Filomat 39:5 (2025), 1509–1522
https://doi.org/10.2298/FIL2505509R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Study of two dimensional α-modified Bernstein bi-variate operators

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Abstract. This research work is the study of two dimensional α-Modified Bernstein sequence of operators. Further, to investigate various approximation results in the space of two dimensional continuous functions in terms of these sequence of operators. Next, we estimate some lemmas in terms of test functions and central moments. In subsequent sections, uniform convergence and order of approximation are studied. Moreover, we prove local approximation results in view of Lipschitz maximal functions. In the last section, approximation results in Bögel-spaces are discussed with the help of mixed-modulus of continuity.

1. Introduction

Approximation theory indeed plays a vital role across various disciplines, providing a framework to represent a complex functions with simple ones. Its applications span from mathematics to engineering, including fields like computational science, data analysis, and computer graphics. In computational aspects, approximation theory aids in describing geometric spaces and solving differential equations. It forms the backbone of numerical analysis, where it helps in devising algorithms for solving mathematical problems numerically. Moreover, in applied mathematics, approximation theory contributes to areas like control theory, where control points and control nets are utilized to study of parametric curves and surfaces. These concepts are fundamental in designing control systems for various engineering applications ([1], [2]). In recent years, with the rise of artificial intelligence, data science and machine learning, approximation theory has found new applications. Techniques from approximation theory are employed in developing algorithms for data analysis, pattern recognition and predictive modeling. They form the basis for constructing models that can approximate complex relationships within data sets. Furthermore, in computer graphics and computer algebra systems, approximation theory is indispensable. It enables the represntation of curves and surfaces using simpler mathematical constructs, facilitating task like rendering realistic images and solving symbolic equations efficiently. Many scientists in medical sciences are also working in terms of these sequences of ([3], [4], [5]).

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A27, 41A35, 41A36.
Keywords. Rate of convergence; Order of approximation; Modulus of continuity; Peetre's K-functional; Bögel functions
Received: 15 August 2024; Revised: 18 November 2024; Accepted: 29 November 2024
Communicated by Snežana Č. Živković-Zlatanivić
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In 1912, Bernstein [6] was the first who presented a sequence of polynomials using probabilities tools, i.e., binomial distribution named as Bernstein polynomials or Bernstein operators:

$$B_s(g;u) = \sum_{\nu=0}^{s} p_{s,\nu}(u) g\left(\frac{\nu}{s}\right),\ s \in \mathbb{N}, \tag{1}$$

where $p_{s,\nu}(u) = \binom{s}{\nu} u^{\nu}(1-u)^{\nu-s}$. He found that $B_s(g;.) \rightrightarrows g$ for every bounded function $g \in C[0,1]$ where $\rightrightarrows$ denotes the convergence is uniform. In recent past, several researchers, *e.g.*, Mursaleen et al. ([7], [8]), Nasiruzamman et al. [9], Acar et al. ([10], [11]), Mohiuddine et al. ([13], [14], [15]), Acu et al. [16], Aslan et al. [12], İçöz et al. ([17]), [18]), Özger et al. ([19], [20]), Ayman et al. [21], Cai et al. ([22], [23]), Cicek et al. [24] and Rao et al. ([25], [26], [27]) introduced new variants in Bernstein operators given in (1) to investigate the better approximation order and rapidity of convergence in various functional spaces in view of different generating functions.

In the recent past, Usta [28] proposed a new generalisation of Bernstein operators as:

$$B_s(g;u) = \frac{1}{s}\sum_{\nu=0}^{s} (\nu - su)^2 u^{\nu-1}(1-u)^{s-\nu-1} g\left(\frac{s}{\nu}\right). \tag{2}$$

He proved some results for uniform convergence for the continuous functions and introduce Voronovskaja type result to estimate covergence speed for the first and second times differentiable functions. Further, to approximate in Lebesgue measurable space, Senapati et al. [29] constructed a Kantorovich version of the operators (2) as follows:

$$K_s^*(g,u) = \frac{s+1}{s}\sum_{\nu=0}^{s} (\nu - su)^2 u^{\nu-1}(1-u)^{s-\nu-1}\int_{\frac{\nu}{s+1}}^{\frac{\nu+1}{s+1}} g(t)dt. \tag{3}$$

Remark 1.1. *The sequence of operators presented in* (3) *are restricted for univariate functions only. Motivated with the above litrature, we present a bi-variate sequence of operators to discuss approximation results for the space of bivariate functions in Lebesgue sence in the subsequent sections.*

2. Construction of modified α- Bernstein bi-Variate Operators

Consider $\mathcal{I}^2 = \{(u_1,u_2) : 0 < u_1 < 1,\ 0 < u_2 < 1\}$ and $C\left(\mathcal{I}^2\right)$ is the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm

$$\|g\|_{C(\mathcal{I}^2)} = \sup_{(u_1,u_2)\in\mathcal{I}^2} |g(u_1,u_2)|.$$

Then, for all $g \in C\left(\mathcal{I}^2\right)$ and $s_1, s_2 \in \mathbb{N}$, we construct a new sequence of bi-variate α-modified Bernstein operators as follows:

$$M_{s_1,s_2}(g;u_1,u_2) = \sum_{\nu_1=0}^{s_1}\sum_{\nu_2=0}^{s_2} P_1(s_1,u_1)P_2(s_2,u_2)\int_{\frac{\nu_1}{s_1+1}}^{\frac{\nu_1+1}{s_1+1}}\int_{\frac{\nu_2}{s_2+1}}^{\frac{\nu_2+1}{s_2+1}} g(t_1,t_2)dt_1dt_2, \tag{4}$$

where

$$P_i(s_i,u_i) = \frac{s_{i+1}}{s_i}\binom{s_i}{\nu_i}(\nu_i - s_iu_i)^2 u_i^{\nu_i-1}(1-u_i)^{s_i-\nu_i-1}. \tag{5}$$

Lemma 2.1. *[29] Let $e_i(t) = t^i$, $i = 0,1,2$. Then, for the operators* (3), *we have*

$$K_s^*(e_0;u) = 1,$$

$$K_s^*(e_1;u) = \left(\frac{s-2}{s+1}\right)u + \frac{3}{2(s+1)},$$
$$K_s^*(e_2;u) = \left(\frac{s^2-7s+6}{(s+1)^2}\right)u^2 + \left(\frac{6s-8}{(s+1)^2}\right)u + \frac{7}{3(s+1)^2}.$$

Lemma 2.2. *Let $e_{i,j}(u_1,u_2) = u_1^i u_2^j$ be two dimensional test functions. Then, for the operator* (4)*, we have*

$$\begin{aligned}
M_{s_1,s_2}^*(e_{0,0};u_1,u_2) &= 1,\\
M_{s_1,s_2}^*(e_{1,0};u_1,u_2) &= \left(\frac{s_1-2}{s_1+1}\right)u_1 + \frac{3}{2(s_1+1)},\\
M_{s_1,s_2}^*(e_{0,1};u_1,u_2) &= \left(\frac{s_2-2}{s_2+1}\right)u_2 + \frac{3}{2(s_2+1)},\\
M_{s_1,s_2}^*(e_{2,0};u_1,u_2) &= \left(\frac{s_1^2-7s_1+6}{(s_1+1)^2}\right)u_1^2 + \left(\frac{6s_1-8}{(s_1+1)^2}\right)u_1 + \frac{7}{3(s_1+1)^2},\\
M_{s_1,s_2}^*(e_{0,2};u_1,u_2) &= \left(\frac{s_2^2-7s_2+6}{(s_2+1)^2}\right)u_2^2 + \left(\frac{6s_2-8}{(s_2+1)^2}\right)u_2 + \frac{7}{3(s_2+1)^2}.
\end{aligned}$$

