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The theory of t absolute randomized truth degree in Goguen n-valued propositional logic system of adding two operators

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Abstract. Using the randomization method of valuation set, we firstly give the definition of t absolute randomized truth degree of propositional formula in Goguen n-valued propositional logic system of adding two operators (t takes $\sim$, Δ), and prove that some inference rules such as MP, HS, intersection inference, union inference and some related properties of t absolute randomized truth degree. Secondly we introduce the concepts of t absolute randomized similarity degree and t absolute randomized pseudo-distance of propositional formulas, prove that some good properties of t absolute randomized similarity degree, meanwhile discuss the continuity problem of operators $\sim, \Delta, \rightarrow, \wedge, \vee$ with respect to t absolute randomized pseudo-distance ρ_D in t absolute randomized logical metric space $(F(S), \rho_D)$. Then we give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of propositional formulas theory Γ and some good properties between them. Finally, we introduce that three different types of approximate reasoning patterns in t absolute randomized logical metric space, and they are proved to be equivalent.

1. Introduction

As we all know, mathematical logic is a formal theory characterized by symbolization, it focuses on formal deduction rather than numerical calculation. However, numerical calculation pays more attention to solving problems and rarely uses formal deduction methods. In order to establish the connection between the two, Wang Guojun created quantitative logic[16, 18–20], which is a combination of mathematical logic and probability calculation.

The idea of introducing probability methods into mathematical logic has gradually emerged since the 1950s, and a monograph on"probabilistic logic"[1] has been published. Later, many scholars have carried out researches on this basis and have made rich achievements. In[4, 10–12, 17], some authors used the randomization method of valuation set to give the randomized truth degree theory of propositional formula in the logic system and to establish the randomized logic metric space. It realizes the integration of probability logic and quantitative logic.

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At present, in the two widely attention logic systems, because of the strong negation in Gödel system and Goguen system, the related research is affected. In order to overcome it, in[2, 3, 5–7], some authors introduced two basic connectives ∼ and Δ, and proposed axiomatic extensions of basic logic system BL is BL_Δ system and $SBL_\sim$ system, in which Δ deduction theorem and strong completeness theorem are both established, so that related research can be carried out smoothly. In[13], the author realized quantitative research of Δ fuzzy logic system in $SBL_\sim$ system. $\text{Gödel}_{\sim,\Delta}$ and $\text{Goguen}_{\sim,\Delta}$ system, as typical representatives of $SBL_\sim$ system, and many scholars have studied in these two systems, among them, in[8, 9, 15], some authors proposed t truth degree theory, k randomized truth degree theory and $\Gamma - k$ randomized truth degree theory in $\text{Goguen}_{\sim,\Delta}$ propositional logic system. In[14], the author evaded infinite product measure in uniformly distributed probability spaces, and introduced concept of absolute truth degree in Lukasiewicz propositional logic. Later, many scholars have carried out researches on this basis. So a subsequent question is whether a similar study of absolute randomization can be carried out in $\text{Goguen}_{\sim,\Delta}$ propositional logic system, so that the absolute randomized truth degree of any formula can be calculated by computer in a finite number of steps, which makes the algorithm implementation of the method in this paper possible.

In this paper, using the randomization method of valuation set, we first put forward the definition of t absolute randomized truth degree of propositional formula in $\text{Goguen}_{\sim,\Delta}$ propositional logic system (t takes ∼, Δ), and prove some inference rules such as MP, HS, intersection inference, union inference of t absolute randomized truth degree. Then we give the concepts of t absolute randomized similarity degree and t absolute randomized pseudo-distance of propositional formulas. We also give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of propositional formulas theory Γ, and introduce three different types of approximate reasoning patterns, which are proved to be equivalent.

The results of this paper generalize the related work in and enrich the quantification research in $\text{Goguen}_{\sim,\Delta}$ propositional logic system. Our work provides the basis for the future study of absolute randomized truth degree in other propositional logic systems.

2. Preliminary

Definition 2.1. ([2]) *The axiom system of BL_Δ is as follows:*

(BL) the axiom system of BL.

(Δ1) $\Delta A \vee \neg \Delta A$.

(Δ2) $\Delta(A \vee B) \to (\Delta A \vee \Delta B)$.

(Δ4) $\Delta A \to \Delta\Delta A$.

(Δ5) $\Delta(A \to B) \to (\Delta A \to \Delta B)$.

The inference rules in BL_Δ is MP rule and Δ rule; the MP rule is from A, $A \to B$, inferred B; and the Δ rule is from A inferred ΔA.

If L is an axiomatic extension of BL, then L_Δ is denoted as an extension of L in the same way that BL is an extension of BL, and the following Δ deduction theorem holds for the BL_Δ system:

Theorem 2.2. ([5]) *(Δ deduction theorem) Let L be an axiomatic extension of BL_Δ. Then for any theory Γ, the formulas A and B, we have*

$$\Gamma, A \vdash B \text{ if and only if } \Gamma \vdash \Delta A \to B.$$

SBL is the axiomatic extension of BL by adding axiom $\neg\neg A \vee \neg A$. SBL_Δ is also an axiomatic extension of SBL.

The $SBL_\sim$ system is a logical system formed by adding the involutive negating connective ∼ on the basis of the SBL system.

Definition 2.3. ([3]) *As an axiomatic extension of SBL, the axiom system in SBL$_\sim$ is as follows:*

(SBL) the axiom system of SBL.

(∼1) $\sim\sim A\to A$.

(∼2) $\neg A\to\sim A$.

(∼3) $\Delta(A\to B)\to\Delta(\sim B\to\sim A)$.

Let $\Delta A=\neg\sim A$ in the SBL$_\sim$ system. Then we can establish the relationship between the SBL$_\Delta$ system and the SBL$_\sim$ system, that is, SBL$_\sim$ has the following equivalent axiom system:

(SBL$_\Delta$) the axiom system of SBL$_\Delta$.

(∼1) $\sim\sim A\to A$.

(∼2) $\Delta(A\to B)\to\Delta(\sim B\to\sim A)$.

The inference rules in SBL$_\Delta$ is MP rule and Δ rule. If L is an axiomatic extension of SBL, then L$_\sim$ is denoted as an extension of L in the same way that SBL$_\sim$ is an extension of SBL, and Gödel$_\sim$ and Goguen$_\sim$ are the two basic types of axiomatic extension of SBL$_\sim$. Because SBL$_\sim$ is also an axiomatic extension of BL$_\Delta$, Δ deduction theorem in SBL$_\sim$ is also holds.

Theorem 2.4. ([3]) *(strong completeness theorem) Let L an axiomatic extension of SBL$_\sim$. Then for theory Γ and formula A, the following two conditions are equivalent:*

(i) $\Gamma\vdash A$.

(ii) For every L-algebra and every model e of theory Γ, there are $e(A)=1$.

Definition 2.5. ([8]) *Let $S=\{p_1,p_2,\ldots\}$ be a countable set, $\sim$ and Δ be two unary operations on S, $\vee$, $\wedge$ and $\to$ be three binary operations on S, respectively, $F(S)$ be a free algebra of type (1,1,2,2,2) generated by S. Then the elements in $F(S)$ are called propositional formulas or formulas, and the elements in S are called atomic formulas.*

Definition 2.6. ([8]) *The Goguen propositional logic system is also called product system, denote as Π. Let $\Pi_{\sim,\Delta}=\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}$. It is stipulated in $\Pi_{\sim,\Delta}$: $\forall x,y\in\Pi_{\sim,\Delta}$, $\sim x=1-x$, $\Delta x=\{^{1,x=1}_{0,x<1}$, $x\vee y=\max\{x,y\}$, $x\wedge y=\min\{x,y\}$, $x\to y=\{^{1,x\le y}_{\frac{y}{x},x>y}$. The system Goguen$_{\sim,\Delta}$ is called an expansion of n-valued product propositional logic system. It is abbreviated as $\Pi_{\sim,\Delta}$.*

Definition 2.7. ([8]) *Let $A=A(p_1,p_2,\ldots,p_m)\in F(S)$. Then A corresponds to an n-valued m-element function $\overline{A}$, in $\Pi_{\sim,\Delta}$, $\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m\to[0,1]$, here $\overline{A}$ $(x_1,\ldots,x_m)$ is formed by the operation symbols $\sim$, Δ, $\vee$, $\wedge$, $\to$ connecting $x_1,\ldots,x_m$, in the same way as $A=A(p_1,p_2,\ldots,p_m)\in F(S)$ is formed by connecting the atomic formulas $p_1,\ldots,p_m$ using the conjunction $\sim$, Δ, $\vee$, $\wedge$, $\to$. Then $\overline{A}$ is called the function induced by the formula A.*

Definition 2.8. ([10]) *Let $N=(1,2,\ldots)$, $D=(p_1,p_2,p_3)$, $0<p_n<1$ $(n=1,2,\ldots)$. Then D is called a randomized sequence in* $(0,1)$.

