac{d}{dx}s(x) = O(1)$, then $s(x)$ is convergent to α.*

It is clear that the condition of slow oscillation is satisfied if the condition $x\frac{d}{dx}s(x) = O(1)$ holds. Indeed,

$$\begin{aligned} |s(t) - s(x)| &= \left| \int_x^t \frac{d}{du} s(u) du \right| \\ &\leq \int_x^t \frac{1}{u} du \\ &\leq \log t - \log x. \end{aligned}$$

Now taking the maximum as $x \leq t \leq \lambda x$ and lim sup as $x \to \infty$ of both sides we conclude that

$$\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |s(t) - s(x)| = 0$$

after taking limit as $\lambda \to 1^+$. So condition (12) is satisfied. In [1], Totur and Çanak benefit from general control modulo and presented the following lemmas.

Lemma 1.5. *For each integer $m \geq 0$, we have the following identity:*

$$x\frac{d}{dx}(\sigma_{(m+1)}(x)) = \tau_{(m)}(x).$$

Lemma 1.6. *For each integer $m \geq 1$, we have the following identity:*

$$\omega_{(m)}(x) = \left(x\frac{d}{dx}\right)_m \tau_{(m-1)}(x)$$

where

$$\begin{aligned}\left(x\frac{d}{dx}\right)_m f(x) &= \left(x\frac{d}{dx}\right)_{m-1}\left(x\frac{d}{dx}f(x)\right)\\ &= \left(x\frac{d}{dx}\right)\left(\left(x\frac{d}{dx}\right)_{m-1} f(x)\right).\end{aligned}$$

Here, for $k = 0$ we have,

$$\left(x\frac{d}{dx}\right)_0 f(x) = f(x)$$

and for $k = 1$ we have,

$$\left(x\frac{d}{dx}\right)_1 f(x) = x\frac{d}{dx}f(x).$$

2. Statistical summability $(C, 1)$ of integrals over $\mathbb{R}_+$ and Tauberian theorems

The notion of statistical convergence was introduced by Fast [3]. Some basic properties of statistical convergence were proved by Schoenberg [13] in 1959. Fridy [5] showed that Hardy's boundedness condition is a Tauberian condition for statistical convergence. In [4], Fridy and Khan extended Hardy's well-known Tauberian theorem and Landau's 'one-sided' Tauberian theorem for the $(C, 1)$ summability method to the case of statistical convergence. Furthermore, they extended classical Tauberian theorem of Hardy and Littlewood to the case of statistical convergence. Móricz [9] studied statistical convergence for double sequences and extended Fridy and Khan's Tauberian condition to slow decrease and slow oscillation in [7]. After that Móricz [8] studied these theorems for the complex-valued functions in nondiscrete setting. In [17], Totur and Çanak generalised the results of Móricz's theorems in [8] by adding some conditions on general control modulo of the oscillatory behaviour of nonnegative integer order $m \geq 0$ of the sequence. Later, the subject of statistical convergence continued to attract the attention of mathematicians. Many studies have been presented by synthesizing statistical convergence with methods such as weighted summability method and logarithmic summability method for the sequences [14, 16]. In addition, some studies of statistical convergence have been done for sequences of fuzzy numbers [11, 18].

Let f be a real- or complex-valued function that is measurable in Lebesgue's sense on some interval (x_0, ∞), where $x_0 \geq 0$. A function f has statistical limit at ∞, if there exists a number α such that for every $\epsilon > 0$,

$$\lim_{v\to\infty} \frac{1}{v-u}|\{x \in (u, v) : |f(x) - \alpha| > \epsilon\}| = 0, \tag{13}$$

where the notion $|\{\cdot\}|$ indicates the Lebesgue measure of the set $\{\cdot\}$. If (13) exists, then we write

$$st - \lim_{x\to\infty} f(x) = \alpha. \tag{14}$$

Also a function $f \in L_{loc}(\mathbb{R}_+)$ is said to be statistically summable $(C, 1)$ to α, if

$$st - \lim_{x\to\infty} \sigma(x) = \alpha \tag{15}$$

[8]. It is clear that the existence of the limit (6) implies the limit (14). However, (14) may imply (6) by adding some suitable conditions on the function f. In [8], Móricz presented following results:

Lemma 2.1. *Let $f \in L_{loc}(\mathbb{R}_+)$ be bounded almost everywhere on $\mathbb{R}_+$ and statistically converges to α, then f statistically $(C, 1)$ converges to α.*

Theorem 2.2. *If a real-valued measurable function f is slowly decreasing on $\mathbb{R}_+$ and statistical summable to a finite number α, then the ordinary limit of f also exists and equals α.*

Theorem 2.3. *If a real-valued function $f \in L_{loc}(\mathbb{R}_+)$ is slowly decreasing and statistical summable $(C, 1)$ to a finite number α, then the ordinary limit of f also exists and equals α.*

Theorem 2.4. *If a complex-valued measurable function f is slowly oscillating on $\mathbb{R}_+$ and statistical summable to a finite number α, then the ordinary limit of f also exists and equals α.*

Theorem 2.5. *If a complex-valued function $f \in L_{loc}(\mathbb{R}_+)$ is slowly oscilating and statistical summable $(C, 1)$ to a finite number α, then the ordinary limit of f also exists and equals α.*

In this work, we give Theorem 3.2 and Theorem 3.4 firstly. So we generalize Theorem 2.2 and Theorem 2.3. After these generalizations, we present Theorem 3.7 and Theorem 3.9 in the complex-valued case. With these results we generalize Theorem 2.4 and Theorem 2.5 respectively. We establish our main theorems by using general control modulo of the oscillatory behaviour of nonnegative integer order $m \geq 0$ of the function $s(x)$.