Proof. In the light of Lemma (2.1) and linearity property, we have

$$\begin{aligned}
M_{s_1,s_2}^*(e_{0,0};u_1,u_2) &= M_{s_1,s_2}^*(e_0;u_1,u_2)M_{s_1,s_2}^*(e_0;u_1,u_2),\\
M_{s_1,s_2}^*(e_{1,0};u_1,u_2) &= M_{s_1,s_2}^*(e_1;u_1,u_2)M_{s_1,s_2}^*(e_0;u_1,u_2),\\
M_{s_1,s_2}^*(e_{0,1};u_1,u_2) &= M_{s_1,s_2}^*(e_0;u_1,u_2)M_{s_1,s_2}^*(e_1;u_1,u_2),\\
M_{s_1,s_2}^*(e_{1,1};u_1,u_2) &= M_{s_1,s_2}^*(e_1;u_1,u_2)M_{s_1,s_2}^*(e_1;u_1,u_2),\\
M_{s_1,s_2}^*(e_{2,0};u_1,u_2) &= M_{s_1,s_2}^*(e_2;u_1,u_2)M_{s_1,s_2}^*(e_0;u_1,u_2),\\
M_{s_1,s_2}^*(e_{0,2};u_1,u_2) &= M_{s_1,s_2}^*(e_0;u_1,u_2)M_{s_1,s_2}^*(e_2;u_1,u_2),
\end{aligned}$$

which proves Lemma (2.2). □

Lemma 2.3. *[29] The central moments for the operators* (3) *are evaluated as:*

$$\begin{aligned}
K_s^*((t-u)^0;u) &= 1,\\
K_s^*((t-u)^1;u) &= \frac{3}{s+1}\left(\frac{1}{2}-u\right),\\
K_s^*((t-u)^2;u) &= \left(\frac{11-3s}{(s+1)^2}\right)u^2 + \left(\frac{3s-11}{(s+1)^2}\right)u + \frac{7}{3(s+1)^2}.
\end{aligned}$$

Lemma 2.4. *Let $\Psi_{u_1,u_2}^{i,j}(t_1,t_2) = \eta_{i,j}(t_1,t_2) = (t_1-u_1)^i(t_2-u_2)^j, (i,j) \in \{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0)\}$ be the central moments functions. Then, for the operators $M_{s_1,s_2}^*(.;.,.)$ introduced by* (4) *satisfies the following identities, for*

$$\begin{aligned}
M_{s_1,s_2}^*(\eta_{0,0};u_1,u_2) &= 1,\\
M_{s_1,s_2}^*(\eta_{1,0};u_1,u_2) &= \frac{3}{s_1+1}\left(\frac{1}{2}-u_1\right),\\
M_{s_1,s_2}^*(\eta_{0,1};u_1,u_2) &= \frac{3}{s_2+1}\left(\frac{1}{2}-u_2\right),\\
M_{s_1,s_2}^*(\eta_{2,0};u_1,u_2) &= \left\{\frac{11-3s_1}{(s_1+1)^2}\right\}u_1^2 + \left\{\frac{3s_1-11}{(s_1+1)^2}\right\} + \frac{7}{3(s_1+1)^2},\\
M_{s_1,s_2}^*(\eta_{0,2};u_1,u_2) &= \left\{\frac{11-3s_2}{(s_2+1)^2}\right\}u_2^2 + \left\{\frac{3s_2-11}{(s_2+1)^2}\right\} + \frac{7}{3(s_2+1)^2}.
\end{aligned}$$

Proof. In the light of Lemma (2.4) and linearity property, we have

$$\begin{aligned}
M^*_{s_1,s_2}(\eta_{0,0}; u_1, u_2) &= M^*_{s_1,s_2}(\eta_0; u_1, u_2) M^*_{s_1,s_2}(\eta_0; u_1, u_2),\\
M^*_{s_1,s_2}(\eta_{1,0}; u_1, u_2) &= M^*_{s_1,s_2}(\eta_1; u_1, u_2) M^*_{s_1,s_2}(\eta_0; u_1, u_2),\\
M^*_{s_1,s_2}(\eta_{0,1}; u_1, u_2) &= M^*_{s_1,s_2}(\eta_0; u_1, u_2) M^*_{s_1,s_2}(\eta_1; u_1, u_2),\\
M^*_{s_1,s_2}(\eta_{2,0}; u_1, u_2) &= M^*_{s_1,s_2}(\eta_2; u_1, u_2) M^*_{s_1,s_2}(\eta_0; u_1, u_2),\\
M^*_{s_1,s_2}(\eta_{0,2}; u_1, u_2) &= M^*_{s_1,s_2}(\eta_0; u_1, u_2) M^*_{s_1,s_2}(\eta_2; u_1, u_2),
\end{aligned}$$

which proves Lemma (2.4). □

Lemma 2.5. *For all $(u_1, u_2) \in \mathcal{I}^2$ and sufficiently large $s_1, s_2 \in \mathbb{N}$ the operators $M^*_{s_1,s_2}(.\,;\,.)$ given by* (4) *satisfy*

$$\begin{aligned}
(1)\quad M^*_{s_1,s_2}(\Psi^{2,0}_{u_1,u_2}; u_1, u_2) &= o\left(\frac{1}{s_1}\right)(u_1+1)^2 \le C_1(u_1+1)^2 \text{ as } s_1, s_2 \to \infty,\\
(2)\quad M^*_{s_1,s_2}(\Psi^{0,2}_{u_1,u_2}; u_1, u_2) &= o\left(\frac{1}{s_2}\right)(u_2+1)^2 \le C_2(u_2+1)^2 \text{ as } s_1, s_2 \to \infty,\\
(3)\quad M^*_{s_1,s_2}(\Psi^{4,0}_{u_1,u_2}; u_1, u_2) &= o\left(\frac{1}{s_1^2}\right)(u_1+1)^4 \le C_3(u_1+1)^4 \text{ as } s_1, s_2 \to \infty,\\
(4)\quad M^*_{s_1,s_2}(\Psi^{0,4}_{u_1,u_2}; u_1, u_2) &= o\left(\frac{1}{s_2^2}\right)(u_2+1)^4 \le C_4(u_2+1)^4 \text{ as } s_1, s_2 \to \infty.
\end{aligned}$$