Definition 2.9. ([4]) *Let $D_0=(p_{01},p_{02},\ldots)$, $D_{\frac{1}{n-1}}=\{p_{\frac{1}{n-1}1},p_{\frac{1}{n-1}2},\ldots\},\ldots$, $D_1=(p_{11},p_{12},\ldots)$ be an n randomized sequences in $(0,1)$, and $p_{0k}+p_{\frac{1}{n-1}k}+\ldots+p_{1k}=1$ $(k=1,2,\ldots)$. Then $D_0,D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}},D_1$ $(n\ge 2)$ is called an n-valued randomized numbers sequence in* $(0,1)$.

Definition 2.10. ([4]) *Suppose that $D_0,D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}},D_1$ $(n\ge2)$ be a series of n randomized numbers in $(0,1)$, $\forall a=(x_1,x_2,\ldots,x_m)\in\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m$. Let $\varphi(\alpha)=Q_1\times\ldots\times Q_m$, here for any $1\le k\le m$, when $x_k=0$, $Q_k=d_{0k}$; when $x_k=\frac{i}{n-1}$, $Q_k=d_{\frac{i}{n-1}k}$ $(i=1,2,\ldots,n-2)$; when $x_k=1$, $Q_k=d_{1k}$. Then we get a mapping*

$$\varphi:\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m\to[0,1],$$

called the D-randomization map of $\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m$.

Proposition 2.11. ([4]) *Let φ be a D-randomization map of $\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m$. Then*

$$\sum\{\varphi(\alpha):\alpha\in\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m\}=1.$$

3. Definition and properties of t absolute randomized truth degree of propositional formula

Definition 3.1. *Let $A = A(p_1, p_2, \ldots, p_m) \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Define*

$$[tA]_1 = \overline{tA}^{-1}(1),$$

$$\mu([tA]_1) = \sum\{\varphi(\alpha) : \alpha \in \overline{tA}^{-1}(1)\},$$

$$\tau_D(tA) = |\mu([tA]_1)|.$$

Then $\tau_D(tA)$ is called the t absolute randomized truth degree of the propositional formula A, where t takes $\sim$ and Δ.

Remark 3.2. *Unless there are another instructions in the text, the following points remain unchanged: (i) Discuss in $\Pi_{\sim,\Delta}$; (ii) Basic grammar, semantic concepts, etc. are the same as classic proposition logic; (iii) p, q, r, z, m, l take $\sim$ and Δ.*

Theorem 3.3. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then*

(i) A is tautology if and only if $\tau_D(\Delta A) = 1$ and $\tau_D(\sim A) = 0$, and A is the contradiction if and only if $\tau_D(\sim A) = 1$.

(ii) A is the contradiction, then $\tau_D(\Delta A) = 0$, but the reverse is not true.

(iii) If $A \approx B$, then $\tau_D(tA) = \tau_D(tB)$.

(iv) If $\models pA \to qB$, then $\tau_D(pA) \leq \tau_D(qB)$.

(v) $\tau_D(\sim tA) = 1 - \tau_D(tA)$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): A is the tautology if and only if $[\Delta A]_1 = \overline{\Delta A}^{-1}(1) = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$. It follows from Definition 3.1 that $\tau_D(\Delta A) = |\mu([\Delta A]_1)| = |\sum\{\varphi(\alpha) : \alpha \in \overline{\Delta A}^{-1}\}| = |\sum\{\varphi(\alpha) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m\}|$. Hence by Proposition 2.11, we get that $\tau_D(\Delta A) = 1$. Carrying out a similar proof we get that A is the tautology if and only if $\tau_D(\sim A) = 0$ and A is the contradiction if and only if $\tau_D(\sim A) = 1$.

(ii): Carrying out a proof similar to that of (1), we can get that when A is the contradiction, then $\tau_D(\Delta A) = 0$. Conversely, if $\tau_D(\Delta A) = 0$, then for any $\alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, we have $\overline{\Delta A}(\alpha) \neq 1$, that is, $\overline{A}(\alpha) \neq 1$. So $\tau_D(\Delta A) = 0$ as long as A is not tautology.

(iii): By $A \approx B$, we have that $[A]_1 = [B]_1$, and hence $[tA]_1 = [tB]_1$. Thus $\overline{tA}^{-1}(1) = \overline{tB}^{-1}(1)$. It follows from Definition 3.1 that $\tau_D(tA) = |\mu([tA]_1)| = |\sum\{\varphi(\alpha) : \alpha \in \overline{tA}^{-1}(1)\}| = |\sum\{\varphi(\alpha) : \alpha \in \overline{tB}^{-1}(1)\}| = \tau_D(tB)$.

(iv): $\models pA \to qB$ if and only if $\overline{pA \to qB}(\alpha) = 1$, if and only if $\overline{pA}(\alpha) \leq \overline{qB}(\alpha)$, $[pA]_1 \leq [qB]_1$ and $\overline{pA}^{-1}(1) \leq \overline{qB}^{-1}(1)$. It follows from Definition 3.1 that $\tau_D(pA) = |\mu([pA]_1)| = |\sum\{\varphi(\alpha) : \alpha \in \overline{pA}^{-1}(1)\}| \leq |\sum\{\varphi(\alpha) : \alpha \in \overline{qB}^{-1}(1)\}| = \tau_D(qB)$.

(v): It follows from Definition 3.1 that $\tau_D(\sim tA) = |\mu([\sim tA]_1)| = |1 - \mu([tA]_1)|$. As $0 \leq \mu([tA]_1) \leq 1$, we have $|1 - \mu([tA]_1)| = |1| - |\mu([tA]_1)| = 1 - |\mu([tA]_1)|$. Thus $\tau_D(\sim tA) = 1 - |\mu([tA]_1)| = 1 - \tau_D(tA)$.

Lemma 3.4. *Let $\forall a, b \in \Pi_{\sim,\Delta}$. Then*

(i) $1 \to qb = qb$.

(ii) $pa \to qb \geq qb$.

Proof. (i): (1) Case 1: $qb = 1$. Then $1 \to qb = 1 \to 1 = 1 = qb$; (2) Case 2: $qb < 1$. Then $1 \to qb = qb$.

So to sum up $1 \to qb = qb$.

(ii): (1) Case 1: $pa \leq qb$. Then $pa \to qb = 1 \geq qb$; (2) Case 2: $pa > qb$. Then $pa \to qb = \frac{qb}{pa} > qb$.

So to sum up $pa \to qb \geq qb$.

Theorem 3.5. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\models pA$, then*

i) $\tau_D(pA \to qB) = \tau_D(pA \wedge qB) = \tau_D(qB)$.

(ii) $\tau_D(qB \to pA) = 1$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$. If $\models pA$, then for any $\alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, we have $\overline{pA}(\alpha) = 1$.

(i): For any $\alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, by Lemma 3.4(i), we have that $\overline{pA \to qB}(\alpha) = (\overline{pA} \to \overline{qB})(\alpha) = \overline{pA}(\alpha) \to \overline{qB}(\alpha) = 1 \to \overline{qB}(\alpha) = \overline{qB}(\alpha)$ and $\overline{pA \wedge qB}(\alpha) = (\overline{pA} \wedge \overline{qB})(\alpha) = \overline{pA}(\alpha) \wedge \overline{qB}(\alpha) = 1 \wedge \overline{qB}(\alpha) = \overline{qB}(\alpha)$. Thus $[pA \to qB]_1 = [qB]_1$ and $[pA \wedge qB]_1 = [qB]_1$. Then $\overline{pA \to qB}^{-1}(1) = \overline{qB}^{-1}(1)$ and $\overline{pA \wedge qB}^{-1}(1) = \overline{qB}^{-1}(1)$. It follows from Definition 3.1 that

$$\begin{aligned}\tau_D(pA \to qB) &= |\mu([pA \to qB]_1)| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{pA \to qB}^{-1}(1)\}| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{pA \wedge qB}^{-1}(1)\}| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{qB}^{-1}(1)\}|.\end{aligned}$$

Thus $\tau_D(pA \to qB) = \tau_D(pA \wedge qB) = \tau_D(qB)$.

(ii): For any $\alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, by Lemma 3.4(ii), we get that $\overline{qB \to pA}(\alpha) = (\overline{qB} \to \overline{pA})(\alpha) \geq \overline{pA}(\alpha) = 1$. Thus $[qB \to pA]_1 = \overline{qB \to pA}^{-1}(1) = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$. It follows from Definition 3.1 and Proposition 2.11 that

$$\begin{aligned}\tau_D(qB \to pA) &= |\mu([qB \to pA]_1)| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{qB \to pA}^{-1}(1)\}| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m| \\ &= 1.\end{aligned}$$

Example 3.6. *Let $A = (\sim p_1 \vee \Delta p_2) \to p_2$, $B = (\sim p_1 \to \sim p_2) \to p_1$, $C = (\Delta p_1 \to \sim p_2) \to \sim p_1$, and $D_0 = \{0.1, 0.2\}$, $D_{\frac{1}{3}} = \{0.2, 0.1\}$, $D_{\frac{2}{3}} = \{0.3, 0.4\}$ and $D_1 = \{0.4, 0.3\}$ be a 4-valued randomized numbers sequence in $(0,1)$. Calculate $\tau_D(\Delta A \wedge \sim B) \to \Delta C$.*

Answer. $\overline{A}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0,1]$, $\overline{A}(x, y) = (\sim x \vee \Delta y) \to y$;

$\overline{B}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0,1]$, $\overline{B}(x, y) = (\sim x \to \sim y) \to x$;

$\overline{C}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0,1]$, $\overline{C}(x, y) = (\Delta x \to \sim y) \to \sim x$.

