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Nonlinear maps preserving the second mixed triple $\eta - *-$ product between von Neumann algebras

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Abstract. Let $\eta \neq \pm 1$ be a non-zero scalar, and let Φ be a not necessarily linear bijection between two von Neumann algebras, one of which has no center abelian projections, satisfying $\Phi(I) = I$ and $\Phi(iI)^* = -\Phi(iI)$ and preserving the second mixed triple $\eta - *-$product. It is showed that Φ is a linear $*-$isomorphism if $|\eta| = 1$ and Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$isomorphism if $|\eta| \neq 1$.

1. Introduction

In recent years, an intense research activity has been addressed to study not necessarily linear mappings between von Neumann algebras preserving the $\eta - *-$product or some of its variants. The origins of the Jordan $\eta - *-$ product go back to [8], where Šemrl introduced and studied the Jordan $(-1) - *-$ product in relation to quadratic functionals. More recently, Bai and Du [1] established that any bijective map between von Neumann algebras without central abelian projections preserving the Jordan $(-1) - *-$product is a sum of a linear and a conjugate linear $*-$isomorphisms.

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and $\Phi : \mathcal{M} \to \mathcal{N}$ be a not necessarily linear bijection between two von Neumann algebras, one of which has no central abelian projections. In [2], Dai and Lu proved that if Φ satisfies $\Phi(AB + \eta BA^*) = \Phi(A)\Phi(B) + \eta\Phi(B)\Phi(A^*)$ for all $A, B \in \mathcal{M}$, then Φ is a linear $*-$ isomorphism if η is not real and Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$ isomorphism if η is real. In [3], Huo et al. proved that if Φ preserves the Jordan triple $\eta - *-$ product and $\Phi(I) = I$, then Φ is a linear $*-$isomorphism if η is not real and Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$isomorphism if η is real. In [11], Zhang et al. established that if $\eta \neq -1$ and Φ satisfies

$$\Phi([A, B]_*^\eta \bullet_\eta C) = [\Phi(A), \Phi(B)]_*^\eta \bullet_\eta \Phi(C),$$

for all $A, B, C \in \mathcal{M}$ and $\Phi(I) = I, \Phi(iI)^* = -\Phi(iI)$, then one of the following statements holds: when $|\eta| = 1$, then Φ is a linear $*-$isomorphism; when $|\eta| \neq 1$, then Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$ isomorphism. More research on the Jordan and Lie derivable mappings can be found in [4-7,9-12].

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Let $\mathcal{M}$ be a $*-$algebra and η be a non-zero scalar. For $A, B, C \in \mathcal{M}$, define the Jordan $\eta - *-$ product of A and B by $A \bullet_\eta B = AB + \eta BA^*$, the Lie $\eta - *-$product of B and C by $[B, C]_*^\eta = BC - \eta CB^*$, respectively. The mixed triple $\eta - *-$products have two cases which are related with the triple $\eta - *-$products $[A, B]_*^\eta \bullet_\eta C$ and $[A \bullet_\eta B, C]_*^\eta$ for all A, B and C in $\mathcal{M}$. In order to distinguish the mixed triple $\eta - *-$ products, the mixed triple $\eta - *-$product $[A \bullet_\eta B, C]_*^\eta$ is called the second mixed triple $\eta - *-$ product. Motivated by these studies, this paper will discuss nonlinear mappings preserving the second mixed triple $\eta - *-$ product between von Neumann algebras.

Let us fix some notations and terminologies. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real number field and the complex number field, respectively. Let i denote the imaginary unit. Throughout, all algebras and spaces are over the complex number field $\mathbb{C}$. A von Neumann algebra $\mathcal{M}$ is a weakly closed, self adjoint algebra of operators on a complex Hilbert H containing the identity operator I. The set $\mathcal{Z}(\mathcal{M}) = \{S \in \mathcal{M} : ST = TS$ for all $T \in \mathcal{M}\}$ is called the center of $\mathcal{M}$. A projection P is called a center abelian projection if $P \in \mathcal{Z}(\mathcal{M})$ and $P\mathcal{M}P$ is abelian. The center carrier of A, denoted by $\overline{A}$, is the smallest center projection P satisfying $PA = A$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{M}) : S = S^*, S \leq A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$. A self-adjoint element A of $\mathcal{M}$ is called positive if its spectrum $\sigma(A)$ consists of non-negative real numbers. Moreover, an element A of $\mathcal{M}$ is positive if and only if there exists B in $\mathcal{M}$ with $A = B^*B$.

