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One sided x-projection, one sided x-idempotent and strongly EP elements in a $*$-ring

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Abstract. In this article, we present the concept of one sided x-projection, explore and investigate left and right x-projection elements, and introduce new characterizations of one sided x-idempotent elements in a $*$-ring. We also propose numerous equivalent conditions for left x-projections within a given ring. Next, several equivalent conditions about Hermitian, one sided x-projection, one sided x-idempotent, and strongly EP elements in a $*$-ring are given.

1. Introduction

Throughout this article, R will be used to signify a ring with unit 1. An involution in R is an anti-isomorphism $* : R \to R, a \mapsto a^*$, satisfying

$$(a+b)^* = a^* + b^*, (ab)^* = b^*a^*, (a^*)^* = a,$$

for all $a, b \in R$.
An element a of R is said to be group invertible if there exists $a^\# \in R$ such that

$$aa^\# = a^\# a, a = aa^\# a, a^\# = a^\# aa^\#.$$

The element $a^\#$ is called group inverse of a if it is uniquely determined by above equations (see [1, 9, 30]). Denote by $R^\#$ the set of all group invertible elements of R.

Let R be a ring with involution and $a \in R$. An element $a^+ \in R$ is said to be the Moore-Penrose inverse (or MP-inverse) [16, 27] of a is a^+, if

$$(a^+a)^* = a^+a, (aa^+)^* = aa^+, a^+aa^+ = a^+, aa^+a = a.$$

There is at most one a^+ such that four equalities hold, if such an element a^+ exists. We denote by R^+ the set of all Moore-Penrose invertible elements of R.

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An element $a \in R$ is called a partial isometry if $a = aa^*a$. Obviously, $a \in R^+$ is a partial isometry [8] if and only if $a^* = a^+$. We denote by R^{PI} the set of all partial isometries of R. An element $a \in R$ is called EP [7, 30] if $a \in R^\# \cap R^+$ and $a^+ = a^\#$. We denote by R^{EP} the set of all EP elements of R. Let $a \in R^\# \cap R^+$, then a is said to be a strongly EP element [35] if $a \in R^{EP}$ is partial isometry. Also, we denote by R^{SEP} the set of all strongly EP elements of R.

In [10], $a \in R$ is called idempotent if $a = a^2$. And a is said to be a Hermitian (or symmetric) if $a^* = a$. We denote the set of all idempotent elements and the set of all Hermitian elements of R by $E(R)$ and R^{Her}, respectively. Projection elements play a fundamental role in various areas of mathematics, including functional analysis, operator algebras, and algebraic topology. An element $a \in R$ is said to be a projection [17] if $a^2 = a = a^*$. Denote the set of all projections of R by $PE(R)$.

For generalized inverse elements of a ring with involution, we refer to [5, 11–13, 15, 18–23, 25, 26, 28, 29, 31, 32, 34]. In [2], Adeniji studied idempotent elements are left identity in the semigroup generated by the left action of each singular element on a symmetric group. In [14], Fernanda et al. discussed a method for computing lifted idempotents. In [33], some equivalent conditions for two projections $p, q \in R$ have been described by Li et al.

In [7, 8], Mosić and Djordjević proposed new characterizations of both Moor-Penrose inverse and group invertible to be PI and EP elements in rings with involution. Yao and Wei [17] studied equivalent conditions based on projections and a-idempotents for an element to be strongly EP elements in a $*$-ring. In [10], Mosić studied characterizations of Hermitian elements in rings with involution, which are Moore-Penrose invertible. In our work, we presented some necessary and sufficient conditions for elements in rings with involution to be a partial isometry, EP element, strongly EP element, and generalized inverse by using solutions of equations (see, [3, 4]). The results in this article are motivated by the above works.

One sided x-projection is significant because it offers a new way to characterize generalized inverse elements and enhances the research on generalized inverses in a $*$-ring. So, this article's main objective is to introduce a new concept of elements in a ring with involution, which we define as one sided x-projections, and to present new characterizations of one sided x-idempotent elements. Furthermore, we investigate the relationship between Hermitian elements and left projection elements, and we present equivalent conditions for projection elements. We also describe the relationship between left idempotent elements and central projection elements in an involution ring.

This article is organized as follows: we study new types of one sided x-projection elements and new properties of one sided x-idempotent elements in a ring with involution. In Section 2, we discuss one sided x-projections in a $*$-ring. In Section 3, we give new characterizations for left and right x-projection elements and left and right x-idempotent elements. Moreover, we discuss some equivalent conditions for left x-projection elements in a $*$-ring. In Section 4, we give several equivalent conditions for Hermitian elements, left and right x-projection elements, left and right x-idempotent elements, and SEP elements.

We will state our results as follows:

2. One sided x-projections

Definition 2.1. *Let R be a $*$-ring and $a,\ x \in R$. Then a is said to be a left (right) x-projection if $a^2 = xa$ ($a^2 = ax$) and $a^* = a$.*

Especially, if a is a left 1-projection, then a is called a projection.