In order to facilitate calculation and understanding, the following chart is made.

x	y	$\overline{A}(x,y)$	$\overline{B}(x,y)$	$\overline{C}(x,y)$	$(\Delta A \wedge \sim B) \to \Delta C$
0	0	0	0	1	1
0	$\frac{1}{3}$	$\frac{1}{3}$	0	1	1
0	$\frac{2}{3}$	$\frac{2}{3}$	0	1	1
0	1	1	1	1	1
$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	1

$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	0
$\frac{1}{3}$	1	1	1	$\frac{2}{3}$	1
$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	1
$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{2}{3}$	1	1	1	$\frac{1}{3}$	1
1	0	1	1	0	1
1	$\frac{1}{3}$	1	1	0	1
1	$\frac{2}{3}$	1	1	0	1
1	1	1	1	1	1

Thus $[(\Delta A \wedge \sim B) \to \Delta C]_1 = \overline{((\Delta A \wedge \sim B) \to \Delta C)}^{-1}(1) = \{(0,0),(0,\frac{1}{3}),(0,\frac{2}{3}),(0,1),(\frac{1}{3},0),(\frac{1}{3},\frac{1}{3}),(\frac{1}{3},1),(\frac{2}{3},0),(\frac{2}{3},1),(1,0),(1,\frac{1}{3}),(1,\frac{2}{3}),(1,1)\}$. It follows from Definition 3.1 that $\tau_D((\Delta A \wedge \sim B) \to \Delta C) = |\mu([(\Delta A \wedge \sim B) \to \Delta C]_1)| = |0.1 \times (0.2+0.1+0.4+0.3) + 0.2 \times (0.2+0.1+0.3) + 0.3 \times (0.2+0.3) + 0.4 \times (0.2+0.1+0.4+0.3)| = 0.77$.

Lemma 3.7. *Let $\forall a, b \in \Pi_{\sim,\Delta}$. Then $qb \vee pa = qb + pa - (qb \wedge pa)$.*

Proof. Let $\lambda_1 = qb \vee pa - qb - pa + (qb \wedge pa)$.

(1) Case 1: $qb \leq pa$. Then $\lambda_1 = pa - qb - pa + qb = 0$, that is, $qb \vee pa = qb + pa - (qb \wedge pa)$.

(2) Case 2: $qb > pa$. Then $\lambda_1 = qb - qb - pa + pa$, that is, $qb \vee pa = qb + pa - (qb \wedge pa)$.

So to sum up $qb \vee pa = qb + pa - (qb \wedge pa)$.

Theorem 3.8. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then $\tau_D(qB \vee pA) = \tau_D(qB) + \tau_D(pA) - \tau_D(qB \wedge pA)$.*

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then for any $\forall \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, by Lemma 3.7, we have that $\overline{qB \vee pA}(\alpha) = \overline{qB}(\alpha) + \overline{pA}(\alpha) - \overline{qB \wedge pA}(\alpha)$. Thus $[qB \vee pA]_1 = [qB]_1 + [pA]_1 - [qB \wedge pA]_1$. Then $\overline{qB \vee pA}^{-1}(1) = \overline{qB}^{-1}(1) + \overline{pA}^{-1}(1) - \overline{qB \wedge pA}^{-1}(1)$. It follows from Definition 3.1 that

$$\begin{aligned}
\tau_D(qB \vee pA) &= |\mu([qB \vee pA]_1)| \\
&= |\sum\{\varphi(\alpha) : \alpha \in \overline{qB \vee pA}^{-1}(1)\}| \\
&= |\sum\{\varphi(\alpha) : \alpha \in \overline{qB}^{-1}(1)\}| + |\sum\{\varphi(\alpha) : \alpha \in \overline{pA}^{-1}(1)\}| \\
&- |\sum\{\varphi(\alpha) : \alpha \in \overline{qB \wedge pA}^{-1}(1)\}|.
\end{aligned}$$

Thus $\tau_D(qB \vee pA) = \tau_D(qB) + \tau_D(pA) - \tau_D(qB \wedge pA)$.

Remark 3.9. *Because p, q take $\sim$ and Δ, the conclusion of Theorem 3.8 has specifically the following four forms.*

(i) $\tau_D(\Delta B \vee \Delta A) = \tau_D(\Delta B) + \tau_D(\Delta A) - \tau_D(\Delta B \wedge \Delta A)$.

(ii) $\tau_D(\Delta B \vee \sim A) = \tau_D(\Delta B) + \tau_D(\sim A) - \tau_D(\Delta B \wedge \sim A)$.

(iii) $\tau_D(\sim B \vee \Delta A) = \tau_D(\sim B) + \tau_D(\Delta A) - \tau_D(\sim B \wedge \Delta A)$.

(iv) $\tau_D(\sim B \vee \sim A) = \tau_D(\sim B) + \tau_D(\sim A) - \tau_D(\sim B \wedge \sim A)$.

Theorem 3.10. *(t absolute randomized truth degree MP rule) Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\tau_D(pA) \geq \alpha$ and $\tau_D(pA \to qB) \geq \beta$, then $\tau_D(qB) \geq \alpha + \beta - 1$.*

Proof. Suppose that A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$. $\forall a, b \in \Pi_{\sim,\Delta}$, we have $qb \geq pa + (pa \to qb) - 1$. Hence $|\overline{qB}^{-1}(1)| \geq |\overline{pA}^{-1}(1)| + |\overline{pA \to qB}^{-1}(1)| - 1$. It follows from Definition 3.1 that

$$\begin{aligned} \tau_D(qB) &= |\mu([qB]_1)| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{qB}^{-1}(1)\}| \\ &\geq |\sum\{\varphi(\alpha) : \alpha \in \overline{pA}^{-1}(1)\}| + |\sum\{\varphi(\alpha) : \alpha \in \overline{pA \to qB}^{-1}(1)\}| - 1. \end{aligned}$$

Thus $\tau_D(qB) \geq \alpha + \beta - 1$.

Corollary 3.11. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\tau_D(pA) = 1$ and $\tau_D(pA \to qB) = 1$, then $\tau_D(qB) = 1$.*

Lemma 3.12. *Let $\forall a, b, c \in \Pi_{\sim,\Delta}$. Then $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.*

Proof. Let $\lambda_2 = (pa \to qb) \to ((qb \to rc) \to (pa \to rc)) - 1$.
(1) Caes 1: $pa \leq qb$.
(1.1) Case 1.1: $qb \leq rc$. Then $\lambda_2 = 1 \to (1 \to 1) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
(1.2) Case 1.2: $qb > rc$.
(1.2.1) Case 1.2.1: $pa \leq rc$. Then $\lambda_2 = 1 \to (\frac{rc}{qb} \to 1) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
(1.2.2) Case 1.2.2: $pa > rc$. Then $\lambda_2 = 1 \to (\frac{rc}{qb} \to \frac{rc}{pa}) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
(2) Case 2: $pa > qb$.
(2.1) Case 2.1: $qb > rc$. Then $\lambda_2 = \frac{qb}{pa} \to (\frac{rc}{qb} \to \frac{rc}{pa}) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
(2.2) Case 2.2: $qb \leq rc$.
(2.2.1) Case 2.2.1: $pa \leq rc$. Then $\lambda_2 = \frac{qb}{pa} \to (1 \to 1) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
(2.2.2) Case 2.2.2: $pa > rc$. Then $\lambda_2 = \frac{qb}{pa} \to (1 \to \frac{rc}{pa}) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.
So to sum up $(pa \to qb) \to ((qb \to rc) \to (pa \to rc)) = 1$.

Theorem 3.13. *(t absolute randomized truth degree HS rule) Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\tau_D(pA \to qB) \geq \alpha$ and $\tau_D(qB \to rC) \geq \beta$, then $\tau_D(pA \to rC) \geq \alpha + \beta - 1$.*

Proof. Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Lemma 3.12, we have that $\models (pA \to qB) \to ((qB \to rC) \to (pA \to rC))$. It follows from Theorem 3.3(iv) that $\tau_D((qB \to rC) \to (pA \to rC)) \geq \tau_D(pA \to qB) \geq \alpha$. Since $\tau_D(qB \to rC) \geq \beta$, by Theorem 3.10, we get that $\tau_D(pA \to rC) \geq \alpha + \beta - 1$.

Corollary 3.14. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\tau_D(pA \to qB) = 1$ and $\tau_D(qB \to rC) = 1$, then $\tau_D(pA \to rC) = 1$.*

Lemma 3.15. *Let $\forall a, b, c \in \Pi_{\sim,\Delta}$. Then $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.*

Proof. Let $\lambda_3 = (pa \to (qb \wedge rc)) - ((pa \to qb) \wedge (pa \to rc))$.
(1) Case 1: $qb \leq rc$.
(1.1) Case 1.1: $pa \leq qb$. Then $\lambda_3 = (pa \to qb) - (1 \wedge 1) = 1 - 1 = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
(1.2) Case 1.2: $pa > qb$.