Lemma 1.1([3], Lemma 1.1) Let $\mathcal{M}$ be a von Neumann algebra without central abelian projections. Then there exists a projection P with $\underline{P} = 0$ and $\overline{P} = I$.

Lemma 1.2([2], Lemma 1.2) Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. Let A be an operator in $\mathcal{M}$ and P a projection with $\overline{P} = I$.

(1) If $ABP = 0$ for all $B \in \mathcal{M}$, then $A = 0$;

(2) If η is a non-zero scalar and $(PT(I - P)) \bullet_\eta A = 0$ for all $T \in \mathcal{M}$, then $A(I - P) = 0$.

2. Additivity

The main result in this section reads as follows.

Theorem 2.1 Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras, one of which has no center abelian projections. Let $\eta \neq \pm 1$ be a non-zero scalar, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a not necessarily linear bijection. Suppose that Φ preserves the second mixed triple $\eta - *-$ product. Then Φ is additive.

In the following, let $\mathcal{M}^a = \{A \in \mathcal{M} : A^* = A\}, \mathcal{N}^a = \{B \in \mathcal{N} : B^* = B\}$. Without loss of generality, we assume that $\mathcal{M}$ has no central abelian projections. It follows from Lemma 1.1 that there exists a projection $P_1 \in \mathcal{M}$ such that $\underline{P_1} = 0$ and $\overline{P_1} = I$. Set $P_2 = I - P_1$. Then P_2 is a projection in $\mathcal{M}$ and $\underline{P_2} = 0$ and $\overline{P_2} = I$. Denote $\mathcal{M}_{kl} = P_k\mathcal{M}P_l, k, l = 1, 2$.

The proof will be organized in some lemmas.

Lemma 2.1 $\Phi(0) = 0$.

Proof. By the surjectivity of Φ, there exists $A \in \mathcal{M}$ such that $\Phi(A) = 0$. Since Φ preserves the second mixed triple $\eta - *-$ product, we have

$$\Phi(0) = \Phi([0 \bullet_\eta A, A]_*^\eta) = [\Phi(0) \bullet_\eta \Phi(A), \Phi(A)]_*^\eta = 0.$$

Lemma 2.2 For every $A_{12} \in \mathcal{M}_{12}, A_{21} \in \mathcal{M}_{21}$, we have

$$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21}).$$

Proof. Since Φ is surjection, there exists an operator $X = \sum_{k,l=1}^2 X_{kl} \in \mathcal{M}$ such that $\Phi(X) = \Phi(A_{12}) + \Phi(A_{21})$. For every $\lambda \in \mathbb{C}$, by $[I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta} P_2}{1+\eta}, A_{12}]_*^\eta = 0$, then $[I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta} P_2}{1+\eta}, A_{12} + A_{21}]_*^\eta = [I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta} P_2}{1+\eta}, A_{21}]_*^\eta$, and $\Phi([I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta} P_2}{1+\eta}, A_{12}]_*^\eta) = 0$.

It follows from Lemma 2.1 that

$$\begin{aligned}
&\Phi([I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, A_{21}]_*^\eta) \\
&= \Phi([I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, A_{21}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, A_{12}]_*^\eta) \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}), \Phi(A_{21})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}), \Phi(A_{12})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}), \Phi(A_{21}) + \Phi(A_{12})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}), \Phi(X)]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{\lambda P_1 + \frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, X])_*^\eta.
\end{aligned}$$

Since Φ is injection, we have $[I \bullet_\eta \frac{\lambda P_1+\frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, X]_*^\eta = [I \bullet_\eta \frac{\lambda P_1+\frac{\overline{\lambda}}{\eta}P_2}{1+\eta}, A_{21}]_*^\eta$, that is, $(\lambda - \eta\overline{\lambda})X_{11} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})X_{21} + (\frac{\overline{\lambda}}{\eta} - \lambda)X_{22} = (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})A_{21}$.