It is evident that (1) a is a projection if and only if a is a left $2a - 1$-projection.

(2) a is a left x-projection if and only if a is a right x^*-projection.

(3) a is a projection if and only if $a = aa^*$.

(4) a is a projection if and only if $a = a^*a$.

Left (right) x-projection is always left (right) x-idempotent. But the converse is not true in general. For example, in $R \in \mathbb{C}^{3\times3}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{pmatrix}$ is a left $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$-idempotent. While it is not left $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$-projection.

Definition 2.2. *Let R be a ring and $x,\ a \in R$. Then a is said to be a left (right) x-idempotent if $a^2 = xa$ ($a^2 = ax$).*

Clearly, $e \in R$ is idempotent if and only if e is a left 1-idempotent. Also, we have $e \in R$ is an idempotent if and only if e is a left $2e - 1$-idempotent.

Proposition 2.3. *Let R be a $*$-ring, $a \in R$ and a is a projection. Then the following statements are equivalent:*
(1) a is a left $2a - a^*$-idempotent;
(2) a is a left $a + 1 - a^*$-idempotent;
(3) $1 - a^*$ is a right $1 - a^* + a$-idempotent.

Proof. "(1) $\Longrightarrow$ (2)" *From the hypothesis, one has $a = a^2 = a^*$ and $a^2 = (2a - a^*)a$, this gives $a^*a = a^2 = a = a^*$, so we have*

$$a^2 = 2a^2 - a^2 = 2a - a^*a = a + a - a^*a = a^2 + a - a^*a = (a + 1 - a^*)a.$$

Hence, a is left $a + 1 - a^*$-idempotent.

"(2) $\Longrightarrow$ (3)" *By the assumption, one yields $a^2 = (a + 1 - a^*)a$, this infers $a = a^*a = a^2 = a^*$. This gives*

$$(1 - a^*)^2 = 1 - 2a^* + (a^*)^2 = 1 - 2a^* + a - a^2 + (a^*)^2$$
$$= 1 - 2a^* + a - a^*a + (a^*)^2 = (1 - a^*)(1 - a^* + a).$$

Therefore, $1 - a^*$ is a right $1 - a^* + a$-idempotent.

"(3) $\Longrightarrow$ (1)" *Using the hypothesis, one gets $(1 - a^*)^2 = (1 - a^*)(1 - a^* + a)$, this leads to $a^* = a^*a$, so we have*

$$a^2 = 2a^2 - a^2 = 2a^2 - a^*a = (2a - a^*)a.$$

Hence, a is left $2a - a^*$-idempotent.

Proposition 2.4. *Let $a \in R^+$. Then the following statements are equivalent:*
(1) $a \in R^{PI}$;
(2) a^*a is a left $2a^*a - a^+a$-projection;
(3) a is a left $aa^* + a - 1$-idempotent.

Proof. "(1) $\Longrightarrow$ (2)" *Since $a \in R^{PI}$, $a^+ = a^*$, this leads to*

$$(a^*a)^2 = (2a^*a - a^*a)a^*a = (2a^*a - a^+a)a^*a.$$

Hence, a^*a is a left $2a^*a - a^+a$-projection.

"(2) $\Longrightarrow$ (3)" *From the hypothesis, one yields*

$$(a^*a)^2 = (2a^*a - a^+a)a^*a,$$

i.e., $a^*a = a^*aa^*a$. Pre-multiplying by $(a^+)^*$, one has $a = aa^*a$, this gives

$$(aa^* + a - 1)a = aa^*a + a^2 - a = a^2.$$

Therefore, a is a left $aa^* + a - 1$-idempotent.

"(3) $\Longrightarrow$ (1)" *Using the assumption, one gets*

$$a^2 = (aa^* + a - 1)a = aa^*a + a^2 - a,$$

i.e., $a = aa^*a$. Hence, $a \in R^{PI}$.

Proposition 2.5. *Let $a \in R^+ \cap R^\#$. Then $a \in R^{EP}$ if and only if aa^+ is a left $a^+a + 1 - aa^+$-projection.*

Proof. " $\Longrightarrow$ " *Since $a \in R^{EP}$, $a^+ = a^\#$, this leads to*

$$(aa^+)^2 = aa^+ = a^\#a^2a^+ = a^\#a^2a^+ + aa^+ - aa^+aa^+$$
$$= a^+a^2a^+ + aa^+ - aa^+aa^+ = (a^+a + 1 - aa^+)aa^+.$$

Hence, aa^+ is a left $a^+a + 1 - aa^+$-projection.

" $\Longleftarrow$ " *The condition "aa^+ is a left $a^+a + 1 - aa^+$-projection" gives*

$$(aa^+)^2 = (a^+a + 1 - aa^+)aa^+,$$

i.e., $aa^+ = a^+a^2a^+$. Therefore, $a \in R^{EP}$ by [24, Corollary 2.13].