(1.2.1) Case 1.2.1: $pa \geq rc$. Then $\lambda_3 = (pa \to qb) - (\frac{qb}{pa} \wedge \frac{rc}{pa}) = \frac{qb}{pa} - \frac{qb}{pa} = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
(1.2.2) Case 1.2.2: $pa < rc$. Then $\lambda_3 = (pa \to qb) - (\frac{qb}{pa} \wedge 1) = \frac{qb}{pa} - \frac{qb}{pa} = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
(2) Case 2: $qb > rc$.
(2.1) Case 2.1: $rc > pa$. Then $\lambda_3 = (pa \to rc) - (1 \wedge 1) = 1 - 1 = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
(2.2) Case 2.2: $rc \leq pa$.
(2.2.1) Case 2.2.1: $qb \leq pa$. Then $\lambda_3 = (pa \to rc) - (\frac{qb}{pa} \wedge \frac{rc}{pa}) = \frac{rc}{pa} - \frac{rc}{pa} = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
(2.2.2) Case 2.2.2: $qb > pa$. Then $\lambda_3 = (pa \to rc) - (1 \wedge \frac{rc}{pa}) = \frac{rc}{pa} - \frac{rc}{pa} = 0$,
that is, $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.
So to sum up $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

Theorem 3.16. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then $\tau_D(pA \to (qB \wedge rC)) = \tau_D(pA \to qB) + \tau_D(pA \to rC) - \tau_D((pA \to qB) \vee (pA \to rC))$.*

Proof. Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Lemma 3.15, we have that $pA \to (qB \wedge rC) \approx (pA \to qB) \wedge (pA \to rC)$. It follows from Theorem 3.3(iii) that $\tau_D(pA \to (qB \wedge rC)) = \tau_D((pA \to qB) \wedge (pA \to rC))$. By Theorem 3.8, we get that $\tau_D(pA \to (qB \wedge rC)) = \tau_D(pA \to qB) + \tau_D(pA \to rC) - \tau_D((pA \to qB) \vee (pA \to rC))$.

Corollary 3.17. *t absolute randomized truth degree intersection inference rule) Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\tau_D(pA \to qB) \geq \alpha$ and $\tau_D(pA \to rC) \geq \beta$, then $\tau_D(pA \to (qB \wedge rC)) \geq \alpha + \beta - 1$.*

Corollary 3.18. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\tau_D(pA \to qB) = 1$ and $\tau_D(pA \to rC) = 1$, then $\tau_D(pA \to (qB \wedge rC)) = 1$.*

Lemma 3.19. *Let $\forall a, b, c \in \Pi_{\sim,\Delta}$. Then $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.*

Proof. Let $\lambda_4 = ((pa \vee qb) \to rc) - ((pa \to rc) \wedge (qb \to rc))$.
(1) Case 1: $pa \leq qb$.
(1.1) Case 1.1: $qb \leq rc$. Then $\lambda_4 = (qb \to rc) - (1 \wedge 1) = 1 - 1 = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
(1.2) Case 1.2: $qb > rc$.
(1.2.1) Case 1.2.1: $pa \leq rc$. Then $\lambda_4 = (qb \to rc) - (1 \wedge \frac{rc}{qb}) = \frac{rc}{qb} - \frac{rc}{qb} = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
(1.2.2) Case 1.2.2: $pa > rc$. Then $\lambda_4 = (qb \to rc) - (\frac{rc}{pa} \wedge \frac{rc}{qb}) = \frac{rc}{qb} - \frac{rc}{qb} = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
(2) Case 2: $pa > qb$.
(2.1) Case 2.1: $qb > rc$. Then $\lambda_4 = (pa \to rc) - (\frac{rc}{pa} \wedge \frac{rc}{qb}) = \frac{rc}{pa} - \frac{rc}{pa} = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
(2.2) Case 2.2: $qb \leq rc$.
(2.2.1) Case 2.2.1: $pa \leq rc$. Then $\lambda_4 = (pa \to rc) - (1 \wedge 1) = 1 - 1 = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
(2.2.2) Case 2.2.2: $pa > rc$. Then $\lambda_4 = (pa \to rc) - (\frac{rc}{pa} \wedge 1) = \frac{rc}{pa} - \frac{rc}{pa} = 0$,
that is, $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.
So to sum up $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.

Theorem 3.20. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then $\tau_D((pA \vee qB) \to rC) = \tau_D(pA \to rC) + \tau_D(qB \to rC) - \tau_D((pA \to rC) \vee (qB \to rC))$.*

Proof. Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Lemma 3.19, we get that $(pA \vee qB) \to rC \approx (pA \to rC) \wedge (qB \to rC)$. It follows from Theorem 3.3(iii) that $\tau_D((pA \vee qB) \to rC) = \tau_D((pA \to rC) \wedge (qB \to rC))$. By Theorem 3.8, we have that $\tau_D((pA \vee qB) \to rC) = \tau_D(pA \to rC) + \tau_D(qB \to rC) - \tau_D((pA \to rC) \vee (qB \to rC))$.

Corollary 3.21. *(t absolute randomized truth degree union inference rule) Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\tau_D(pA \to rC) \geq \alpha$ and $\tau_D(qB \to rC) \geq \beta$, then $\tau_D((pA \vee qB) \to rC) \geq \alpha + \beta - 1$.*

Corollary 3.22. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\tau_D(pA \to rC) = 1$ and $\tau_D(qB \to rC) = 1$, then $\tau_D((pA \vee qB) \to rC) = 1$.*

Lemma 3.23. *Let $\forall a, b \in \Pi_{\sim,\Delta}$. then*

(i) $pa \to qb = pa \to (pa \wedge qb)$.

(ii) $pa \to qb = (pa \vee qb) \to qb$.

Proof. (i): Let $\lambda_5 = (pa \to qb) - (pa \to (pa \wedge qb)$.

(1) Case 1: $pa \leq qb$. Then $\lambda_5 = 1 - 1 = 0$, that is, $pa \to qb = pa \to (pa \wedge \wedge qb)$.

(2) Case 2: $pa > qb$. Then $\lambda_5 = \frac{qb}{pa} - \frac{qb}{pa} = 0$, that is, $pa \to qb = pa \to (pa \wedge \wedge qb)$.

So to sum up $pa \to qb = pa \to (pa \wedge \wedge qb)$.

(ii): Let $\lambda_6 = (pa \to qb) - ((pa \vee qb) \to qb)$.

(1) Case 1: $pa \leq qb$. Then $\lambda_6 = 1 - 1 = 0$, that is, $pa \to qb = (pa \vee qb) \to qb$.

(2) Case 2: $pa > qb$. Then $\lambda_6 = \frac{qb}{pa} - \frac{qb}{pa} = 0$, that is, $pa \to qb = (pa \vee qb) \to qb$.

So to sum up $pa \to qb = (pa \vee qb) \to qb$.

Theorem 3.24. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) $\tau_D(pA \to qB) = \tau_D(pA \to (pA \wedge qB))$.

(ii) $\tau_D(pA \to qB) = \tau_D((pA \vee qB) \to qB)$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Lemma 3.23(i), we get that $pA \to qB \approx pA \to (pA \wedge qB)$. It follows from Theorem 3.3(iii) that $\tau_D(pA \to qB) = \tau_D(pA \to (pA \wedge qB))$.

(ii): By Lemma 3.23(ii), we have that $pA \to qB \approx (pA \vee qB) \to qB$. It follows from Theorem 3.3(iii) that $\tau_D(pA \to qB) = \tau_D((pA \vee qB) \to qB)$.

4. *t* absolute randomized similarity degree and *t* absolute randomized pseudo-distance of propositional formulas

Definition 4.1. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Define*

$$\xi_D(pA, qB) = \tau_D((pA \to qB) \wedge (qB \to pA)).$$

Then $\xi_D(pA, qB)$ is called the t absolute randomized similarity degree between propositional formulas A and B.

Lemma 4.2. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\models pA$ and $\models qB$, then $\models pA \wedge qB$.*

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Since $\models (pA \otimes qB) \to (pA \wedge qB)$ and $\models ((pA \otimes qB) \to (pA \wedge qB)) \to (pA \to (qB \to (pA \wedge qB)))$, by MP rule, we get that $\models pA \to (qB \to (pA \wedge qB))$. Also since $\models pA$ and $\models qB$, it follows from double MP rule that $\models pA \wedge qB$.

Theorem 4.3. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) If $A \approx B$, then $\xi_D(tA, tB) = 1$.

(ii) $\xi_D(pA, qB) = \xi_D(qB, pA)$.

(iii) $\xi_D(pA \vee qB, pA) = \tau_D(qB \to pA)$.