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\,(g; \delta_{s_1}, \delta_{s_2}) = \sup\{\mid g(t,s) - g(u_1,u_2) \mid : (t,s), (u_1,u_2) \in \mathcal{I}^2\},$$

with $\mid t - u_1 \mid \le \delta_{s_1}$, $\mid s - u_2 \mid \le \delta_{s_2}$ with the partial modulus of continuity defined as:

$$\omega_1\,(g; \delta_{s_1}) = \sup_{0<u_2<1}\ \sup_{|x_1-x_2|\le\delta_{s_1}} \{\mid g(x_1,u_2) - g(x_2,u_2) \mid\},$$

$$\omega_2\,(g; \delta_{s_2}) = \sup_{0<u_1<1}\ \sup_{|x_1-x_2|\le\delta_{s_2}} \{\mid g(u_1,x_1) - g(u_1,x_2) \mid\}.$$

Theorem 2.6. *For any $g \in C(\mathcal{I}^2)$ we have*

$$\mid M^*_{s_1,s_2}(g; u_1, u_2) - g(u_1, u_2) \mid \le 2\Big(\omega_1(g; \delta_{s_1}) + \omega_2(g; \delta_{s_2})\Big).$$

Proof. In order to give the proof of Theorem 2.6, in general, we use well-known Cauchy-Schwarz inequality. Thus, we see that

$$\begin{aligned}
\mid M^*_{s_1,s_2}(g; u_1, u_2) &- g(u_1, u_2) \mid\\
&\le M^*_{s_1,s_2}\,(\mid g(t,s) - g(u_1,u_2) \mid; u_1, u_2)\\
&\le M^*_{s_1,s_2}\,(\mid g(t,s) - g(u_1,s) \mid; u_1, u_2)\\
&+ M^*_{s_1,s_2}\,(\mid g(u_1,s) - g(u_1,u_2) \mid; u_1, u_2)\\
&\le M^*_{s_1,s_2}\,(\omega_1(g; \mid t-u_1 \mid); u_1, u_2) + M^*_{s_1,s_2}\,(\omega_2(g; \mid s-u_2 \mid); u_1, u_2)\\
&\le \omega_1(g; \delta_{s_1})\left(1 + \delta_{s_1}^{-1} M^*_{s_1,s_2}(\mid t-u_1 \mid; u_1, u_2)\right)\\
&+ \omega_2(g; \delta_{s_2})\left(1 + \delta_{s_2}^{-1} M^*_{s_1,s_2}(\mid s-u_2 \mid; u_1, u_2)\right)\\
&\le \omega_1(g; \delta_{s_1})\left(1 + \frac{1}{\delta_{s_1}}\sqrt{M^*_{s_1,s_2}((t-u_1)^2; u_1, u_2)}\right)
\end{aligned}$$

$$+ \quad \omega_2(g;\delta_{s_2})\left(1+\frac{1}{\delta_{s_2}}\sqrt{M^*_{s_1,s_2}((s-u_2)^2;u_1,u_2)}\right).$$

If we choose $\delta^2_{s_1} = M^*_{s_1,s_2}((t-u_1)^2;u_1,u_2)$ and $\delta^2_{s_2} = M^*_{s_1,s_2}((s-u_2)^2;u_1,u_2)$, then we easily to reach our desired results. □

Here, we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1,\rho_2 \in [0,\infty)$, Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ defined by

$$\begin{aligned}\mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{g : \sup(1+t)^{\rho_1}(1+s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)\right)\\ &\le M\frac{1}{(1+u_1)^{\rho_1}}\frac{1}{(1+u_2)^{\rho_2}}\Big\},\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)=\left\{\frac{|\, g(t,s)-g(u_1,u_2)\,|}{|\, t-u_1\,|^{\rho_1}|\, s-u_2\,|^{\rho_2}};\quad (t,s),(u_1,u_2)\in\mathcal{I}^2\right\}. \tag{6}$$

Theorem 2.7. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E\times E)$. Then, for any $\rho_1,\rho_2 \in [0,\infty)$, there exists $M > 0$ such that*

$$\begin{aligned}|\, M^*_{s_1,s_2}(g;u_1,u_2) &- g(u_1,u_2)\,|\\ &\le M\Big\{\Big((d(u_1,E))^{\rho_1}+\left(\delta^2_{s_1}\right)^{\frac{\rho_1}{2}}\Big)\Big((d(u_2,E))^{\rho_2}+\left(\delta^2_{s_2}\right)^{\frac{\rho_2}{2}}\Big)\\ &+ (d(u_1,E))^{\rho_1}(d(u_2,E))^{\rho_2}\Big\},\end{aligned}$$

where δ_{s_1} and δ_{s_2} defined by Theorem 2.6.

Proof. Take $|\, u_1-x_0\,|= d(u_1,E)$ and $|\, y_2-y_0\,|= d(u_2,E)$. For any $(u_1,u_2)\in\mathcal{I}^2$, and $(x_0,y_0)\in E\times E$ we let $d(u_1,E)=\inf\{|\, u_1-u_2\,|: u_2\in E\}$. Thus, we can write here

$$|\, g(t,s)-g(u_1,u_2)\,|\le M\,|\, g(t,s)-g(x_0,y_0)\,|+|\, g(x_0,y_0)-g(u_1,u_2)\,|. \tag{7}$$

Apply $M^*_{s_1,s_2}(.;.,.)$, we obtain

$$\begin{aligned}|\, M^*_{s_1,s_2}(g;u_1,u_2) &- g(u_1,u_2)\,|\\ &\le M^*_{s_1,s_2}\left(|\, g(u_1,u_2)-g(x_0,y_0)\,|+|\, g(x_0,y_0)-g(u_1,u_2)\,|\right)\\ &\le MM^*_{s_1,s_2}\left(|\, t-x_0\,|^{\rho_1}|\, s-y_0\,|^{\rho_2};u_1,u_2\right)+M\,|\, u_1-x_0\,|^{\rho_1}|\, u_2-y_0\,|^{\rho_2}.\end{aligned}$$

For all $A,B\ge 0$ and $\rho\in[0,\infty)$ we know inequality $(A+B)^\rho \le A^\rho + B^\rho$, thus

$$|\, t-x_0\,|^{\rho_1}\le|\, t-u_1\,|^{\rho_1}+|\, u_1-x_0\,|^{\rho_1},$$

$$|\, s-y_0\,|^{\rho_1}\le|\, s-y_2\,|^{\rho_2}+|\, u_2-y_0\,|^{\rho_2}.$$

Therefore

$$\begin{aligned}|\, M^*_{s_1,s_2}(g;u_1,u_2)-g(u_1,u_2)\,| &\le MM^*_{s_1,s_2}\left(|\, t-u_1\,|^{\rho_1}|\, s-y_2\,|^{\rho_2};u_1,u_2\right)\\ &+ M\,|\, u_1-x_0\,|^{\rho_1}M^*_{s_1,s_2}\left(|\, s-u_2\,|^{\rho_2};u_1,u_2\right)\\ &+ M\,|\, u_2-y_0\,|^{\rho_2}M^*_{s_1,s_2}\left(|\, t-u_1\,|^{\rho_1};u_1,u_2\right)\\ &+ 2M\,|\, u_1-x_0\,|^{\rho_1}|\, y_2-y_0\,|^{\rho_2}M^*_{s_1,s_2}\left(\nu_{0,0};u_1,u_2\right).\end{aligned}$$