3. Main results

Firstly, we consider real-valued functions and prove the following Tauberian theorems.
First of all, we establish the convergence of $s(x)$ from the statistical convergence of $\sigma_{(k)}(x)$.

Remark 3.1. *The following theorem generalizes Theorem 2.3. It is clear that if we take $r = 1$ in Theorem 3.2, Theorem 2.3 is obtained.*

Theorem 3.2. *Let $s(x) \in L^1_{loc}(\mathbb{R}_+)$ be a real-valued function. If the statistical limit α of $\sigma_{(r)}(x)$ exists at ∞ for some nonnegative integer r and $s(x) \in \mathcal{SD}$, then the ordinary limit of $s(x)$ also exists at ∞ and equals α.*

Proof. From the assumption of $s(x) \in \mathcal{SD}$ we have $\sigma_{(r)}(x) \in \mathcal{SD}$ for each integer $r \geq 1$. By taking $r = k$ in the statistical convergence of $\sigma_{(r)}(x)$ and using Theorem 2.2, we get

$$\lim_{x\to\infty} \sigma_{(k)}(x) = \alpha.$$

Now taking $r = k - 1$ in the statistical convergence of $\sigma_{(r)}(x)$ and using Theorem 1.1 again, we obtain

$$\lim_{x\to\infty} \sigma_{(k-1)}(x) = \alpha.$$

If we continue in this vein, we get that

$$\lim_{x\to\infty} \sigma_{(1)}(x) = \alpha$$

and it's mean that $s(x)$, summable $(C, 1)$ to α. Hence we conclude that the ordinary limit of $s(x)$ exists at ∞ and equals α by Theorem 1.1. □

Remark 3.3. *The next theorem generalizes Theorem 2.2. Indeed if* $s(x) \in \mathcal{SD}$*, then we get* $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SD}$ *by the identity (7). However the assumption* $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SD}$ *does not imply* $s(x) \in \mathcal{SD}$.

Theorem 3.4. *Let* $s(x)$ *be a real-valued and measurable function with the assumption of* $s(x) = O(1)$*. If the statistical limit* α *of* $s(x)$ *exists at* ∞ *and* $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SD}$ *for some nonnegative integer* r*, then the ordinary limit of* $s(x)$ *also exists at* ∞ *and equals* α.

Proof. By Lemma 2.1, we have

$$st - \lim_{x\to\infty} \sigma_{(1)}(\omega_{(r)}(x)) = 0. \tag{16}$$

It follows from assumption of $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SD}$ for some nonnegative integer r and (16) that

$$\lim_{x\to\infty} \sigma_{(1)}(\omega_{(r)}(x)) = 0$$

with the help of Theorem 2.2. Using Lemma 1.5 and Lemma 1.6 with the identity (10), we can establish that

$$\sigma_{(1)}(\omega_{(r)}(x)) = x\frac{d}{dx}\sigma_{(2)}(\omega_{(r-1)}(x)).$$

Therefore we obtain that

$$x\frac{d}{dx}\sigma_{(2)}(\omega_{(r-1)}(x)) = O_L(1). \tag{17}$$

And the equality (17) yields $\sigma_{(2)}(\omega_{(r-1)}(x)) \in \mathcal{SD}$. It follows from Kronecker identity (7) and the hypothesis of $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SD}$ that $\sigma_{(1)}(\omega_{(r-1)}(x)) \in \mathcal{SD}$. Continuing in this vein, we obtain that $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SD}$. By applying Theorem 2.2 to this result and (16) with $r = 0$ we get

$$\lim_{x\to\infty} \sigma_{(1)}(\omega_{(0)}(x)) = 0. \tag{18}$$

Now we conclude that $\sigma_{(1)}(x)$ is slowly decreasing. Also if we use Lemma 2.1 again, we have

$$st - \lim_{x\to\infty} \sigma_{(1)}(x) = \alpha. \tag{19}$$

Hence we obtain that

$$\lim_{x\to\infty} \sigma_{(1)}(x) = \alpha \tag{20}$$

by Theorem 2.2 and (19). Combining (18) and (20) with the help of Kronecker identity (7), we get that the ordinary limit of $s(x)$ exists at ∞ and equals α. □

Corollary 3.5. *Let* $s(x)$ *be a real-valued and measurable function with the assumption of* $s(x) = O(1)$*. If the statistical limit* α *of* $s(x)$ *exists at* ∞ *and* $\omega_{(r)}(x) = O_L(1)$ *for some nonnegative integer* r*, then the ordinary limit of* $s(x)$ *also exists at* ∞ *and equals* α.