(iv) $\xi_D(pA \wedge qB, pA) = \tau_D(pA \to qB)$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): As $A \approx B$, we have $tA \approx tB$. Thus $\models tA \to tB$ and $\models tB \to tA$. By Lemma 4.2, we have that $\models (tA \to tB) \wedge (tB \to tA)$. Thus $[(tA \to qB) \wedge (tB \to tA)]_1 = \overline{(tA \to qB) \wedge (tB \to tA)}^{-1}(1) = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$. It follows from Definition 4.1 and Proposition 2.11 that

$$\begin{aligned} \xi_D(tA, tB) &= \tau_D((tA \to tB) \wedge (tB \to tA)) \\ &= |\mu([(tA \to tB) \wedge (tB \to tA)]_1)| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \overline{(tA \to tB) \wedge (tB \to tA)}^{-1}(1)\}| \\ &= |\sum\{\varphi(\alpha) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m\}| \\ &= 1. \end{aligned}$$

(ii): $\forall a, b \in \Pi_{\sim,\Delta}$, we have $(pa \to qb) \wedge (qb \to pa) = (qb \to pa) \wedge (pa \to qb)$. Thus $(pA \to qB) \wedge (qB \to pA) \approx (qB \to pA) \wedge (pA \to qB)$. Then by Theorem 3.3(iii), we get that $\tau_D((pA \to qB) \wedge (qB \to pA)) = \tau_D((pA \to qB) \wedge (qB \to pA))$. It follows from Definition 4.1 that $\xi_D(pA, qB) = \xi_D(qB, pA)$.

(iii): By Lemma 3.19, we have that $(pA \vee qB) \to pA \approx (pA \to pA) \wedge (qB \to pA) = qB \to pA$. It follows from Lemma 3.15 that $pA \to (pA \vee qB) \approx (pA \to pA) \vee (pA \to qB) = pA \to pA$. Then by Definition 4.1, we get that

$$\begin{aligned} \xi_D(pA \vee qB, pA) &= \tau_D(((pA \vee qB) \to pA) \wedge (pA \to (pA \vee qB))) \\ &= \tau_D(((pA \to pA) \wedge (qB \to pA)) \wedge ((pA \to pA) \vee (pA \to qB))) \\ &= \tau_D((qB \to pA) \wedge (pA \to pA)) \\ &= \tau_D(qB \to pA). \end{aligned}$$

(iv): By Lemma 3.19, we have that $(pA \wedge qB) \to pA \approx (pA \to pA) \vee (qB \to pA) = pA \to pA$. It follows from Lemma 3.15 that $pA \to (pA \wedge qB) \approx (pA \to pA) \wedge (pA \to qB) = pA \to qB$. Then by Definition 4.1, we get that

$$\begin{aligned} \xi_D(pA \wedge qB, pA) &= \tau_D(((pA \wedge qB) \to pA) \wedge (pA \to (pA \wedge qB))) \\ &= \tau_D(((pA \to pA) \vee (qB \to pA)) \wedge ((pA \to pA) \wedge (pA \to qB))) \\ &= \tau_D((pA \to pA) \wedge (pA \to qB)) \\ &= \tau_D(pA \to qB). \end{aligned}$$

Theorem 4.4. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\models pA$, then*

(i) $\xi_D(pA, qB) = \tau_D(qB)$.

(ii) $\xi_D(pA \vee qB, pA) = 1$.

(iii)$\xi_D(pA \wedge qB, pA) = \tau_D(qB)$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Definition 4.1, we get that $\xi_D(pA, qB) = \tau_D((pA \to qB) \wedge (qB \to pA))$. It follows from Theorem 3.8 that $\xi_D(pA, qB) = \tau_D(pA \to qB) + \tau_D(qB \to pA) - \tau_D((pA \to qB) \vee (qB \to pA))$. Thus $\xi_D(pA, qB) = \tau_D(pA \to qB) + \tau_D(qB \to pA) - 1$. Since $\models pA$, by Theorem 3.5, we have that $\xi_D(pA, qB) = \tau_D(qB) + 1 - 1 = \tau_D(qB)$.

(ii): By Theorem 4.3(iii), we get that $\xi_D(pA \vee qB, pA) = \tau_D(qB \to pA)$. Since $\models pA$, it follows from Theorem 3.5(ii) that $\xi_D(pA \vee qB, pA) = 1$.

(iii): By Theorem 4.3(iv), we have that $\xi_D(pA \wedge qB, pA) = \tau_D(pA \to qB)$. Since $\models pA$, it follows from Theorem 3.5(i) that $\xi_D(pA \wedge qB, pA) = \tau_D(qB)$.

Corollary 4.5. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\models qB$, then*

(i) $\xi_D(pA, qB) = \tau_D(pA)$.

(ii) $\xi_D(pA \vee qB, pA) = \tau_D(pA)$.

(iii)$\xi_D(pA \wedge qB, pA) = 1.$.

Lemma 4.6. *Let $A, B, C, D \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\models pA \to qB$ and $\models rC \to zD$, then $\models (pA \wedge rC) \to (qB \wedge zD)$.*

Proof. Let A, B, C, D contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Since $\models (pA \wedge rC) \to pA$ and $\models pA \to qB$, by HS rule, we get that $\models (pA \wedge rC) \to qB$. Also since $\models (pA \wedge rC) \to rC$ and $\models rC \to zD$, it follows from HS rule that $\models (pA \wedge rC) \to zD$. By Lemma 4.2, we have that $\models ((pA \wedge rC) \to qB) \wedge ((pA \wedge rC) \to zD))$. It follows from Lemma 3.15 that $\models (((pA \wedge rC) \to qB) \wedge ((pA \wedge rC) \to zD)) \to ((pA \wedge rC) \to (qB \wedge zD))$. Then by MP rule, we get that $\models (pA \wedge rC) \to (qB \wedge zD)$.

Lemma 4.7. *Let $\forall a, b, c \in \Pi_{\sim,\Delta}$. Then $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.*

Proof. Let $\lambda_7 = (pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) - 1$.

(1) Case 1: $pa \leq qb$.

(1.1) Case 1.1: $qb \leq rc$. Then $\lambda_7 = 1 \to (rc \to rc) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(1.2) Case 1.2: $qb > rc$.

(1.2.1) Case 1.2.1: $pa \leq rc$. Then $\lambda_7 = 1 \to (rc \to qb) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(1.2.2) Case 1.2.2: $pa > rc$. Then $\lambda_7 = 1 \to (pa \to qb) - 1 = 1 - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2) Case 2: $pa > qb$.

(2.1) Case 2.1: $qb > rc$. Then $\lambda_7 = \frac{qb}{pa} \to (pa \to qb) - 1 = (\frac{qb}{pa} \to \frac{qb}{pa}) - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2.2) Case 2.2: $qb \leq rc$.

(2.2.1) Case 2.2.1: $pa \leq rc$. Then $\lambda_7 = \frac{qb}{pa} \to (rc \to rc) - 1 = (\frac{qb}{pa} \to 1) - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2.2.2) Case 2.2.2: $pa > rc$. Then $\lambda_7 = \frac{qb}{pa} \to (pa \to rc) - 1 = (\frac{qb}{pa} \to \frac{rc}{pa}) - 1 = 0$,
that is, $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

So to sum up $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

Lemma 4.8. *Let $\forall a, b, c \in \Pi_{\sim,\Delta}$. Then $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.*

Proof. Let $\lambda_8 = (pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) - 1$.

(1) Case 1: $pa \leq qb$.

(1.1) Case 1.1: $qb \leq rc$. Then $\lambda_8 = 1 \to (pa \to qb) - 1 = (1 \to 1) - 1 = 0$,
that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

(1.2) Case 1.2: $qb > rc$.

(1.2.1) Case 1.2.1: $pa \leq rc$. Then $\lambda_8 = 1 \to (pa \to rc) - 1 = (1 \to 1) - 1 = 0$,
that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

(1.2.2) Case 1.2.2: $pa > rc$. Then $\lambda_8 = 1 \to (rc \to rc) - 1 = (1 \to 1) - 1 = 0$, that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

(2) Case 2: $pa > qb$.

(2.1) Case 2.1: $qb > rc$. Then $\lambda_8 = \frac{qb}{pa} \to (rc \to rc) - 1 = (\frac{qb}{pa} \to 1) - 1 = 0$, that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

(2.2) Case 2.2: $qb \leq rc$.

(2.2.1) Case 2.2.1: $pa \leq rc$. Then $\lambda_8 = \frac{qb}{pa} \to (pa \to qb) - 1 = (\frac{qb}{pa} \to \frac{qb}{pa}) - 1 = 0$, that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

(2.2.2) Case 2.2.2: $pa > rc$. Then $\lambda_8 = \frac{qb}{pa} \to (rc \to qb) - 1 = (\frac{qb}{pa} \to \frac{qb}{rc}) - 1 = 0$, that is, $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

So to sum up $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.

Theorem 4.9. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) $\xi_D(pA \vee rC, qB \vee rC) \geq \xi_D(pA, qB)$.

(ii) $\xi_D(pA \wedge rC, qB \wedge rC) \geq \xi_D(pA, qB)$.

(iii) $\xi_D(pA \to rC, qB \to rC) \geq \xi_D(pA, qB)$.