On apply Hölder inequality on $M^*_{s_1,s_2}$, we get

$$M^*_{s_1,s_2}\left(|\, t-u_1\,|^{\rho_1}|\, s-u_2\,|^{\rho_2};u_1,u_2\right)$$

$$
\begin{aligned}
&= \mathcal{U}^{\alpha_1}_{n_1,k}(| t-u_1 |^{\rho_1};u_1,u_2)\,\mathcal{V}^{\alpha_2}_{n_2,l}(| s-u_2 |^{\rho_2};u_1,u_2)\\
&\leq \left(M^*_{s_1,s_2}(| t-u_1 |^2;u_1,u_2)\right)^{\frac{\rho_1}{2}}\left(M^*_{s_1,s_2}(\nu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_1}{2}}\\
&\times \left(M^*_{s_1,s_2}(| s-u_2 |^2;u_1,u_2)\right)^{\frac{\rho_2}{2}}\left(M^*_{s_1,s_2}(\nu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_2}{2}}.
\end{aligned}
$$

Thus, we can obtain

$$
\begin{aligned}
| M^*_{s_1,s_2}(g;u_1,u_2) &- g(u_1,u_2) |\\
&\leq M\left(\delta^2_{s_1}\right)^{\frac{\rho_1}{2}}\left(\delta^2_{s_2}\right)^{\frac{\rho_2}{2}} + 2M\,(d(u_1,E))^{\rho_1}\,(d(u_2,E))^{\rho_2}\\
&+ M\,(d(u_1,E))^{\rho_1}\left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}} + L\,(d(u_2,E))^{\rho_2}\left(\delta^2_{s_1}\right)^{\frac{\rho_1}{2}}.
\end{aligned}
$$

We have complete the proof. □

Theorem 2.8. *Let $g \in C'(\mathcal{I}^2)$. Then, for all $(u_1,u_2)\in\mathcal{I}^2$, operator $M^*_{s_1,s_2}(.;.,.)$ satisfying*

$$
| M^*_{s_1,s_2}(g;u_1,u_2) - g(u_1,u_2) | \leq \| g_{u_1} \|_{C(\mathcal{I}^2)}\left(\delta^2_{s_1}\right)^{\frac{1}{2}} + \| g_{u_2} \|_{C(\mathcal{I}^2)}\left(\delta^2_{s_2}\right)^{\frac{1}{2}},
$$

where δ_{s_1} and δ_{s_2} are defined by Theorem 2.6.

Proof. Let $g \in C'(\mathcal{I}^2)$, and for any fixed $(u_1,u_2)\in\mathcal{I}^2$, we have

$$
g(t,s) - g(u_1,u_2) = \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho + \int_{y_2}^{s} g_\nu(u_1,\nu)\mathrm{d}\nu.
$$

On apply $M^*_{s_1,s_2}(.;.,.)$

$$
M^*_{s_1,s_2}(g(t,s);u_1,u_2) - g(u_1,u_2) = M^*_{s_1,s_2}\left(\int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho;u_1,u_2\right) + M^*_{s_1,s_2}\left(\int_{u_2}^{s} g_\nu(u_1,\nu)\mathrm{d}\nu;u_1,u_2\right).
$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$
| \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho | \leq \int_{u_1}^{t} | g_\varrho(\varrho,s)\mathrm{d}\varrho | \leq \| g_{u_1} \|_{C(\mathcal{I}^2)} | t-u_1 | \tag{8}
$$

$$
| \int_{y_2}^{s} g_\nu(u_1,\nu)\mathrm{d}\nu | \leq \int_{y_2}^{s} | g_\nu(u_1,\nu)\mathrm{d}\nu | \leq \| g_{y_2} \|_{C(\mathcal{I}^2)} | s-y_2 |. \tag{9}
$$

In the view of (8), (8) and (9) we can obtain

$$
\begin{aligned}
| M^*_{s_1,s_2}(g(u_1,u_2);u_1,u_2) &- g(u_1,u_2) |\\
&\leq M^*_{s_1,s_2}\left(\left| \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho\right|;u_1,u_2\right)\\
&+ M^*_{s_1,s_2}\left(\left| \int_{y_2}^{s} g_\nu(u_1,\nu)\mathrm{d}\nu\right|;u_1,u_2\right)\\
&\leq \| g_{u_1} \|_{C(\mathcal{I}^2)} M^*_{s_1,s_2}(| t-u_1 |;u_1,u_2)\\
&+ \| g_{u_2} \|_{C(\mathcal{I}^2)} M^*_{s_1,s_2}(| s-u_2 |;u_1,u_2)\\
&\leq \| g_{u_1} \|_{C(\mathcal{I}^2)}\left(M^*_{s_1,s_2}((t-u_1)^2;u_1,u_2)M^*_{s_1,s_2}(1;u_1,u_2)\right)^{\frac{1}{2}}\\
&+ \| g_{u_2} \|_{C(\mathcal{I}^2)}\left(M^*_{s_1,s_2}((s-u_2)^2;u_1,u_2)M^*_{s_1,s_2}(1;u_1,u_2)\right)^{\frac{1}{2}}\\
&= \| g_{u_1} \|_{C(\mathcal{I}^2)}\left(\delta^2_{s_1}\right)^{\frac{1}{2}} + \| g_{u_2} \|_{C(\mathcal{I}^2)}\left(\delta^2_{s_2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Theorem 2.9. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$R^{\alpha_1,\alpha_2}_{s_1,s_2}(f;u_1,u_2) = M^*_{s_1,s_2}(g;u_1,u_2) + f(u_1,u_2) - f\left(\mathcal{U}^{\alpha_1}_{n_1,k}(\nu_{1,0};u_1,u_2), \mathcal{V}^{\alpha_2}_{n_2,l}(\nu_{0,1};u_1,u_2)\right),$$

where, from Lemma (2.2),

$$\mathcal{U}^{\alpha_1}_{n_1,k}(\nu_{1,0};u_1,u_2) = \left(\frac{s_1-2}{s_1+1}\right)u_1 + \frac{3}{2(s_1+1)} \quad \text{and} \quad \mathcal{V}^{\alpha_2}_{n_2,l}(\nu_{0,1};u_1,u_2) = \left(\frac{s_2-2}{s_2+1}\right)u_2 + \frac{3}{2(s_2+1)}$$

Then, for all $g \in C'(\mathcal{I}^2)$, $R^{\alpha_1,\alpha_2}_{s_1,s_2}(.;.,.)$ satisfying

$$\begin{aligned} R^{\alpha_1,\alpha_2}_{s_1,s_2}(g;t,s) - g(u_1,u_2) \leq & \left\{\delta^2_{s_1} + \delta^2_{s_2} + \left(\left(\frac{s_1-2}{s_1+1}\right)u_1 + \frac{3}{2(s_1+1)} - u_1\right)^2 \right. \\ & + \left. \left(\left(\frac{s_2-2}{s_2+1}\right)u_2 + \frac{3}{2(s_2+1)} - u_2\right)^2\right\} \| g \|_{C^2(\mathcal{I}^2)} . \end{aligned}$$