If $|\eta| \neq 1$, multiplying the above equation by P_2 from left side and P_1 from right side, we obtain $\overline{\lambda}(\frac{1}{\eta} - \eta)X_{21} = \overline{\lambda}(\frac{1}{\eta} - \eta)A_{21}$. It follows from $|\eta| \neq 1$ that $\frac{1}{\eta} - \eta \neq 0$ and $\overline{\lambda}X_{21} = \overline{\lambda}A_{21}$. Hence by the arbitrariness of λ, $X_{21} = A_{21}$. So $(\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})X_{11} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})X_{22} = 0$ and $\overline{\lambda}X_{11} + \overline{\lambda}X_{22} = 0$. It follows from the arbitrariness of λ that $X_{11} = 0$ and $X_{22} = 0$.

If $|\eta| = 1$, then $\frac{1}{\eta} - \eta = 0$ and $(\lambda - \eta\overline{\lambda})X_{11} + (\frac{\overline{\lambda}}{\eta} - \lambda)X_{22} = 0$.

Multiplying the above equation by P_1 from left side, we get $(\lambda - \eta\overline{\lambda})X_{11} = 0$. So by the arbitrariness of λ, $X_{11} = 0$.

Multiplying the above equation by P_2 from right side, we get $(\frac{\overline{\lambda}}{\eta} - \lambda)X_{22} = 0$. So by the arbitrariness of λ, $X_{22} = 0$.

It follows from $[A_{12} \bullet_\eta (\lambda P_1), I]_*^\eta = 0$ and Lemma 2.1 that

$$\begin{aligned}
&\Phi([A_{21} \bullet_\eta (\lambda P_1), I]_*^\eta) \\
&= \Phi([A_{21} \bullet_\eta (\lambda P_1), I]_*^\eta) + \Phi([A_{12} \bullet_\eta (\lambda P_1), I]_*^\eta) \\
&= [\Phi(A_{21}) \bullet_\eta \Phi(\lambda P_1), \Phi(I)]_*^\eta + [\Phi(A_{12}) \bullet_\eta \Phi(\lambda P_1), \Phi(I)]_*^\eta \\
&= [(\Phi(A_{21}) + \Phi(A_{12})) \bullet_\eta \Phi(\lambda P_1), \Phi(I)]_*^\eta \\
&= [\Phi(X) \bullet_\eta \Phi(\lambda P_1), \Phi(I)]_*^\eta \\
&= \Phi([X \bullet_\eta (\lambda P_1), I]_*^\eta).
\end{aligned}$$

Since Φ is injection, this implies that $[X \bullet_\eta (\lambda P_1), I]_*^\eta = [A_{21} \bullet_\eta (\lambda P_1), I]_*^\eta$. Then $(\lambda - \eta^2\lambda)X_{21} + \eta(\lambda - \overline{\lambda})X_{21}^* = (\lambda - \eta^2\lambda)A_{21} + \eta(\lambda - \overline{\lambda})A_{21}^*$. Thus we get $X_{21} = A_{21}$. Similarly, we can prove $X_{12} = A_{12}$.

Therefore, $\Phi(A_{12} + A_{21}) = \Phi(X) = \Phi(A_{12}) + \Phi(A_{21})$.

Lemma 2.3 For every $A_{kk} \in \mathcal{M}_{kk}, A_{kl} \in \mathcal{M}_{kl}, 1 \leq k \neq l \leq 2$, we have

$$\Phi(A_{kk} + A_{kl}) = \Phi(A_{kk}) + \Phi(A_{kl}).$$

Proof By the surjectivity of Φ, we can find an operator $X=\sum_{k,l=1}^{2}X_{kl}\in\mathcal{M}$ such that $\Phi(X)=\Phi(A_{kk})+\Phi(A_{kl})$. For every $\lambda\in\mathbb{C}$, by $[I\bullet_\eta \frac{\lambda P_l}{1+\eta},A_{kk}]_*^\eta=0$ and Lemma 2.1, we have

$$\begin{aligned}
&\Phi([I\bullet_\eta \frac{\lambda P_l}{1+\eta},A_{kl}]_*^\eta)\\
&=\Phi([I\bullet_\eta \frac{\lambda P_l}{1+\eta},A_{kl}]_*^\eta)+\Phi([I\bullet_\eta \frac{\lambda P_l}{1+\eta},A_{kk}]_*^\eta)\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}),\Phi(A_{kl})]_*^\eta+[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}),\Phi(A_{kk})]_*^\eta\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}),\Phi(A_{kl})+\Phi(A_{kk})]_*^\eta\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}),\Phi(X)]_*^\eta\\
&=\Phi([I\bullet_\eta \frac{\lambda P_l}{1+\eta},X]_*^\eta).
\end{aligned}$$

Since Φ is injection, we have $[I\bullet_\eta \frac{\lambda P_l}{1+\eta},X]_*^\eta=[I\bullet_\eta \frac{\lambda P_l}{1+\eta},A_{kl}]_*^\eta$ and $\lambda X_{lk}+(\lambda-\eta\overline{\lambda})X_{ll}-\eta\overline{\lambda}X_{kl}=-\eta\overline{\lambda}A_{kl}$.