Proposition 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then the following statements are equivalent:*

(1) $a \in R^{SEP}$;

(2) $aa^{\#}$ is a left $a^*a + 1 - aa^{\#}$-idempotent;

(3) a^*a is a left $aa^{\#} + 1 - a^{+}a$-projection;

(4) aa^* is a left $a^*a - aa^{+} + aa^*$-projection.

Proof. "(1) $\Longrightarrow$ (2)" *Assume that $a \in R^{SEP}$. Then $a^* = a^{+} = a^{\#}$, this infers*

$$(aa^{\#})^2 = a^{\#}a^2a^{\#} + aa^{\#} - aa^{\#}aa^{\#} = (a^{\#}a + 1 - aa^{\#})aa^{\#} = (a^*a + 1 - aa^{\#})aa^{\#}.$$

Hence, $aa^{\#}$ is a left $a^*a + 1 - aa^{\#}$-idempotent.

"(2) $\Longrightarrow$ (3)" *By the assumption, one obtains*

$$(aa^{\#})^2 = (a^*a + 1 - aa^{\#})aa^{\#},$$

i.e., $aa^{\#} = a^*a$, which implies that

$$(aa^{\#} + 1 - a^{+}a)a^*a = aa^{\#}a^*a - a^*a + a^{+}aa^*a = aa^{\#}a^*a = (a^*a)^2.$$

Therefore, a^*a is a left $aa^{\#} + 1 - a^{+}a$-projection.

"(3) $\Longrightarrow$ (4)" *Applying the hypothesis, one gets*

$$(a^*a)^2 = (aa^{\#} + 1 - a^{+}a)a^*a,$$

i.e., $a^*aa^*a = aa^{\#}a^*a$. Post-multiplying by a^{+}, one has $a^*aa^* = aa^{\#}a^*$. Pre-multiplying by $a^{+}a$, one gives $a^*aa^* = a^*$, it follows that $a^* = aa^{\#}a^*$ and $a = aa^*a$, this infers $a = a(aa^{\#})^*$. By [10, Theorem 1.1.3], $a \in R^{EP}$. Now we have

$$a^{+} = a^{+}aa^{+} = a^{+}aa^*aa^{+} = a^*, \text{ so } a^*a^2a^* = a^{+}a^2a^* = aa^* = (aa^*a)a^* = (aa^*)^2.$$

Hence, aa^* is a left $a^*a - aa^{+} + aa^*$-projection.

"(4) $\Longrightarrow$ (1)" *The condition "aa^* is a left $a^*a - aa^{+} + aa^*$-projection" leads to*

$$(aa^*)^2 = (a^*a - aa^{+} + aa^*)aa^*,$$

i.e., $a^*a^2a^* = aa^*$. Post-multiplying by $(a^{+})^*$, one has $a^*a^2 = a$. Therefore, $a \in R^{SEP}$ by [8, Theorem 2.3].

Proposition 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Nor}$ if and only if aa^* is a left $a^*a - aa^{+} + 1$-projection.*

Proof. " $\Longrightarrow$ " *Suppose that $a \in R^{Nor}$. Then $aa^* = a^*a$, it follows that*

$$(aa^*)^2 = a^*a^2a^* = a^*a^2a^* - aa^{+}aa^* + aa^* = (a^*a - aa^{+} + 1)aa^*.$$

Hence, aa^* is a left $a^*a - aa^{+} + 1$-projection.

" $\Longleftarrow$ " *From the assumption, one has*

$$(aa^*)^2 = (a^*a - aa^{+} + 1)aa^*,$$

i.e., $aa^*aa^* = a^*a^2a^*$. Post-multiplying by $(a^{+})^*$, one yields $aa^*a = a^*a^2$. Therefore, $a \in R^{Nor}$ by [10, Theorem 1.3.2].

Proposition 2.8. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if a^*a is a left $2a^*a - a^2$-projection.*

Proof. " $\Longrightarrow$ " *Since $a \in R^{Her}$, $a = a^*$, this gives*

$$(a^*a)^2 = 2a^*aa^*a - a^*aa^*a = 2a^*aa^*a - a^2a^*a = (2a^*a - a^2)a^*a.$$

Thus, a^*a is a left $2a^*a - a^2$-projection.

" $\Longleftarrow$ " *Using the assumption, one has*

$$(a^*a)^2 = (2a^*a - a^2)a^*a,$$

i.e., $a^*aa^*a = a^2a^*a$. Post-multiplying by $a^{+}(a^{+})^*$, one yields $a^*a = a^2$. Hence, $a \in R^{Her}$ by [10, Theorem 1.4.1].

Example 2.9. *Let R be a commutative ring and set* $V_2(R) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in R\right\}$. *Choose* $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in V_2(R)$. *Set* $A^* = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. *Then $V_2(R)$ is a $*$-ring.*

Clearly, $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ *is a left* $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$*-projection if and only if*

(1) $2b = 0$;

(2) a is a left x-idempotent of R;

(3) $xb + ya = 0$.