Proof. In the light of operator $R^{\alpha_1,\alpha_2}_{s_1,s_2}(.;.,.)$ and Lemma (2.2), we obtain

$$R^{\alpha_1,\alpha_2}_{s_1,s_2}(1;u_1,u_2) = 1, \quad R^{\alpha_1,\alpha_2}_{s_1,s_2}(t-u_1;u_1,u_2) = 0 \quad \text{and} \quad R^{\alpha_1,\alpha_2}_{s_1,s_2}(s-y_2;u_1,u_2) = 0.$$

For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned} g(t,s) - g(u_1,u_2) = & \frac{\partial g(u_1,u_2)}{\partial u_1}(t-u_1) + \int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda \\ & + \frac{\partial g(u_1,u_2)}{\partial u_2}(s-u_2) + \int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi. \end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{s_1,s_2}$, we see that

$$\begin{aligned} & R^{\alpha_1,\alpha_2}_{s_1,s_2}(g(t,s);u_1,u_2) - R^{\alpha_1,\alpha_2}_{s_1,s_2}(g(u_1,u_2) \\ & = R^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial \lambda^2}\mathrm{d}\lambda;u_1,u_2\right) + R^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi;u_1,u_2\right) \\ & = M^*_{s_1,s_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial \lambda^2}\mathrm{d}\lambda;u_1,u_2\right) + M^*_{s_1,s_2}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi;u_1,u_2\right) \\ & - \int_{u_1}^{\left(\frac{s_1-2}{s_1+1}\right)u_1+\frac{3}{2(s_1+1)}}\left(\left(\frac{s_1-2}{s_1+1}\right)u_1 + \frac{3}{2(s_1+1)} - \lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda \\ & - \int_{u_2}^{\left(\frac{s_2-2}{s_2+1}\right)u_2+\frac{3}{2(s_2+1)}}\left(\left(\frac{s_2-2}{s_2+1}\right)u_2 + \frac{3}{2(s_2+1)} - \psi\right)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi. \end{aligned}$$

From hypothesis, we easily obtain

$$\left|\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right| \leq \int_{u_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right| \leq \| g \|_{C^2(\mathcal{I}^2)} (t-u_1)^2,$$

$$\left|\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi\right| \leq \int_{y_2}^{s}\left|(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi\right| \leq \| g \|_{C^2(\mathcal{I}^2)} (s-u_2)^2,$$

$$\left|\int_{u_1}^{\left(\frac{s_1-2}{s_1+1}\right)u_1+\frac{3}{2(s_1+1)}}\left(\left(\frac{s_1-2}{s_1+1}\right)u_1 + \frac{3}{2(s_1+1)} - \lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right|$$

$$\leq \ \| g \|_{C^2(\mathcal{I}^2)} \left(\left(\frac{s_2-2}{s_2+1} \right) u_2 + \frac{3}{2(s_2+1)} - u_1 \right)^2$$

$$\Big| \int_{y_2}^{\left(\frac{s_2-2}{s_2+1}\right)u_2+\frac{3}{2(s_2+1)}} \left(\left(\frac{s_2-2}{s_2+1} \right) u_2 + \frac{3}{2(s_2+1)} - \psi \right) \frac{\partial^2 g(u_1,\psi)}{\partial \psi^2} \mathrm{d}\psi \Big|$$

$$\leq \ \| g \|_{C^2(\mathcal{I}^2)} \left(\left(\frac{s_2-2}{s_2+1} \right) u_2 + \frac{3}{2(s_2+1)} - y_2 \right)^2.$$

Thus,

$$\begin{aligned} | R^{\alpha_1,\alpha_2}_{s_1,s_2}(g;t,s) - g(u_1,u_2) | \ \leq \ & \Big\{ M^*_{s_1,s_2}((t-u_1)^2;u_1,u_2) + M^*_{s_1,s_2}((s-y_2)^2;u_1,u_2) \\ & + \left(\left(\frac{s_1-2}{s_1+1} \right) u_1 + \frac{3}{2(s_1+1)} - u_1 \right)^2 \\ & + \left(\left(\frac{s_2-2}{s_2+1} \right) u_2 + \frac{3}{2(s_2+1)} - y_2 \right)^2 \Big\} \| g \|_{C^2(\mathcal{I}^2)} . \end{aligned}$$

We complete the proof of desired Theorem 2.9. □

3. Some approximation results in Bögel space

Take any function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ for a real compact intervals of $\mathcal{I}_1 \times \mathcal{I}_2$. For all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ suppose $\Delta^*_{(t,s)} g(u_1,u_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)} g(u_1,u_2) = g(t,s) - g(t,y_2) - g(u_1,s) + g(u_1,u_2).$$

If at any point $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ the function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ defined on $\mathcal{I}_1 \times \mathcal{I}_2$, then $\lim_{(t,s)\to(u_1,u_2)} \Delta^*_{(t,s)} g(u_1,u_2) = 0$.

If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1 \times \mathcal{I}_2)$ on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(u_1,u_2),\ t\neq u_1,\ s\neq y_2} \frac{1}{(t-u_1)(s-y_2)} \left(\Delta^*_{(t,s)} \right) = D_B g(u_1,u_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(u_1,u_2)$ and be $D_\varphi(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is g, B−differentiable on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$, then for all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ there exists positive constant M such that $| \Delta^*_{(t,s)} g(u_1,u_2) | \leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1 \times \mathcal{I}_2$, whene $\mathcal{I}_1 \times \mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1 \times \mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1 \times \mathcal{I}_2$ which equipped the norm on B as $\| g \|_B = \sup_{(t,s),(u_1,u_2)\in\mathcal{I}_1\times\mathcal{I}_2} | \Delta^*_{(t,s)} g(u_1,u_2) |$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2) = \sup\{\Delta^*_{(t,s)} g(u_1,u_2) :| t-u_1 | \leq \delta_1,\ | s-y_2 | \leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0,\infty) \times [0,\infty)$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [30, 31].

Let $(u_1, u_2) \in \mathcal{I}^2$ and $s_1, s_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $M^*_{s_1,s_2}(.;.,.)$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(t,s);u_1,u_2) = M^*_{s_1,s_2}\Big(g(t,u_2) + g(u_1,s) - g(t,s);u_1,u_2\Big). \tag{10}$$

Theorem 3.1. *For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that*

$$M^*_{s_1,s_2}(h;u_1,u_2) = \sum_{\nu_1=0}^{s_1}\sum_{\nu_2=0}^{s_2} P^{*,\alpha}_1(s_1,u_1)P^{*,\alpha}_2(s_2,u_2)\int_{\frac{\nu_1}{s_1+1}}^{\frac{\nu_1+1}{s_1+1}}\int_{\frac{\nu_2}{s_2+1}}^{\frac{\nu_2+1}{s_2+1}} g(t_1,t_2)dt_i, \tag{11}$$

where

$$P^{*,\alpha}_i(s_i,u_i) = \frac{s_{i+1}}{s_i}\binom{s_i}{\nu_i}(\nu_i - s_iu_i)^2u_i^{\nu_i-1}(1-\nu_i)^{s_i}. \tag{12}$$

$$| K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(t,s);u_1,u_2) - g(u_1,u_2) | \leq 4\omega_B(g;\delta_{s_1},\delta_{s_2}),$$

where δ_{s_1} and δ_{s_2} are defined by Theorem 2.6.