Multiplying the above equation by P_k from left side, then $-\eta\overline{\lambda}X_{kl}=-\eta\overline{\lambda}A_{kl}$. By the arbitrariness of λ, $X_{kl}=A_{kl}$. Consequently, $\lambda X_{lk}+(\lambda-\eta\overline{\lambda})X_{ll}=0$. Thus we get $X_{lk}=0$ and $X_{ll}=0$.

It follows from $[I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},A_{kl}]_*^\eta=0$ and Lemma 2.1 that

$$\begin{aligned}
&\Phi([I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},A_{kk}]_*^\eta)\\
&=\Phi([I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},A_{kk}]_*^\eta)+\Phi([I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},A_{kl}]_*^\eta)\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta}),\Phi(A_{kk})]_*^\eta+[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta}),\Phi(A_{kl})]_*^\eta\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta}),\Phi(A_{kk})+\Phi(A_{kl})]_*^\eta\\
&=[\Phi(I)\bullet_\eta \Phi(\frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta}),\Phi(X)]_*^\eta\\
&=\Phi([I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},X]_*^\eta).
\end{aligned}$$

Since Φ is injection, we get $[I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},X]_*^\eta=[I\bullet_\eta \frac{\lambda P_k+\frac{\overline{\lambda}}{\eta}P_l}{1+\eta},A_{kk}]_*^\eta$. Substituting $X_{lk}=0$ and $X_{ll}=0$ into the above equation, we get $(\lambda-\eta\overline{\lambda})X_{kk}=(\lambda-\eta\overline{\lambda})A_{kk}$. So $X_{kk}=A_{kk}$. Therefore, $\Phi(A_{kk}+A_{kl})=\Phi(X)=\Phi(A_{kk})+\Phi(A_{kl})$.

Similarly, $\Phi(A_{ll}+A_{kl})=\Phi(A_{ll})+\Phi(A_{kl})$.

Lemma 2.4 For every $A_{kk}\in\mathcal{M}_{kk}$, $A_{lk}\in\mathcal{M}_{lk}$ and $A_{kl}\in\mathcal{M}_{kl}$, $1\leq k\neq l\leq 2$, we have

$$\Phi(A_{kk}+A_{kl}+A_{lk})=\Phi(A_{kk})+\Phi(A_{kl})+\Phi(A_{lk}).$$

Proof Since Φ is surjective, there exists an operator $X=\sum_{k,l=1}^{2}X_{kl}\in\mathcal{M}$ such that $\Phi(X)=\Phi(A_{kk})+\Phi(A_{kl})+$

$\Phi(A_{lk})$. For every $\lambda \in \mathbb{C}$, it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
&\Phi(\lambda X_{lk} - \eta\overline{\lambda}X_{kl} + (\lambda - \eta\overline{\lambda})X_{ll}) \\
&= \Phi([I \bullet_\eta \frac{\lambda P_l}{1+\eta}, X]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(X)]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(A_{kk}) + \Phi(A_{kl}) + \Phi(A_{lk})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(A_{kk})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(A_{kl})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(A_{lk})]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{\lambda P_l}{1+\eta}, A_{kk}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_l}{1+\eta}, A_{kl}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_l}{1+\eta}, A_{lk}]_*^\eta) \\
&= \Phi(-\eta\overline{\lambda}A_{kl}) + \Phi(\lambda A_{lk}) \\
&= \Phi(\lambda A_{lk} - \eta\overline{\lambda}A_{kl}).
\end{aligned}$$