Proof. *" $\Longrightarrow$ " From the assumption, we have $A^2 = BA$ and $A^* = A$, i.e.,* $\begin{pmatrix} a^2 & ab + ba \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} xa & xb + ya \\ 0 & xa \end{pmatrix}$ *and* $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. *This gives $b = -b$, and so*

(1) $2b = 0$;

(2) $a^2 = xa$;

(3) $xb + ya = ab + ba = 2ab = a(2b) = 0$.

" $\Longleftarrow$ " The equalities $2b = 0$, $a^2 = xa$, and $xb + ya = 0$ imply

$2b = b + b = 0$, $b = -b$, *so* $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} = A^*$ *and*

$$A^2 = \begin{pmatrix} a^2 & ab + ba \\ 0 & a^2 \end{pmatrix} = BA = \begin{pmatrix} xa & xb + ya \\ 0 & xa \end{pmatrix} = \begin{pmatrix} xa & 0 \\ 0 & xa \end{pmatrix}.$$

Therefore, $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ *is a left* $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$*-projection.*

Especially, in $V_2(\mathbb{R})$, $0 \neq A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a left $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$-projection if and only if A is a projection and $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = B$.

Example 2.10. *Let $R = T_2(\mathbb{R})$ and* $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. *Taking* $A^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$. *Then R is a $*$-ring.*

Clearly, $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ *is idempotent;*

$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ *is a left* $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$*-idempotent, but it is not a left* $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$*-projection;*

$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ *is a left* $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$*-idempotent, but* $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ *is not idempotent;*

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *is a left* $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$*-projection.*

3. The properties of one sided x-projection

In this section, we establish and investigate new properties of left and right x-projection elements and left and right x-idempotent elements in a ring with involution.

In the following proposition, we study the relationship between Hermitian elements and left x-projection elements by using the concept of a ring with involution.

Proposition 3.1. *Let R be a $*$-ring and $e, a \in R$. Then*

(1) $a \in R^{Her}$ if and only if a is a left a-projection element; it is also a right a-projection element.

(2) Let $x \in R^{Her}$. Then a is a left x-projection element if and only if $a - x$ is a right $-x$-projection element.

(3) a^*a is a left a^*a-projection element; it is also a right a^*a-projection element.

(4) aa^* is a left aa^*-projection element; it is also a right aa^*-projection element.

(5) $a^* + a$ is a left $a^* + a$-projection element; it is also a right $a^* + a$-projection element.

Proof. It is routine.

Recall that an element $a \in R$ is said to be regular if there exists $x \in R$ such that $axa = a$. Such x is called an inner inverse of a.

Proposition 3.2. *Let R be a $*$-ring and $a, x \in R$. If a is a left x-projection element, then*

(1) $x^2a = axa$;

(2) If $a^2 = 1$, then $x = a$;

(3) x is an inner inverse of a if and only if $a^3 = a$;

(4) a is a left $x + ax - x^2$-projection element.

Proof. (1) The condition "a is a left x-projection element" gives $a^2 = xa$, $a^* = a$, this infers

$$x^2a = x(xa) = xa^2 = (xa)a = (a^2)a = aa^2 = axa.$$

(2) Since $a^2 = 1$, this gives a is invertible. Hence, $a^2 = xa$, this infers $x = a$.

(3) " $\Longrightarrow$ " Suppose that $a = axa$, $a^2 = xa$ and $a^* = a$. Then we get

$$a = axa = a(xa) = a(a^2) = a^3.$$

" $\Longleftarrow$ " Applying the hypothesis, we have $a^3 = a$ and $a^2 = xa$, this gives

$$a = a^3 = a(a^2) = a(xa) = axa.$$

Hence, x is an inner inverse of a.

(4) The assumptions $a^2 = xa$ and (1) give

$$a^2 = xa + axa - x^2a = (x + ax - x^2)a.$$

Hence, a is a left $x + ax - x^2$-projection element.

Noting that $(1 - e)(1 - 2e) = (1 - e)(1 - e)$ for any $e \in R$. Then we have the following proposition.

Proposition 3.3. *Let R be a $*$-ring and $e \in R$. Then the following statements are equivalent:*

(1) e is a projection element;

(2) $1 - e$ is a left $1 - 2e$-projection element;

(3) $1 - e$ is a right $1 - 2e$-projection element;

(4) e is a left $e + e^* - 1$-projection element;

(5) e is a right $e + e^* - 1$-idempotent element;

(6) $1 - e$ is a left $1 - e - e^*$-idempotent.

Proposition 3.4. *Let $a \in R^{reg}$. If a is a left x-projection element, then a is a left $x + u - uau$-projection element.*

Proof. Since $a \in R^{reg}$, $a = aua$. By the hypothesis, we get $a^2 = xa$ and $a = a^*$. Now we have

$$a^2 = xa = xa + ua - ua = xa + ua - uaua = (x + u - uau)a.$$

Therefore, a is a left $x + u - uau$-projection element.