Proof. Let $(t,s),(u_1,u_2) \in \mathcal{I}^2$. For all $s_1, s_2 \in \mathbb{N}$ and $\delta_{s_1}, \delta_{s_2} > 0$, it follows that

$$\begin{aligned} | \Delta^*_{(u_1,u_2)}g(t,s) | &\leq \omega_B(g;| t - s_1 | | s - u_2 |) \\ &\leq \left(1 + \frac{t-u_1}{\delta_{s_1}}\right)\left(1 + \frac{s-u_2}{\delta_{s_2}}\right)\omega_B(g;\delta_{s_1},\delta_{s_2}). \end{aligned}$$

From (10) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned} | K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(t,s);u_1,u_2) &- g(u_1,u_2) | \\ &\leq M^*_{s_1,s_2}\left(| \Delta^*_{(u_1,u_2)}g(t,s) |;u_1,u_2\right) \\ &\leq \Big(M^*_{s_1,s_2}(\phi_{0,0};u_1,u_2) + \frac{1}{\delta_{s_1}}\left(M^*_{s_1,s_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}} \\ &+ \frac{1}{\delta_{s_2}}\left(M^*_{s_1,s_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}} \\ &+ \frac{1}{\delta_{s_1}}\left(M^*_{s_1,s_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}} \\ &\times \frac{1}{\delta_{s_2}}\left(M^*_{s_1,s_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}}\Big)\omega_B(g;\delta_{n_1},\delta_{n_2}). \end{aligned}$$

In the view of Theorem 2.6 we easily get our results. □

If we let $x = (t,s),\ y = (u_1,u_2) \in \mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip^{\xi}_M = \left\{g \in C(\mathcal{I}^2) : | \Delta^*_{(u_1,u_2)}g(x,y) | \leq M \parallel x - y \parallel^{\xi},\right\}$$

where M is a positive constant, $0 < \xi \leq 1$, and Euclidean norm $\parallel x - y \parallel = \sqrt{(t-u_1)^2 + (s-u_2)^2}$.

Theorem 3.2. *For all $g \in Lip^{\xi}_M$ operator $K^{\alpha_1,\alpha_2}_{s_1,s_2}(.;.,.)$ satisfying*

$$| K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(x,y);u_1,u_2) - g(u_1,u_2) | \leq M\{\delta^2_{s_1} + \delta^2_{s_2}\}^{\frac{\xi}{2}},$$

where δ_{s_1} and δ_{s_2} are defined by Theorem 2.6.

Proof. We easily see that

$$\begin{aligned} K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(x,y);u_1,u_2) &= M^*_{s_1,s_2}(g(u_1,y)+g(x,u_2)-g(x,s);u_1,u_2)\\ &= M^*_{s_1,s_2}\left(g(u_1,u_2)-\Delta^*_{(u_1,u_2)}g(x,s);u_1,u_2\right)\\ &= g(u_1,u_2)-M^*_{s_1,s_2}\left(\Delta^*_{(u_1,u_2)}g(x,s);u_1,u_2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{s_1,s_2}(g(x,y);u_1,u_2) &- g(u_1,u_2)\mid\\ &\leq M^*_{s_1,s_2}\left(\mid \Delta^*_{(u_1,u_2)}g(x,y)\mid ;u_1,u_2\right)\\ &\leq MM^*_{s_1,s_2}\left(\| x-y\|^{\xi};u_1,u_2\right)\\ &\leq M\{M^*_{s_1,s_2}\left(\| x-y\|^{2};u_1,u_2\right)\}^{\frac{\xi}{2}}\\ &\leq M\{M^*_{s_1,s_2}\left((t-u_1)^2;u_1,u_2\right)+M^*_{s_1,s_2}\left((s-u_2)^2;u_1,u_2\right)\}^{\frac{\xi}{2}}. \end{aligned}$$

□

Theorem 3.3. *If $g\in D_{\varphi}(\mathcal{I}^2)$ and $D_Bg\in B(\mathcal{I}^2)$, then*

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{s_1,s_2}(g;u_1,u_2) &- g(u_1,u_2)\mid\\ &\leq C\Big\{3\| D_Bg\|_{\infty}+\omega_{mixed}(D_Bg;\delta_{s_1},\delta_{s_2})\Big\}(u_1+1)(u_2+1)\\ &+ \Big\{1+\sqrt{C_2}(u_1+1)+\sqrt{C_1}(u_2+1)\Big\}\\ &\times \omega_{mixed}(D_Bg;\delta_{n_1},\delta_{n_2})(u_1+1)(u_2+1), \end{aligned}$$

where δ_{s_1}, δ_{s_2} defined by Theorem 2.6 and C is any positive constant.

Proof. Suppose $\rho\in(u_1,t)$, $\xi\in(y_2,s)$ and

$$\Delta^*_{(u_1,u_2)}g(t,s)=(t-u_1)(s-u_2)D_Bg(\rho,\xi),$$

$$D_Bg(\rho,\xi)=\Delta^*_{(u_1,u_2)}D_Bg(\rho,\xi)+D_Bg(\rho,y)+D_Bg(x,\xi)-D_Bg(u_1,u_2).$$

For all $D_Bg\in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned} &\mid K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Delta^*_{(u_1,u_2)}g(t,s);u_1,u_2\right)\mid\\ =\ &\mid K^{\alpha_1,\alpha_2}_{s_1,s_2}((t-u_1)(s-u_2)D_Bg(\rho,\xi);u_1,u_2)\mid\\ \leq\ & K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\mid t-u_1\|s-u_2\|\Delta^*_{(u_1,u_2)}D_Bg(\rho,\xi)\mid ;u_1,u_2\right)\\ +\ & K^{\alpha_1,\alpha_2}_{s_1,s_2}\Big(\mid t-u_1\|s-u_2\mid(\mid D_Bg(\rho,u_2)\mid\\ +\ &\mid D_Bg(u_1,\xi)\mid+\mid D_Bg(u_1,u_2)\mid);u_1,u_2\Big)\\ \leq\ & K^{\alpha_1,\alpha_2}_{s_1,s_2}\Big(\mid t-u_1\|s-u_2\mid\omega_{mixed}(D_Bg;\mid\rho-u_1\mid,\mid\xi-u_2\mid);u_1,u_2\Big)\\ +\ & 3\| D_Bg\|_{\infty}K^{\alpha_1,\alpha_2}_{s_1,s_2}(\mid t-u_1\|s-u_2\mid;u_1,u_2). \end{aligned}$$