Since Φ is injection, we get $\lambda X_{lk} - \eta\overline{\lambda}X_{kl} + (\lambda - \eta\overline{\lambda})X_{ll} = \lambda A_{lk} - \eta\overline{\lambda}A_{kl}$. Thus $X_{kl} = A_{kl}$, $X_{lk} = A_{lk}$ and $X_{ll} = 0$. It follows from Lemma 2.3 and $[I \bullet_\eta \frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}, A_{lk}]_*^\eta = 0$ that

$$\begin{aligned}
&\Phi((\frac{\overline{\lambda}}{\eta} - \lambda)X_{kk} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})X_{kl}) \\
&= \Phi([I \bullet_\eta \frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}, X]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(\frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}), \Phi(X)]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}), \Phi(A_{kk}) + \Phi(A_{kl}) + \Phi(A_{lk})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}), \Phi(A_{kk})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}), \Phi(A_{kl})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}), \Phi(A_{kk})]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}, A_{kk}]_*^\eta) + \Phi([I \bullet_\eta \frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}, A_{kl}]_*^\eta) + \Phi([I \bullet_\eta \frac{\frac{\overline{\lambda}}{\eta}P_k + \lambda P_l}{1+\eta}, A_{lk}]_*^\eta) \\
&= \Phi((\frac{\overline{\lambda}}{\eta} - \lambda)A_{kk}) + \Phi((\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})A_{kl}) \\
&= \Phi((\frac{\overline{\lambda}}{\eta} - \lambda)A_{kk} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})A_{kl}).
\end{aligned}$$

This implies that $(\frac{\overline{\lambda}}{\eta} - \lambda)X_{kk} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})X_{kl} = (\frac{\overline{\lambda}}{\eta} - \lambda)A_{kk} + (\frac{\overline{\lambda}}{\eta} - \eta\overline{\lambda})A_{kl}$. Thus, we have $X_{kk} = A_{kk}$.

Therefore, $\Phi(A_{kk} + A_{kl} + A_{lk}) = \Phi(X) = \Phi(A_{kk}) + \Phi(A_{kl}) + \Phi(A_{lk})$.

Lemma 2.5 For every $A_{kl} \in \mathcal{M}_{kl}, k, l = 1, 2$, we have

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Proof By the surjectivity of Φ, we can find $X = \sum_{k,l=1}^{2} X_{kl} \in \mathcal{M}$ such that $\Phi(X) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) +$

$\Phi(A_{22})$. For every $\lambda \in \mathbb{C}$, it follows from $[I \bullet_\eta \frac{\lambda P_1}{1+\eta}, A_{22}]_*^\eta = 0$ and Lemma 2.4 that

$$\begin{aligned}
&\Phi((\lambda - \eta\overline{\lambda})X_{11} + \lambda X_{12} - \eta\overline{\lambda}X_{21}) \\
&= \Phi([I \bullet_\eta \frac{\lambda P_1}{1+\eta}, X]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(X)]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(A_{11})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(A_{12})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(A_{21})]_*^\eta \\
&\quad + [\Phi(I) \bullet_\eta \Phi(\frac{\lambda P_1}{1+\eta}), \Phi(A_{22})]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{\lambda P_1}{1+\eta}, A_{11}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_1}{1+\eta}, A_{12}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_1}{1+\eta}, A_{21}]_*^\eta) + \Phi([I \bullet_\eta \frac{\lambda P_1}{1+\eta}, A_{22}]_*^\eta) \\
&= \Phi((\lambda - \eta\overline{\lambda})A_{11}) + \Phi(\lambda A_{12}) + \Phi(-\eta\overline{\lambda}A_{21}) \\
&= \Phi((\lambda - \eta\overline{\lambda})A_{11} + \lambda A_{12} - \eta\overline{\lambda}A_{21}).
\end{aligned}$$

This implies that $(\lambda - \eta\overline{\lambda})X_{11} + \lambda X_{12} - \eta\overline{\lambda}X_{21} = (\lambda - \eta\overline{\lambda})A_{11} + \lambda A_{12} - \eta\overline{\lambda}A_{21}$.

Multiplying the above equation by P_1 from left side and P_1 from right side, we have $(\lambda - \eta\overline{\lambda})X_{11} = (\lambda - \eta\overline{\lambda})A_{11}$. By the arbitrariness of λ, $X_{11} = A_{11}$. So $\lambda X_{12} - \eta\overline{\lambda}X_{21} = \lambda A_{12} - \eta\overline{\lambda}A_{21}$.