Proof. By using the identity

$$\omega_{(r)}(x) = x\frac{d}{dx}\sigma_{(1)}(\omega_{(r-1)}(x)),$$

we obtain that $\sigma_{(1)}(\omega_{(r-1)}(x)) \in \mathcal{SD}$ and $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SD}$. Then, the proof ends since the conditions in Theorem 3.4 are met. □

Secondly, we consider complex-valued functions and give the following Tauberian theorems.

In the following theorem, we obtain the convergence of $s(x)$ from the statistical convergence of $\sigma_{(k)}(x)$.

Remark 3.6. *The following theorem generalizes Theorem 2.5. Obviously, if we take $r = 1$ in Theorem 3.7, then we obtain Theorem 2.5.*

Theorem 3.7. *Let $s(x) \in L^1_{loc}(\mathbb{R}_+)$ be a complex-valued function. If the statistical limit α of $\sigma_{(r)}(x)$ exists at ∞ for some nonnegative integer r and $s(x) \in \mathcal{SO}$, then the ordinary limit of $s(x)$ exists at ∞ and equals α.*

Proof. By the assumption of $s(x) \in \mathcal{SO}$ we have $\sigma_{(r)}(x) \in \mathcal{SO}$ for each integer $r \geq 1$. Now taking $r = k$ in the statistical convergence of $\sigma_{(r)}(x)$ and using Theorem 2.4, we get

$$\lim_{x\to\infty} \sigma_{(k)}(x) = \alpha.$$

After taking $r = k - 1$ in the statistical convergence of $\sigma_{(r)}(x)$ and using Theorem 1.3, we conclude that

$$\lim_{x\to\infty} \sigma_{(k-1)}(x) = \alpha.$$

If we continue in this way, we obtain that

$$\lim_{x\to\infty} \sigma_{(1)}(x) = \alpha.$$

$s(x)$ is summable $(C, 1)$ to α in other words. Eventually, the conditions in Theorem 1.3 are satisfied. Hence we obtain that the ordinary limit of $s(x)$ exists at ∞ and equals α. □

Remark 3.8. *The next theorem generalizes Theorem 2.4. Indeed if we take $r = 0$ in the next theorem, we get $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SO}$ and from the slow oscillation of $s(x)$ and the identity (7) we can conclude this. However the assumption $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SO}$ does not imply $s(x) \in \mathcal{SO}$.*

Theorem 3.9. *Let $s(x)$ be a complex-valued and measurable function with the assumption of $s(x) = O(1)$. If the statistical limit α of $s(x)$ exists at ∞. and $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SO}$ for some nonnegative integer r, then the ordinary limit of $s(x)$ also exists at ∞ and equals α.*

Proof. From Lemma 2.1, we have (16) and it follows from assumption of $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SO}$ for some nonnegative integer r and (16) that

$$\lim_{x\to\infty} \sigma_{(1)}(\omega_{(r)}(x)) = 0$$

with the help of Theorem 2.4. By benefit from identity

$$\sigma_{(1)}(\omega_{(r)}(x)) = x\frac{d}{dx}\sigma_{(2)}(\omega_{(r-1)}(x)),$$

we get that

$$x\frac{d}{dx}\sigma_{(2)}(\omega_{(r-1)}(x)) = O(1). \tag{21}$$

And the equality (21) yields $\sigma_{(2)}(\omega_{(r-1)}(x)) \in \mathcal{SO}$. It follows from Kronecker identity (7) and the assumption of $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SO}$ that $\sigma_{(1)}(\omega_{(r-1)}(x)) \in \mathcal{SO}$. Continuing in this way, we have $\sigma_{(1)}(\omega_{(0)}(x)) \in \mathcal{SO}$. By applying Theorem 2.4 to this result and (16) with $r = 0$ we obtain (18). Hence $\sigma^{(1)}(x)$ is slowly oscillating. By Lemma 2.1, we have (19). Therefore we conclude that (20) by Theorem 2.4 and (19). Finally using (18) and (20) in the Kronecker identity (7), we obtain that the ordinary limit of $s(x)$ exists at ∞ and equals α. □

Corollary 3.10. *Let $s(x)$ be a complex-valued and measurable function with the assumption of $s(x) = O(1)$. If the statistical limit α of $s(x)$ exists at ∞ and $\omega_{(r)}(x) = O(1)$ for some nonnegative integer r, then the ordinary limit of $s(x)$ also exists at ∞ and equals α.*

Proof. By using the identity

$$\omega_{(r)}(x) = x\frac{d}{dx}\sigma_{(1)}(\omega_{(r-1)}(x)),$$

we obtain that $\sigma_{(1)}(\omega_{(r-1)}(x)) \in \mathcal{SO}$ and $\sigma_{(1)}(\omega_{(r)}(x)) \in \mathcal{SO}$. Then, the proof ends since the conditions in Theorem 3.9 are met. □

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