Here ω_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned} \omega_{mixed}(D_Bg;\mid\rho-u_1\mid,\mid\xi-u_2\mid) &\leq \omega_{mixed}(D_Bg;\mid t-u_1\mid,\mid s-u_2\mid)\\ &\leq \left(1+\delta_{s_1}^{-1}\mid t-u_1\mid\right)\left(1+\delta_{s_2}^{-1}\mid s-u_2\mid\right)\omega_{mixed}(D_Bg;\mid\delta_{s_1},\delta_{s_2}). \end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned} \mid K^{*}_{s_1,s_2}(g;u_1,u_2) \quad - \quad & g(u_1,u_2) \mid \\ = \quad & \mid \Delta^{*}_{(u_1,u_2)} g(t,s);u_1,u_2 \mid \\ \leq \quad & 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \Big(K^{\alpha_1,\alpha_2}_{s_1,s_2}(\mid t-u_1 \parallel s-u_2 \mid;u_1,u_2) \\ + \quad & \delta_{s_1}^{-1} K^{\alpha_1,\alpha_2}_{s_1,s_2}\left((t-u_1)^2 \mid s-u_2 \mid;u_1,u_2\right)\Big) \\ + \quad & \delta_{s_2}^{-1} K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\mid t-u_1 \mid (s-u_2)^2;u_1,u_2\right) \\ + \quad & \delta_{s_1}^{-1}\delta_{s_2}^{-1} K^{\alpha_1,\alpha_2}_{s_1,s_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right) \varpi_{mixed}(D_B g;\delta_{s_1},\delta_{s_2}); \end{aligned}$$

$$\begin{aligned} \mid K^{*}_{s_1,s_2}(g;u_1,u_2) \quad - \quad & g(u_1,u_2) \mid \\ = \quad & \mid \Delta^{*}_{(u_1,u_2)} g(t,s);u_1,u_2 \mid \\ \leq \quad & 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \Big\{ \left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_1}^{-1}\left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{4,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_2}^{-1}\left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,4}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_1}^{-1}\delta_{s_2}^{-1} K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,2}_{u_1,u_2};u_1,u_2\right)\Big\} \varpi_{mixed}(D_B g;\delta_{s_1},\delta_{s_2}). \end{aligned}$$

Which follows that

$$\begin{aligned} \mid K^{*}_{s_1,s_2}(g;u_1,u_2) \quad - \quad & g(u_1,u_2) \mid \\ = \quad & 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,0}_{u_1,u_2};u_1,u_2\right) K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{0,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \Big\{ \left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,0}_{u_1,u_2};u_1,u_2\right) K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{0,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_1}^{-1}\left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{4,0}_{u_1,u_2};u_1,u_2\right) K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{0,2}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_2}^{-1}\left(K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,0}_{u_1,u_2};u_1,u_2\right) K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{0,4}_{u_1,u_2};u_1,u_2\right)\right)^{\frac{1}{2}} \\ + \quad & \delta_{s_1}^{-1}\delta_{s_2}^{-1} K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{2,0}_{u_1,u_2};u_1,u_2\right) K^{\alpha_1,\alpha_2}_{s_1,s_2}\left(\Psi^{0,2}_{u_1,u_2};u_1,u_2\right)\Big\} \varpi_{mixed}(D_B g;\delta_{s_1},\delta_{s_2}). \end{aligned}$$

From Lemma 2.5, we demonstrate

$$\begin{aligned} \mid K^{*}_{s_1,s_2}(g;u_1,u_2) - g(u_1,u_2) \mid \quad \leq \quad & 3 \parallel D_B g \parallel_\infty \left(\sqrt{C_1 C_2}(u_1+1)(u_2+1)\right) \\ + \quad & \Big\{ \left(\sqrt{C_1 C_2}(u_1+1)(y_2+1)\right) \end{aligned}$$

$$+ \quad \delta_{n_1}^{-1}\left(\sqrt{C_2}\sqrt{o\left(\frac{1}{n_1}\right)}(u_1+1)^2(u_2+1)\right)$$
$$+ \quad \delta_{n_2}^{-1}\left(\sqrt{C_1}\sqrt{o\left(\frac{1}{n_2}\right)}(u_2+1)^2(u_1+1)\right)$$
$$+ \quad \delta_{n_1}^{-1}\delta_{n_2}^{-1}\left(\sqrt{o\left(\frac{1}{n_1}\right)}\sqrt{o\left(\frac{1}{n_2}\right)}(u_1+1)(u_2+1)\right)\Bigg\}\varpi_{mixed}\left(D_B g;\delta_{s_1},\delta_{s_2}\right).$$

Which complete the proof of Theorem 3.3. □

4. Numerical and Graphical Analysis of the equation (4)

In this section, we analyze the convergence behavior of the operator described in equation (4) for the function $g(u_1,u_2) = u_1^5 u_2(u_1-1)sin\,4\pi u_2$. Moreover, the numerical behavior of the equation (4) are discussed in table table(1) with the help of different values of $s_1 = s_2 = 15, 20, 25$ and a common error formula $E_{s_1,s_2}(g;u_1,u_2) = |M_{s_1,s_2}(g;u_1,u_2) - g(u_1,u_2)|$ to find the error between different values of $s_1, s_2 = 15, 20, 25$. Furthermore, Figures (1) and (2) show graphs illustrating the operator's convergence and error, as described in equation (4).

u_1, u_2	$E_{15,15}(g;u_1,u_2)$	$E_{20,20}(g;u_1,u_2)$	$E_{25,25}(g;u_1,u_2)$
0.2	0.00629825	0.00352466	0.000911841
0.3	0.0024016	0.000268322	0.000293837
0.4	0.00233554	0.00202209	0.000956192
0.5	0.000743578	0.00275673	0.0.00113165
0.6	0.0228959	0.0.0191484	0.0178506

Table 1: Error Approximation Table

.

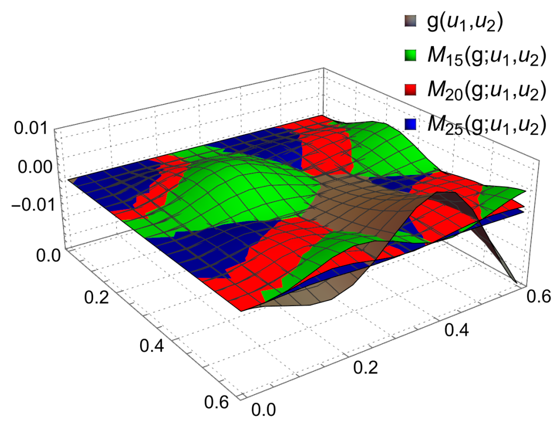

Figure 1: Convergence of Operator $M_{s_1,s_2}(g;u_1,u_2)$ for $s_1, s_2 = 15, 20, 25$

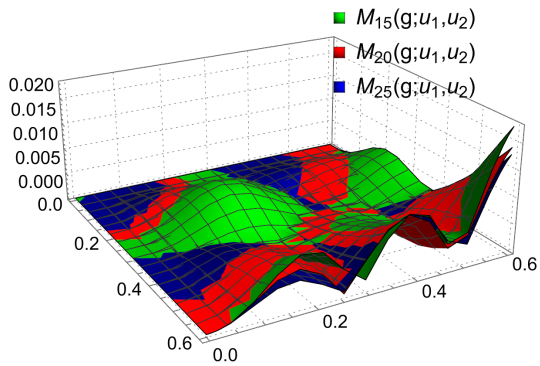

Figure 2: Error Approximation $E_{s_1,s_2}(g;u_1,u_2) = |M_{s_1,s_2}(g;u_1,u_2) - g(u_1,u_2)|$

.