Proposition 3.5. *Let R be a $*$-ring and $x, a \in R$. If a is a left x-projection element, then*

(1) ax^*xa is a left ax^*x^2-projection element.
(2) axx^*a is a left axx^*x-projection element.
(3) xax^* is a left xa^2-projection element.
(4) xax^* is a right a^3-projection element.
(5) ax^*xa is a left a^2x^2-projection element.
(6) ax^*xa is a right $(a^*)^2xa$-projection element.
(7) ax^*xa is a right $(a^*)^2a^2$-projection element.
(8) xa^2x^* is a left a^4-projection element.
(9) xa^2x^* is a right a^4-projection element.
(10) xax^* is a left a^3-projection element.

Proof. (1) The assumption "a is a left x-projection element" gives $a^2 = xa$ and $a = a^*$, this infers

$$(ax^*xa)^2 = ax^*xa^2x^*xa = ax^*x(xa)x^*xa = (ax^*x^2)(ax^*xa),$$

and

$$(ax^*xa)^* = a^*x^*xa^* = ax^*xa.$$

Hence, ax^*xa is a left ax^*x^2-projection element.

(2) From the hypothesis $a^2 = xa$ and $a = a^*$, we obtain

$$(axx^*a)^2 = axx^*a^2xx^*a = axx^*xaxx^*a = (axx^*x)(axx^*a),$$

and

$$(axx^*a)^* = a^*xx^*a^* = axx^*a.$$

Therefore, axx^*a is a left axx^*x-projection element.

(3) The assumptions $a^2 = xa$ and $a = a^*$ yield

$$a^2 = (a^2)^* = (xa)^* = a^*x^* = ax^*.$$

This infers

$$(xax^*)^2 = x(ax^*)xax^* = xa^2(xax^*),$$

and

$$(xax^*)^* = (x^*)^*a^*x^* = xax^*.$$

Hence, xax^* is a left xa^2-projection element.

(4) The equalities $a^2 = xa$, $a = a^*$ and (3) imply

$$(xax^*)^2 = (xax^*)(xax^*) = xax^*(a^2)x^* = xax^*a(ax^*) = xax^*(a^3),$$

and

$$(xax^*)^* = xax^*.$$

Hence, xax^* is a right a^3-projection element.

(5) The conditions $a^2 = xa$, $a = a^*$ and (3) give

$$(ax^*xa)^2 = (ax^*xa)(ax^*xa) = (ax^*)(xa)(ax^*xa) = a^2x(a^2)(x^*xa) = (a^2x^2)(ax^*xa),$$

and

$$(ax^*xa)^* = ax^*xa.$$

Thus, ax^*xa is a left a^2x^2-projection element.

(6) By the hypothesis and (3), we get

$$(ax^*xa)^2 = (ax^*xa)(ax^*)xa = ax^*xa(a^2xa) = ax^*xa((a^*)^2xa),$$

and

$$(ax^*xa)^* = ax^*xa.$$

Hence, ax^*xa is a right $(a^*)^2xa$-projection element.

(7) The equalities $a = a^*$, $a^2 = xa$ and $a^2 = ax^*$ give

$$(ax^*xa)^2 = (ax^*xa)((a^*)^2xa) = ax^*xa((a^*)^2a^2),$$

and

$$(ax^*xa)^* = ax^*xa.$$

Hence, ax^*xa is a right $(a^*)^2a^2$-projection element.

(8) The assumptions $a^2 = xa$, $a = a^*$, and $a^2 = ax^*$ imply

$$(xa^2x^*)^2 = (xa^2x^*)(xa^2x^*) = (xa)(ax^*)(xa^2x^*) = (a^2)(a^2)(xa^2x^*) = a^4(xa^2x^*),$$

and

$$(xa^2x^*)^* = x(a^*)^2x^* = xa^2x^*.$$

Thus, xa^2x^* is a left a^4-projection element.

(9) The conditions $a^2 = xa = ax^*$ and $a = a^*$ lead to

$$(xa^2x^*)^2 = (xa^2x^*)(xa^2x^*) = (xa^2x^*)((xa)(ax^*)) = (xa^2x^*)a^4,$$

and

$$(xa^2x^*)^* = xa^2x^*.$$

Therefore, xa^2x^* is a right a^4-projection element.

(10) The equalities $a = a^*$, $a^2 = xa$, and $a^2 = ax^*$ imply

$$(xax^*)^2 = (xax^*)(xax^*) = (a^2)x^*xax^* = a(ax^*)xax^* = (a^3)xax^*,$$

and

$$(xax^*)^* = xax^*.$$

Hence, xax^* is a left a^3-projection element.

Let $e \in E(R)$. If $xe = exe$ for any $x \in R$, then e is said to be a left semicentral idempotent element. If $ex = exe$ for any $x \in R$, then e is said to be a right simicentral idempotent element.

Example 3.6. *Let $R = T_2(\mathbb{R})$ and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. Taking $A^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$. Then R is a $*$-ring.*

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$ and $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Clearly, XAX^* is a left (right) A^3-projection.