Proof. Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Lemma 4.7, we get that $\models (pA \to qB) \to ((pA \vee rC) \to (qB \vee rC))$ and $\models (qB \to pA) \to ((qB \vee rC) \to (pA \vee rC))$. It follows from Lemma 4.6 that $\models ((pA \to qB) \wedge (qB \to pA)) \to (((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC)))$. Then by Theorem 3.3(iv), we have that $\tau_D((pA \to qB) \wedge (qB \to pA)) \leq \tau_D(((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC)))$. It follows from Definition 4.1 that

$$\begin{aligned} \xi_D(pA \vee rC, qB \vee rC) &= \tau_D(((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC))) \\ &\geq \tau_D((pA \to qB) \wedge (qB \to pA)) \\ &= \xi_D(pA, qB). \end{aligned}$$

(ii): By Lemma 4.8, we get that $\models (pA \to qB) \to ((pA \wedge rC) \to (qB \wedge rC))$ and $\models (qB \to pA) \to ((qB \wedge rC) \to (pA \wedge rC))$. It follows from Lemma 4.6 that $\models ((pA \to qB) \wedge (qB \to pA)) \to (((pA \wedge rC) \to (qB \wedge rC)) \wedge ((qB \wedge rC) \to (pA \wedge rC)))$. Then by Theorem 3.3(iv), we have that $\tau_D((pA \to qB) \wedge (qB \to pA)) \leq \tau_D(((pA \wedge rC) \to (qB \wedge rC)) \wedge ((qB \wedge rC) \to (pA \wedge rC)))$. It follows from Definition 4.1 that

$$\begin{aligned} \xi_D(pA \wedge rC, qB \wedge rC) &= \tau_D(((pA \wedge rC) \to (qB \wedge rC)) \wedge ((qB \wedge rC) \to (pA \wedge rC))) \\ &\geq \tau_D((pA \to qB) \wedge (qB \to pA)) \\ &= \xi_D(pA, qB). \end{aligned}$$

(iii): By Lemma 3.12, we get that $\models (pA \to qB) \to ((qB \to rC) \to (pA \to rC))$ and $\models (qB \to pA) \to ((pA \to rC) \to (qB \to rC))$. It follows from Lemma 4.6 that $\models ((pA \to qB) \wedge (qB \to pA)) \to (((qB \to rC) \to (pA \to rC)) \wedge ((pA \to rC) \to (qB \to rC)))$. Then by Theorem 3.3(iv), we have that $\tau_D((pA \to qB) \wedge (qB \to pA)) \leq \tau_D(((qB \to rC) \to (pA \to rC)) \wedge ((pA \to rC) \to (qB \to rC)))$. It follows from Definition 4.1 that

$$\begin{aligned} \xi_D(qB \to rC, pA \to rC) &= \tau_D(((qB \to rC) \to (pA \to rC)) \wedge ((pA \to rC) \to (qB \to rC))) \\ &\geq \tau_D((pA \to qB) \wedge (qB \to pA)) \\ &= \xi_D(pA, qB). \end{aligned}$$

Thus by Theorem 4.3(ii), we get that $\xi_D(pA \to rC, qB \to rC) = \xi_D(qB \to rC, pA \to rC) \geq \xi_D(pA, qB)$.

Corollary 4.10. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) $\xi_D(pA \vee qB, pA \vee rC) \geq \xi_D(qB, rC)$.

(ii) $\xi_D(pA \wedge qB, pA \wedge rC) \geq \xi_D(qB, rC)$.

(iii) $\xi_D(pA \to qB, pA \to rC) \geq \xi_D(qB, rC)$.

Theorem 4.11. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then $\xi_D(pA, qB) = \tau_D(pA \to qB) + \tau_D(qB \to pA) - 1 \geq \tau_D(pA) + \tau_D(qB) - 1$.*

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Definition 4.1 and Theorem 3.8, we have that

$$\begin{aligned}\xi_D(pA, qB) &= \tau_D((pA \to qB) \wedge (qB \to pA)) \\ &= \tau_D(pA \to qB) + \tau_D(qB \to pA) - \tau_D((pA \to qB) \vee (qB \to pA)) \\ &= \tau_D(pA \to qB) + \tau_D(qB \to pA) - 1.\end{aligned}$$

Since $\models pA \to (qB \to pA)$ and $\models qB \to (pA \to qB)$, it follows from Theorem 3.3(iv) that $\tau_D(pA) \leq \tau_D(qB \to pA)$ and $\tau_D(qB) \leq \tau_D(pA \to qB)$. Thus $\xi_D(pA, qB) = \tau_D(pA \to qB) + \tau_D(qB \to pA) - 1 \geq \tau_D(pA) + \tau_D(qB) - 1$.

Theorem 4.12. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then $\xi_D(pA, rC) \geq \xi_D(pA, qB) + \xi_D(qB, rC) - 1$.*

Proof. Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Theorem 4.11, we get that $\xi_D(pA, qB) + \xi_D(qB, rC) - 1 = [\tau_D(pA \to qB) + \tau_D(qB \to pA) - 1] + [\tau_D(qB \to rC) + \tau_D(rC \to qB) - 1] - 1 = [\tau_D(pA \to qB) + \tau_D(qB \to rC) - 1] + [\tau_D(rC \to qB) + \tau_D(qB \to pA) - 1] - 1$. It follows from Theorem 3.13 that $\xi_D(pA, qB) + \xi_D(qB, rC) - 1 \leq \tau_D(pA \to rC) + \tau_D(rC \to pA) - 1 = \xi_D(pA, rC)$.

Definition 4.13. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, stipulate that $\rho_D : F(S) \times F(S) \to [0,1]$. Define*

$$\rho_D(pA, qB) = 1 - \xi_D(pA, qB).$$

Then ρ_D is called the t absolute randomized pseudo-distance on $F(S)$, and $(F(S), \rho_D)$ is called the t absolute randomized logical metric space.

Remark 4.14. *Let A, B, C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then*

(i) By Definition 4.13 and Theorem 4.3(i), we have that $\rho_D(pA, pA) = 1 - \xi_D(pA, pA) = 0$.

(ii) It follows from Definition 4.13 and Theorem 4.3(ii) that $\rho_D(pA, qB) = \rho_D(qB, pA)$.

(iii) By Definition 4.13 and Theorem 4.12, we get that $\rho_D(pA, rC) = 1 - \xi_D(pA, rC) \leq 1 - [\xi_D(pA, qB) + \xi_D(qB, rC) - 1] = 1 - \xi_D(pA, qB) + 1 - \xi_D(qB, rC) = \rho_D(pA, qB) + \rho_D(qB, rC)$.

Thus $\rho_D(pA, qB)$ is the t absolute randomized pseudo-distance between propositional formulas A and B, that is, t absolute randomized truth degree can form three properties satisfying t absolute randomized pseudo-distance. Then it can form t absolute randomized logical metric space. So Definition 4.13 is reasonable.

Theorem 4.15. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then*

(i) $\rho_D(pA \vee qB, pA) = 1 - \tau_D(qB \to pA)$.

(ii) $\rho_D(pA \wedge qB, pA) = 1 - \tau_D(pA \to qB)$.

Proof. It is easy to prove Theorem 4.15 by Definition 4.13 and Theorem 4.3(iii) (iv).

Theorem 4.16. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$. If $\models pA$, then*

(i) $\rho_D(pA, qB) = 1 - \tau_D(qB)$.

(ii) $\rho_D(pA \vee qB, pA) = 0$.

(iii) $\rho_D(pA \wedge qB, pA) = 1 - \tau_D(qB)$.

Proof. It is easy to prove Theorem 4.16 by Definition 4.13 and Theorem 4.4.

Corollary 4.17. *Let $A, B \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. If $\models qB$, then*

(i) $\rho_D(pA, qB) = 1 - \tau_D(pA)$.

(ii) $\rho_D(pA \vee qB, pA) = 1 - \tau_D(pA)$.

(iii) $\rho_D(pA \wedge qB, pA) = 0$.

Theorem 4.18. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) $\rho_D(pA, qB) \geq \rho_D(pA \vee rC, qB \vee rC)$.

(ii) $\rho_D(pA, qB) \geq \rho_D(pA \wedge rC, qB \wedge rC)$.

(iii) $\rho_D(pA, qB) \geq \rho_D(pA \to rC, qB \to rC)$.

Proof. It is easy to prove Theorem 4.18 by Definition 4.13 and Theorem 4.9.

Corollary 4.19. *Let $A, B, C \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then*

(i) $\rho_D(qB, rC) \geq \rho_D(pA \vee qB, pA \vee rC)$.

(ii) $\rho_D(qB, rC) \geq \rho_D(pA \wedge qB, pA \wedge rC)$.

(iii) $\rho_D(qB, rC) \geq \rho_D(pA \to qB, pA \to rC)$.

Theorem 4.20. *Let $A, B, \in F(S)$, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$. Then $\rho_D(pA, qB) = 2 - \tau_D(pA \to qB) - \tau_D(qB \to pA) \leq 2 - \tau_D(pA) - \tau_D(qB)$.*

Proof. It is easy to prove Theorem 4.20 by Definition 4.13 and Theorem 4.11.