5. Conclusion and Remarks

In this research article, we propose a new bivariate sequence of α-Modified Bernstein Bi-Variate operators with aid of non negative parameter α. Further, We study the bivariate properties of α-Modified Bernstein operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by use of mixed-modulus of continuity.

6. Conflict of interest

The authors declared that they have no conflict of interest.

7. Data availability statement

Data sharing not applicable

References

[1] Khan, K. and Lobiyal, D. K.: *Bézier curves based on Lupaş (p,q)-analogue of Bernstein functions in CAGD*, Comput. Appl. Math., **317**, 458-477, (2017).
[2] Khan, K., Lobiyal, D. K. and Kilicman, A.: *Bézier Curves and Surface Based on Modified Bernstein Polynomials*, Azerb. J. Math., **9**(1), 1-20 (2019).
[3] Izadbakhsh, A., Kalat, A. A. and Khorashadizadeh, S.:*Observer-based adaptivecontrol for HIV infection therapy using the Baskakov operator*, Biomed. Signal Process. Control., **6**5, 102343, (2021).
[4] Uyan, H., Aslam A. O., Karateke, S. and Büyükyazici, I.: *Interpolation for neural network operators activated with a generalized logistic-type function*, (2024).
[5] Zhang, Q., Mu, M. and Wang, X.: *Modified Robotic Manipulator Controller Based on Bernstein-Kantorovich-Stancu Operator*, Micromachines **14** (1), 44, (2022).
[6] Bernstein, S. N.: *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow., **1**3 (2), 1–2, (1912).
[7] Mursaleen M., Khan F., Khan A and Kiliçman A.: *Some approximation results for generalized Kantorovich-type operators*, J. Inequalities Appl., **2**013 (1), 585, (2013).
[8] Mursaleen M, Ansari K. J. and Khan A.: *Some approximation results by (p,q)-analogue of Bernstein–Stancu operators*, Appl. Math. Comput., **2**64, 392–402, (2015).

[9] Nasiruzzaman, M, Rao, N, Wazir, S, Kumar, R.: *Approximation on parametric extension of Baskakov-Durrmeyer operators on weighted spaces*, J. Inequal. Appl., **2019**, Article Id: 103, (2019).
[10] Acar, T., Aral, A and Mohiuddine, S. A.: *Approximation by bivariate* (p, q)*-Bernstein-Kantorovich operators*, Iran. J. Sci. Technol. Trans. Sci., **42**, 655-662 (2018).
[11] Acar, T and Kajla, A.: *Degree of Approximation for bivariate generalized Bernstein type operators*, Results Math., **73** (79) (in press), (2018).
[12] Aslan, R. and Mursaleen, M.: *Some approximation results on a class of new type* λ*-Bernstein polynomials*, J. Math. Inequal, **16** (2), 445-462, (2022).
[13] Mohiuddine, S. A., Acar, T and Alotaibi, A.: *Construction of a new family of Bernsteion Kantorovich operators*, Math. Meth. Appl. Sci., **40** , 7749-7759, (2017).
[14] Mohiuddine S. A., Özger F.: *Approximation of functions by Stancu variant of Bernstein–Kantorovich operators based on shape parameter* α, Rev. R. Acad. Cienc. Exactas, Fís. Nat. Madr. Serie A. Matemáticas (RACSAM), 114:70, (2020).
[15] Mohiuddine S.A., Ahmad N., Özger F., Alotaibi A., Hazarika B.: *Approximation by the parametric generalization of Baskakov–Kantorovich operators linking with Stancu operators*, Iran J Sci Technol Trans Sci, **45**, 593-605, (2021).
[16] Acu, A. M., Agrawal, P. and Kumar, D.: *Approximation properties of modified q-Bernstein-Kantorovich operators*, Commun. Fac. Sci. Univ. Ank. Ser. Al Math. Stat. **68** (2), 2170-2197, (2019).
[17] İçöz, G and Çekim, B.: *Dunkl generalization of Szász operators via q-calculus*, J. Inequal. Appl. 2015, 2015:284, 11 pp.
[18] İçöz, G and Çekim, B.: *Stancu-type generalization of Dunkl analogue of Szász-Kantorovich operators*. Math. Met. Appl. Sci. 2015, DOI: 10.1002/mma.3602.
[19] Özger, F.: *Weighted Statistical Approximation Properties of Univariate and Bivariate-Kantorovich Operators*, Filomat, **33** (11), 3473–3486, (2019).
[20] Özger, F., Ersoy, M. T. and Özger, Z. Ö.: *Existence of solutions: Investigating Fredholm integral equations via a fixed-point theorem*, Axioms, **13**, 261, (2024).
[21] Mursaleen, M. A., Heshamuddin, M., Rao, N., Sinha B. K., and Yadav, A. K.,: *Hermite polynomials linking Szász–Durrmeyer operators*, Comp. Appl. Math., **43** (223), 407-421, (2024).
[22] Cai, Q. B. and Aslan, R.,: *Note on a new construction of Kantorovich form q-Bernstein operators related to shape parameter* λ, Comput. Model. Eng. Sci, **130** (3), 1479-1493, (2022).
[23] Cai, Q. B., Aslan, R., Özger, F., and Srivastava, H. M.,: *Approximation by a new Stancu variant of generalized* (λ, μ)*-Bernstein operators*, Alex. Eng. J., **107**, 205-214, (2024).
[24] Cicek, H., Aslan, R., and Izgi, A.: *Approximation by univariate, bivariate and GBS operators of Chlodowsky-Kantorovich operators*, New trend math. sci., **8** (2), 44-54, (2020).
[25] Rao, N., Mursaleen, M. A., and Aslan, R., *A note on a general sequence of* λ*-Szász Kantorovich type operators*, Commut. appl. Math., **43** (8), 428, (2024).
[26] Rao, N., Malik P.: α*-Baskakov-Durrmeyer type operators and their approximation properties*, Filomat, **37** (3), 935-948, (2023).
[27] Rao, N., Yadav, A. K., Mursaleen, M., Sinha B. K., and Jha, N. K.: *Szász-Beta operators via Hermite Polynomial*, J. King Saud Univ. Comput. Inf. Sci., 103120, (2024).
[28] Usta, F.: *On new modification of Bernstein operators theory and applications,* Iran. J. Sci. Technol. Trans. A: Sci. **44** (4), 1119–1124, (2020).
[29] Senapati, A., Kumar, A. and Som, T.: *Convergence analysis of modified Bernstein–Kantorovich type operators,* Rend. Circ. Mat. Palermo. Series 2, **72** (7), 3749-3764, (2023).
[30] Bögel, K.: *Mehrdimensionale differentiation von Funktionen mehrerer veränderlichen*, J. Reine Angew. Math., **170**, 197–217 (1934).
[31] Bögel, K.: *Über die mehrdimensionale differentiation*, Jahresber. Dtsch. Math.Ver., **65**, 45–71 (1935).