The following theorems, 3.7 and 3.8, describe central projections in a beautiful way that makes them easier to understand and more useful in rings. It provides a valuable tool for understanding the structure of $*$-rings and their associated projections. So, we characterize the relationship between the left x-idempotent elements and the central projection elements in a $*$-ring.

Theorem 3.7. *Let R be a $*$-ring and $e \in E(R)$. If for any $a \in R$, ea^* is a left a-idempotent element, then e is the central projection element.*

Proof. For any $a \in R$, we have

$$ea^*ea^* = aea^*,$$

$$e(a+1)^*e(a+1)^* = (a+1)e(a+1)^*.$$

Hence, $ea^*e = ae$. Choose $a = e^*$, then $e = e^*e$, so e is a projection element. For any $x \in R$, take $g = e + (1 - e)xe$. Then $eg = e$, $ge = g$, and $g^2 = g$. From the equation, we know that eg^* is a left g-idempotent, so $eg^*eg^* = geg^*$. Hence,

$$gg^* = geg^* = (geg^*)^* = (eg^*eg^*)^* = gege = g.$$

Therefore, g is a projection element, so we have

$$g = g^* = (ge)^* = e^*g^* = eg = e.$$

Thus, $(1 - e)xe = 0$ holds for any $x \in R$, that is, e is a left simicentral element. So e^* is a right semicentral idempotent element, but e is a projection element, so e is the central element.

Similarly, we have the below theorem.

Theorem 3.8. *Let R be a $*$-ring and $e \in E(R)$. If for any $a \in R$, ea^* is a right a-idempotent element, then e is the central projection element.*

Proposition 3.9. *Let R be a $*$-ring and $x, y, a \in R$. If x, y are all left a-projection elements, then $x + y$ is a left a-projection element if and only if $xy + yx = 0$.*
Proof. *" $\Longrightarrow$ " The conditions $x^2 = ax$, $y^2 = ay$, $x^* = x$, and $y^* = y$ imply*

$$(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = ax + xy + yx + ay = a(x + y) + xy + yx.$$

Since $x + y$ is a left a-projection, $(x + y)^2 = a(x + y)$. Hence $xy + yx = 0$.
" $\Longleftarrow$ " The assumption gives $x^2 = ax$, $y^2 = ay$, and $xy + yx = 0$, this infers

$$(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = x^2 + y^2 = ax + ay = a(x + y).$$

Thus, $x + y$ is a left a-projection.

4. Left x-idempotent elements and generalized inverse

In this section, we present several equivalent conditions for the Hermitian elements, left and right x-projections, left and right x-idempotents, and strongly EP elements.

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then the following statements are equivalent:*
(1) $a \in R^{Her}$;
(2) $a^* - a$ is a left $-a$-projection element;
(3) $a^* - a$ is a right $-a$-projection element;
(4) $a^* - a$ is a right $-a$-idempotent element;
(5) $a^* - a$ is a left $-a$-idempotent element.
Proof. *"(1) $\Longrightarrow$ (2)" Since $a \in R^{Her}$, $a^* = a$, this leads to*

$$(a^* - a)^2 = a^*a^* - a^*a - aa^* + a^2 = a^*a^* - a^*a^* - aa^* + a^2 = -a(a^* - a),$$

and

$$(a^* - a)^* = a - a^* = a^* - a.$$

Hence, $a^* - a$ is a left $-a$-projection element.
"(2) $\Longrightarrow$ (3)" From the assumption, one gets $(a^* - a)^2 = -a(a^* - a)$, that is, $a^*a^* = a^*a$. Applying the involution, one obtains $a^2 = a^*a$. By [10, Theorem 1.4.1], $a = a^*$. Thus, $(a^* - a)^2 = 0 = (a^* - a)(-a)$.
"(3) $\Longrightarrow$ (4)" It is clear.
"(4) $\Longrightarrow$ (5)" By the hypothesis, one has $aa^* = a^*a^*$. Applying the involution, one yields $aa^* = a^2$. By [10, Theorem 1.4.1], $a = a^*$. Now we have $(a^* - a)^2 = 0 = -a(a^* - a)$.
"(5) $\Longrightarrow$ (1)" Applying the condition, one yields $a^*a^* = a^*a$. By "(2) $\Longrightarrow$ (3)", $a = a^*$. Hence, $a \in R^{Her}$.