Corollary 4.21. *If the t absolute randomized truth degree of each formula is 1, then the t absolute randomized pseudo-distance between them is 0.*

Theorem 4.22. *Let $(F(S), \rho_D)$ be the t absolute randomized logical metric space, and ρ_D be the t absolute randomized pseudo-distance on $F(S)$. Then*

(i) The binary operator $\to$ is continuous with respect to the t absolute randomized pseudo- distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(ii) The binary operator $\vee$ is continuous with respect to the t absolute randomized pseudo- distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(iii) The binary operator $\wedge$ is continuous with respect to the t absolute randomized pseudo- distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(iv) The unitary operator Δ is not continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(v) The unitary operator $\sim$ is not continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

Proof. Let A, B, C, D, A_n, B_n contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Remark 4.14(iii), we have that $\rho_D(pA \to rC, qB \to zD) \leq \rho_D(pA \to rC, qB \to rC) + \rho_D(qB \to rC, qB \to zD)$. It follows from Theorem 4.18(iii) that $\rho_D(pA \to rC, qB \to rC) \leq \rho_D(pA, qB)$. By Corollary 4.19(iii), we get that $\rho_D(qB \to rC, qB \to zD) \leq \rho_D(rC, zD)$. Thus $\rho_D(pA \to rC, qB \to zD) \leq \rho_D(pA, qB) + \rho_D(rC, zD)$. If $\lim\limits_{n\to\infty} \rho_D(mA_n, pA) = 0$ and $\lim\limits_{n\to\infty} \rho_D(lB_n, qB) = 0$, then $\lim\limits_{n\to\infty} \rho_D(mA_n \to lB_n, pA \to qB) \leq \lim\limits_{n\to\infty} \rho_D(mA_n, pA) + \lim\limits_{n\to\infty} \rho_D(lB_n, qB) = 0$.

Therefore, the binary operator $\to$ is continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(ii): By Remark 4.14(iii), we get that $\rho_D(pA \vee rC, qB \vee zD) \leq \rho_D(pA \vee rC, qB \vee rC) + \rho_D(qB \vee rC, qB \vee zD)$. It follows from Theorem 4.18(i) that $\rho_D(pA \vee rC, qB \vee rC) \leq \rho_D(pA, qB)$. By Corollary 4.19(i), we have that $\rho_D(qB \vee rC, qB \vee zD) \leq \rho_D(rC, zD)$. Thus $\rho_D(pA \vee rC, qB \vee zD) \leq \rho_D(pA, qB) + \rho_D(rC, zD)$. If $\lim_{n\to\infty} \rho_D(mA_n, pA) = 0$ and $\lim_{n\to\infty} \rho_D(lB_n, qB) = 0$, then $\lim_{n\to\infty} \rho_D(mA_n \vee lB_n, pA \vee qB) \leq \lim_{n\to\infty} \rho_D(mA_n, pA) + \lim_{n\to\infty} \rho_D(lB_n, qB) = 0$.

Therefore, the binary operator $\vee$ is continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(iii): By Remark 4.14(iii), we get that $\rho_D(pA \wedge rC, qB \wedge zD) \leq \rho_D(pA \wedge rC, qB \wedge rC) + \rho_D(qB \wedge rC, qB \wedge zD)$. It follows from Theorem 4.18(ii) that $\rho_D(pA \wedge rC, qB \wedge rC) \leq \rho_D(pA, qB)$. By Corollary 4.19(ii), we have that $\rho_D(qB \wedge rC, qB \wedge zD) \leq \rho_D(rC, zD)$. Thus $\rho_D(pA \wedge rC, qB \wedge zD) \leq \rho_D(pA, qB) + \rho_D(rC, zD)$. If $\lim_{n\to\infty} \rho_D(mA_n, pA) = 0$ and $\lim_{n\to\infty} \rho_D(lB_n, qB) = 0$, then $\lim_{n\to\infty} \rho_D(mA_n \wedge lB_n, pA \wedge qB) \leq \lim_{n\to\infty} \rho_D(mA_n, pA) + \lim_{n\to\infty} \rho_D(lB_n, qB) = 0$.

Therefore, the binary operator $\wedge$ is continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(iv): $\forall a, b \in \Pi_{\sim,\Delta}$, when $pa \neq 1$, we have $pa \to qb \leq \Delta pa \to \Delta qb$, when $qb \neq 1$, we have $qb \to pa \leq \Delta qb \to \Delta pa$.

So when $pa \neq 1$ and $qb \neq 1$, we have $\models (pA \to qB) \to (\Delta pA \to \Delta qB)$ and $\models (qB \to pA) \to (\Delta qB \to \Delta pA)$. Then by Theorem 3.3(iv), we get that $\tau_D(pA \to qB) \leq \tau_D(\Delta pA \to \Delta qB)$ and $\tau_D(qB \to pA) \leq \tau_D(\Delta qB \to \Delta pA)$. It follows from Theorem 4.20 that $\rho_D(\Delta pA, \Delta qB) = 2 - \tau_D(\Delta pA \to \Delta qB) - \tau_D(\Delta qB \to \Delta pA) \leq 2 - \tau_D(pA \to qB) - \tau_D(qB \to pA) = \rho_D(pA, qB)$. If $\lim_{n\to\infty} \rho_D(mA_n, pA) = 0$, then $\lim_{n\to\infty} \rho_D(\Delta mA_n, \Delta pA) \leq \lim_{n\to\infty} \rho_D(mA_n, pA) = 0$.

Thus the unitary operator Δ is continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$ only when $pa \neq 1$ and $qb \neq 1$.

Therefore, the unitary operator Δ is not continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

(v): As $(\neg pA \to \neg qB) \approx (qB \to pA)$ and $(\neg qB \to \neg pA) \approx (pA \to qB)$, by Lemma 4.6, we get that $(\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA) \approx (qB \to pA) \wedge (pA \to qB)$. It follows from Theorem 3.3(iii) that $\tau_D((\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA)) = \tau_D((qB \to pA) \wedge (pA \to qB))$. Thus $\rho_D(\neg pA, \neg qB) = 1 - \xi_D(\neg pA, \neg qB) = 1 - \tau_D((\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA)) = 1 - \tau_D((qB \to pA) \wedge (pA \to qB)) = 1 - \xi_D(qB, pA) = \rho_D(qB, pA) = \rho_D(pA, qB)$. If $\lim_{n\to\infty} \rho_D(mA_n, pA) = 0$, then $\lim_{n\to\infty} \rho_D(\neg mA_n, \neg pA) = \lim_{n\to\infty} \rho_D(mA_n, pA) = 0$.

Therefore, the unitary operator $\neg$ is continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

Since $\Delta = \neg \sim$, it follows from Theorem 4.22(iv) that the unitary operator $\sim$ is not continuous with respect to the t absolute randomized pseudo-distance ρ_D in the t absolute randomized logical metric space $(F(S), \rho_D)$.

Remark 4.23. *The above 5 connectives are the most basic connectives in $\Pi_{\sim,\Delta}$, and other connectives can be transformed through these 5 connectives. Therefore, the continuity problem of other connectives will not be discussed in this paper.*

5. *t* absolute randomized divergence degree and *t* absolute randomized consistency degree of propositional formulas theory Γ

Definition 5.1. *Let $A, B \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0, 1)$, and Γ be the theory in $F(S)$. Define*

$$div_D(\Gamma) = \sup\{\rho_D(pA, qB) | A, B \in D(\Gamma)\}.$$

Then $div_D(\Gamma)$ is called the t absolute randomized divergence degree of theory Γ.

Example 5.2. *Calculate t absolute randomized divergence degree of theory $\Gamma = \{qB, \sim qB\}$.*

Answer. As $\vdash qB \to (\sim qB \to tA)$ is true for every $A \in F(S)$, we have $D(\Gamma) = F(S)$. Then $div_D(\Gamma) = 1$.

Definition 5.3. *Let $A, B \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and Γ be the theory in $F(S)$. Define*

$$i_D(\Gamma) = 1 - \min\{\lceil 1 - \rho_D(pA, qB) \rceil | A, B \in D(\Gamma)\}.$$

Then $i_D(\Gamma)$ is called the t absolute randomized polar index of theory Γ.

Remark 5.4. *(i) $\lceil 1 - \rho_D(pA, qB) \rceil = \{^{1, 0 \leq \rho_D(pA,qB)<1}_{0, \rho_D(pA,qB)=1}$; (ii) $i_D(\Gamma)$ can only take 0 and 1.*

Definition 5.5. *Let $A, B \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and Γ be the theory in $F(S)$. Define*

$$\eta_D(\Gamma) = 1 - \frac{1}{2} div_D(\Gamma)(1 + i_D(\Gamma)).$$

Then $\eta_D(\Gamma)$ is called the t absolute randomized consistency degree of theory Γ.

Theorem 5.6. *Let $A, B \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and Γ be the theory in $F(S)$. Then*

(i) Γ is consistent if and only if $i_D(\Gamma) = 0$.

(ii) Γ is inconsistent if and only if $i_D(\Gamma) = 1$.

Proof. Let A, B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): Γ is consistent if and only if $0 \leq \rho_D(pA, qB) < 1$, and $0 \leq \rho_D(pA, qB) < 1$ if and only if $i_D(\Gamma) = 0$.

(ii): Γ is inconsistent if and only if $\rho_D(pA, qB) = 1$, and $\rho_D(pA, qB) = 1$ if and only if $i_D(\Gamma) = 1$.