Theorem 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then the following statements are equivalent:*
(1) $a \in R^{Her}$;
(2) a^*a *is a left* a^2*-projection element;*
(3) aa^* *is a right* a^2*-projection element;*
(4) a^*a *is a right* $(a^*)^2$*-projection element;*
(5) aa^* *is a left* $(a^*)^2$*-projection element;*
(6) a^*a *is a left* a^2*-idempotent element;*
(7) aa^* *is a right* a^2*-idempotent element;*
(8) a^*a *is a right* $(a^*)^2$*-idempotent element;*
(9) aa^* *is a left* $(a^*)^2$*-idempotent element.*

Proof. "(1) $\Longrightarrow$ (2)" *Since* $a \in R^{Her}$, $a = a^*$ *and* $a^*a = a^2$, *this infers*

$$(a^*a)^* = a^*(a^*)^* = a^*a,$$

and

$$(a^*a)^2 = a^*aa^*a = a^2a^*a.$$

Hence, a^*a *is a left* a^2*-projection element.*

"(2) $\Longrightarrow$ (3)" *The hypothesis gives* $(a^*a)^2 = a^2a^*a$, *i.e.,* $a^*aa^*a = a^2a^*a$. *Post-multiplying by* $a^+(a^+)^*$, *one has* $a^*a = a^2$. *By [10, Theorem 1.4.1],* $a^* = a$. *So, we have* $(aa^*)^2 = (aa^*)a^2$. *Hence,* aa^* *is a right* a^2*-projection element.*

"(3) $\Longrightarrow$ (4)" *By the condition "*aa^* *is a right* a^2*-projection element," we get* $aa^*aa^* = aa^*a^2$. *Pre-multiplying by* $(a^+)^*a^+(a^+)^*a^+$, *one has* $aa^+ = (a^+)^*a$. *Applying the involution, one obtains* $aa^+ = a^*a^+$. *By [10, Theorem 1.4.1],* $a^* = a$. *Thus,* $(a^*a)^2 = a^*a(a^*)^2$ *and so* a^*a *is a right* $(a^*)^2$*-projection element.*

"(4) $\Longrightarrow$ (5)" *From the assumption, one gives* $a^*aa^*a = a^*aa^*a^*$. *Pre-multiplying by* $a^+(a^+)^*$, *one gets* $a^*a = a^*a^*$. *Applying the involution, one obtains* $a^*a = a^2$. *By [10, Theorem 1.4.1],* $a^* = a$. *Thus,* $(aa^*)^2 = (a^*)^2aa^*$, *which implies* aa^* *is a left* $(a^*)^2$*-projection element.*

"(5) $\Longrightarrow$ (6)" *Using the condition "*aa^* *is a left* $(a^*)^2$*-projection element," one yields* $aa^*aa^* = a^*a^*aa^*$. *Post-multiplying by* $(a^+)^*a^+$, *one has* $aa^* = a^*a^*$. *Applying the involution, one yields* $aa^* = a^2$. *By Theorem 4.1,* $a^* = a$. *Therefore,* $(a^*a)^2 = a^2a^*a$, *and so* a^*a *is a left* a^2*-idempotent element.*

"(6) $\Longrightarrow$ (7)" *It is an immediate result of* "(2) $\Longrightarrow$ (3)".

"(7) $\Longrightarrow$ (8)" *It is an immediate result of* "(4) $\Longrightarrow$ (5)".

"(8) $\Longrightarrow$ (9)" *Also, it is an immediate result of* "(5) $\Longrightarrow$ (6)".

"(9) $\Longrightarrow$ (1)" *Applying the assumption, one has* $aa^*aa^* = a^*a^*aa^*$. *Post-multiplying by* $(a^+)^*a^+$, *one obtains* $aa^* = a^*a^*$. *By Theorem 4.1,* $aa^+ = a^*a^+$. *Hence,* $a \in R^{Her}$ *by [10, Theorem 1.4.1].*

Lemma 4.3. *([6]) Let* $a \in R^{\#} \cap R^{+}$. *Then*
(1) $(a^+)^*aa^{\#} = (a^+)^* = a^{\#}a(a^+)^*$;
(2) $(a^{\#})^*a^*a^+ = a^+a^*(a^{\#})^* = a^+$.

Theorem 4.4. *Let* $a \in R^{\#} \cap R^{+}$. *Then the following statements are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(a^{\#})^*aa^*a^+$ *is a left* $(a^{\#})^*a^+$*-projection element;*
(3) $(a^{\#})^*aa^*a^+$ *is a left* $(a^+)^*a^+$*-projection element;*
(4) $(a^+)^*a^+$ *is a left* $(a^{\#})^*aa^*a^+$*-projection element;*
(5) $(a^+)^*a^+$ *is a left* $(a^{\#})^*aa^*a^+$*-idempotent element.*