Theorem 5.7. *Let $A, B \in F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and Γ be the theory in $F(S)$. Then*

(i) Γ is completely consistent if and only if $\eta_D(\Gamma) = 1$.

(ii) Γ is consistent if and only if $\frac{1}{2} \leq \eta_D(\Gamma) \leq 1$.

(iii) Γ is consistent and fully divergent if and only if $\eta_D(\Gamma) = \frac{1}{2}$.

(iv) Γ is inconsistent if and only if $\eta_D(\Gamma) = 0$.

Proof. It is easy to prove Theorem 5.7 by Definition 5.3, 5.5 and Theorem 5.6.

6. Approximate reasoning in $(F(S), \rho_D)$

Definition 6.1. *Let $A \in F(S)$, $\Gamma \subset F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and $\varepsilon > 0$.*

(i) If $\rho_D(pA, D(\Gamma)) = \inf\{\rho_D(pA, qB) | qB \in D(\Gamma)\} < \varepsilon$, then A is called the conclusion that I-type absolute randomized errors less than ε of Γ, denote as $A \in D^1_{\varepsilon D}(\Gamma)$.

(ii) If $1 - \sup\{\tau_D(qB \to pA)) | qB \in D(\Gamma)\} < \varepsilon$, then A is called the conclusion that II-type absolute randomized errors less than ε of Γ, denote as $A \in D^2_{\varepsilon D}(\Gamma)$.

(iii) If $\inf\{H(D(\Gamma), D(\Sigma))\Sigma \subset F(S), \Sigma \vdash pA\} < \varepsilon$, then A is called the conclusion that III-type absolute randomized errors less than ε of Γ, denote as $A \in D^3_{\varepsilon D}(\Gamma)$. Here H is the Hausdorff distance.

Now we show the equivalences of these three approximate reasoning patterns.

Theorem 6.2. *Let $A \in F(S)$, $\Gamma \subset F(S)$, $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and $\varepsilon > 0$. If $A \in D^1_{\varepsilon D}(\Gamma)$, then $A \in D^2_{\varepsilon D}(\Gamma)$.*

Proof. Let A contain the same atomic formulas $p_1, p_2, \dots, p_m$. Since

$$\begin{aligned}\rho_D(pA, D(\Gamma)) &= \inf\{\rho_D(pA, qB) | qB \in D(\Gamma)\}\\ &= \inf\{\rho_D(qB, pA) | qB \in D(\Gamma)\}\\ &= \inf\{1 - \xi_D(qB, pA) | qB \in D(\Gamma)\}\\ &= 1 - \sup\{\xi_D(qB, pA) | qB \in D(\Gamma)\}\\ &= 1 - \sup\{\tau_D((qB \to pA) \wedge (pA \to qB)) | qB \in D(\Gamma)\}\\ &\geq 1 - \sup\{\tau_D(qB \to pA) | qB \in D(\Gamma)\}.\end{aligned}$$

When $A \in D^1_{\varepsilon D}(\Gamma)$, we get that $\rho_D(pA, D(\Gamma)) < \varepsilon$. Thus $1 - \sup\{\tau_D(qB \to pA) | qB \in D(\Gamma)\} < \varepsilon$, that is, $A \in D^2_{\varepsilon D}(\Gamma)$.

Theorem 6.3. *Let $A \in F(S)$, $\Gamma \subset F(S)$, $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and $\varepsilon > 0$. If $A \in D^2_{\varepsilon D}(\Gamma)$, then $A \in D^3_{\varepsilon D}(\Gamma)$.*

Proof. Let A contain the same atomic formulas $p_1, p_2, \dots, p_m$. When $A \in D^2_{\varepsilon D}(\Gamma)$, we get $1 - \sup\{\tau_D(qB \to pA) | qB \in D(\Gamma)\} < \varepsilon$. By $\vdash qB \to (qB \vee pA)$ and MP rule, we have that $(qB \vee pA) \in D(\Gamma)$. It follows from Theorem 4.15(i) that $\rho_D(pA, D(\Gamma)) \leq \rho_D(pA, qB \vee pA) = 1 - \tau_D(qB \to pA) < \varepsilon$.

Let $\Sigma_0 = \Gamma \cup \{pA\}$. Then $\Sigma_0 \subset F(S)$, $\Sigma_0 \vdash pA$ and $D(\Gamma) \subset D(\Sigma_0)$. The following is divided into two aspects to prove $H(D(\Gamma), D(\Sigma_0)) < \varepsilon$.

On the one hand, $\forall qB_0 \in D(\Gamma)$, we have $\rho_D(qB_0, D(\Sigma_0)) = 0$. Thus $H_0(D(\Gamma), D(\Sigma_0)) = \sup\{\rho_D(qB_0, D(\Sigma_0)) | qB_0 \in D(\Gamma)\} = 0 < \varepsilon$.

On the other hand, $\forall pA_0 \in D(\Sigma_0)$ and $qB_0 \in D(\Gamma)$, $\exists \{qB_1, qB_2, \dots, qB_l\}$ and $\{qB_1, qB_2, \dots, qB_y\} \subset \Gamma$ such that $\{qB_1, qB_2, \dots, qB_l\} \vdash qB_0$ and $\{qB_1, qB_2, \dots, qB_y, pA\} \vdash pA_0$. Thus $\vdash (qB_1 \wedge qB_2 \wedge \dots \wedge qB_l) \to qB_0$ and $\vdash (qB_1 \wedge qB_2 \wedge \dots \wedge qB_y \wedge pA) \to pA_0$.

Let $qB^* = qB_1 \wedge qB_2 \wedge \dots \wedge qB_y \wedge pA$ and $qB = qB^* \vee pA_0$. Then $\vdash qB^* \to qB_0$ and $\vdash qB^* \to pA_0$. Thus $\vdash qB^* \to pA_0 \wedge qB^*$ and $\vdash (qB^* \to (qB^* \wedge pA)) \to (qB^* \to (pA_0 \wedge qB^*))$. Then it follows from Theorem 3.3(iii), Theorem 4.15(i) and Theorem 3.24(i) that $\rho_D(pA_0, qB) \leq 1 - \tau_D(qB^* \to (qB^* \wedge pA)) = 1 - \tau_D(qB^* \to pA)$. Also since $\vdash qB^* \to qB_0$, using a similar method to the above one can get that $\tau_D(qB_0 \to pA) \leq \tau_D(qB^* \to pA)$. So $\rho_D(pA_0, qB) \leq 1 - \tau_D(qB_0 \to pA) \leq 1 - \tau_D((qB_0 \to pA) \wedge (pA \to qB_0)) = \rho_D(pA, qB_0)$.

Thus $H_0(D(\Sigma_0), D(\Gamma)) = \sup\{\rho_D(pA_0, D(\Gamma)) | pA_0 \in D(\Sigma_0)\} \leq \rho_D(pA, D(\Gamma)) < \varepsilon$, that is, $H(D(\Gamma), D(\Sigma_0)) < \varepsilon$.

Therefore, $\inf\{H(D(\Gamma), D(\Sigma)) | \Sigma \in F(S), \Sigma \vdash pA\} \leq H(D(\Gamma), D(\Sigma_0)) < \varepsilon$, that is, $A \in D^3_{\varepsilon D}(\Gamma)$.

Theorem 6.4. *Let $A \in F(S)$, $\Gamma \subset F(S)$, $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and $\varepsilon > 0$. If $A \in D^3_{\varepsilon D}(\Gamma)$, then $A \in D^1_{\varepsilon D}(\Gamma)$.*

Proof. Let A contain the same atomic formulas $p_1, p_2, \dots, p_m$. When $A \in D^3_{\varepsilon D}(\Gamma)$, we get that $\Sigma \in F(S)$, $\Sigma \vdash pA$ and $H(D(\Gamma), D(\Sigma)) < \varepsilon$. Thus $pA \in D(\Sigma)$. Therefore, $\rho_D(pA, D(\Gamma)) \leq H(D(\Gamma), D(\Sigma)) < \varepsilon$, that is, $A \in D^1_{\varepsilon D}(\Gamma)$.

7. Conclusions and further work

In this paper, using the randomization method of valuation set, we first put forward the definition of t absolute randomized truth degree of propositional formula in Goguen$_{\sim,\Delta}$ propositional logic system (t takes $\sim, \Delta$), and prove that some inference rules such as MP, HS, intersection inference, union inference of t absolute randomized truth degree. Then we give the concepts of t absolute randomized similarity degree and t absolute randomized pseudo-distance of propositional formulas. We also give the concepts of t absolute randomized divergence degree and absolute randomized consistency degree of propositional formulas theory Γ, and introduce three different types of approximate reasoning patterns, which are proved to be equivalent.

Based on the work in this paper, the following three problems deserve further research:

(i) What are the properties of absolute randomized truth degree in other multivalued propositional logic systems?

(ii) What are the properties of $\Gamma - t$ absolute randomized truth degree in Goguen$_{\sim,\Delta}$ propositional logic system?

(iii) What are the properties of Γ absolute randomized truth degree in other multivalued propositional logic systems?

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