Proof. "(1) $\Longrightarrow$ (2)" *Since* $a \in R^{SEP}$, $a^+ = a^* = a^{\#}$. *Noting that* $(a^{\#})^* = (a^{\#})^*aa^+$. *Now we have*

$$((a^{\#})^*aa^*a^+)^2 = (a^{\#})^*aa^*a^+(a^{\#})^*aa^*a^+ = (a^{\#})^*aa^+a^+(a^{\#})^*aa^*a^+$$

$$= ((a^{\#})^*a^+)((a^{\#})^*aa^*a^+),$$

and

$$((a^{\#})^*aa^*a^+)^* = (a^+)^*aa^*a^{\#} = (a^{\#})^*aa^*a^+.$$

Hence, $(a^{\#})^*aa^*a^+$ *is a left* $(a^{\#})^*a^+$*-projection element.*

"(2) $\Longrightarrow$ (3)" The condition "$(a^\#)^*aa^*a^+$ is a left $(a^\#)^*a^+$-projection element" gives

$$(a^\#)^*aa^*a^+(a^\#)^*aa^*a^+ = (a^\#)^*a^+(a^\#)^*aa^*a^+,$$

and

$$((a^\#)^*aa^*a^+)^* = (a^+)^*aa^*a^\# = (a^\#)^*aa^*a^+.$$

Post-multiplying $(a^+)^*aa^*a^\# = (a^\#)^*aa^*a^+$ by $aa^\#$, one has $(a^\#)^*aa^*a^+ = (a^\#)^*aa^*a^+aa^\#$. Applying the involution, one gives $(a^+)^*aa^*a^\# = (aa^\#)^*(a^+)^*aa^*a^\#$. Post-multiplying $(a^+)^*aa^*a^\# = (aa^\#)^*(a^+)^*aa^*a^\#$ by $a(a^+)^*a^\#a^*$, one gets $aa^+ = (aa^\#)^*$. Applying the involution, one yields $aa^+ = aa^\#$. By [36, Corollary 2.12], $a \in R^{EP}$, so $(a^+)^* = (a^\#)^*$. Therefore, $(a^\#)^*aa^*a^+(a^\#)^*aa^*a^+ = (a^+)^*a^+((a^\#)^*aa^*a^+)$. Hence, $(a^\#)^*aa^*a^+$ is a left $(a^+)^*a^+$-projection element.

"(3) $\Longrightarrow$ (4)" From the assumption, one has

$$(a^\#)^*aa^*a^+(a^\#)^*aa^*a^+ = (a^+)^*a^+((a^\#)^*aa^*a^+).$$

Pre-multiplying by a^+a, one obtains $(a^+)^*a^+(a^\#)^*aa^*a^+ = a^+a(a^+)^*a^+(a^\#)^*aa^*a^+$. Post-multiplying by $a(a^\#)^*a^*(a^+)^*a^+a^*a$, one gets $(a^+)^* = a^+a(a^+)^*$. Applying the involution, one yields $a^+ = a^+a^+a$. By [36, Corollary 2.12], $a \in R^{EP}$. It follows that

$$(a^\#)^*aa^*a^\#(a^\#)^*aa^*a^\# = (a^\#)^*a^\#((a^\#)^*aa^*a^\#).$$

Post-multiplying by $a^2a^+(a^+)^*a^+a^*a^+a^3a^+$, one yields $(a^\#)^*aa^* = (a^\#)^*$. Pre-multiplying by a^+a^*, this infers $a^* = a^+$. Hence, $a \in R^{PI}$, so $a^+ = a^* = a^\#$, which implies that

$$(a^\#)^*aa^*a^+(a^+)^*a^+ = (a^\#)^*aa^*a^+(a^\#)^*aa^+a^+ = (a^\#)^*aa^+a^+(a^\#)^*a^+$$

$$= (a^\#)^*a^+(a^+)^*a^+ = (a^+)^*a^+(a^+)^*a^+,$$

and

$$((a^+)^*a^+)^* = (a^+)^*a^+.$$

Therefore, $(a^+)^*a^+$ is a left $(a^\#)^*aa^*a^+$-projection element.

"(4) $\Longrightarrow$ (5)" It is clear.

"(5) $\Longrightarrow$ (1)" Using the hypothesis, one gives

$$(a^+)^*a^+(a^+)^*a^+ = (a^\#)^*aa^*a^+(a^+)^*a^+.$$

Pre-multiplying by a^+a, one has $(a^+)^*a^+(a^+)^*a^+ = a^+a(a^+)^*a^+(a^+)^*a^+$. Post-multiplying by aa^*aa^*, one yields $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$ by [24, Corollary 2.13]. It follows that

$$(a^\#)^*a^\#(a^\#)^*a^\# = (a^\#)^*aa^*a^\#(a^\#)^*a^\#.$$

Post-multiplying by a^2a^+, one has $(a^\#)^*a^\#(a^\#)^* = (a^\#)^*aa^*a^\#(a^\#)^*$. Applying the involution, one gets $a^\#(a^\#)^*a^\# = a^\#(a^\#)^*aa^*a^\#$. Post-multiplying by a^2a^+ and pre-multiplying by a^+a^2, one obtains $(a^\#)^* = (a^\#)^*aa^*$. Applying the involution, one yields $a^\# = aa^*a^\#$. By [10, Theorem 1.5.2], $a \in R^{PI}$. Therefore, $a \in R^{SEP}$.

Similarly, we have the following theorem.

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then the following statements are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^2a^+ - (a^+)^*$ is a left a-projection element;
(3) $a^2a^+ - (a^+)^*$ is a left a-idempotent element.

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