Multiplying the above equation by P_1 from left side, we have $\lambda X_{12} = \lambda A_{12}$. By the arbitrariness of λ, $X_{12} = A_{12}$. So $-\eta\overline{\lambda}X_{21} = -\eta\overline{\lambda}A_{21}$. Note that η is a non-zero scalar. Then we get $X_{21} = A_{21}$. Similarly, we can prove $X_{22} = A_{22}$.

Therefore, $\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(X) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

Lemma 2.6 For every $A_{kl}, B_{kl} \in \mathcal{M}_{kl}, 1 \le k \ne l \le 2$, we have

$$\Phi(A_{kl} + B_{kl}) = \Phi(A_{kl}) + \Phi(B_{kl}).$$

Proof By $B_{kl} + A_{kl} + (-\eta A_{kl}^*) + (-\eta B_{kl}A_{kl}^*) = [I \bullet_\eta \frac{P_k + A_{kl}}{1+\eta}, P_l + B_{kl}]_*^\eta$ and Lemmas 2.4, 2.3 and 2.2, we get

$$\begin{aligned}
&\Phi(A_{kl} + B_{kl}) + \Phi(-\eta A_{kl}^*) + \Phi(-\eta B_{kl}A_{kl}^*) \\
&= \Phi(A_{kl} + B_{kl} + (-\eta A_{kl}^*) + (-\eta B_{kl}A_{kl}^*)) \\
&= \Phi([I \bullet_\eta \frac{P_k + A_{kl}}{1+\eta}, P_l + B_{kl}]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(\frac{P_k + A_{kl}}{1+\eta}), \Phi(P_l + B_{kl})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{P_k}{1+\eta}) + \Phi(\frac{A_{kl}}{1+\eta}), \Phi(P_l) + \Phi(B_{kl})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{P_k}{1+\eta}), \Phi(P_l)]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{A_{kl}}{1+\eta}), \Phi(P_l)]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{P_k}{1+\eta}), \Phi(B_{kl})]_*^\eta \\
&\quad + [\Phi(I) \bullet_\eta \Phi(\frac{A_{kl}}{1+\eta}), \Phi(B_{kl})]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{P_k}{1+\eta}, P_l]_*^\eta) + \Phi([I \bullet_\eta \frac{A_{kl}}{1+\eta}, P_l]_*^\eta) + \Phi([I \bullet_\eta \frac{P_k}{1+\eta}, B_{kl}]_*^\eta) + \Phi([I \bullet_\eta \frac{A_{kl}}{1+\eta}, B_{kl}]_*^\eta) \\
&= \Phi(A_{kl}) + \Phi(-\eta A_{kl}^*) + \Phi(B_{kl}) + \Phi(-\eta B_{kl}A_{kl}^*).
\end{aligned}$$

That is, $\Phi(A_{kl} + B_{kl}) = \Phi(A_{kl}) + \Phi(B_{kl})$.

Lemma 2.7 For every $A_{kk}, B_{kk} \in \mathcal{M}_{kk}, k = 1, 2$, we have

$$\Phi(A_{kk} + B_{kk}) = \Phi(A_{kk}) + \Phi(B_{kk}).$$

Proof Since Φ is surjective, there exists an operator $X = \sum_{k,l=1}^{2} X_{kl} \in \mathcal{M}$ such that $\Phi(\frac{X}{1+\eta}) = \Phi(\frac{A_{kk}}{1+\eta}) + \Phi(\frac{B_{kk}}{1+\eta})$. For every $\lambda \in \mathbb{C}$ and $k \neq l$, it follows from Lemma 2.1 that

$$\begin{aligned}
&\Phi(\frac{(\lambda - \eta\overline{\lambda})X_{ll} + \lambda X_{lk} - \eta\overline{\lambda}X_{kl}}{1+\eta}) \\
&= \Phi([P_l \bullet_\eta \frac{\lambda P_l}{1+\eta}, \frac{X}{1+\eta}]_*^\eta) = [\Phi(P_l) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(\frac{X}{1+\eta})]_*^\eta \\
&= [\Phi(P_l) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(\frac{A_{kk}}{1+\eta}) + \Phi(\frac{B_{kk}}{1+\eta})]_*^\eta \\
&= [\Phi(P_l) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(\frac{A_{kk}}{1+\eta})]_*^\eta + [\Phi(P_l) \bullet_\eta \Phi(\frac{\lambda P_l}{1+\eta}), \Phi(\frac{B_{kk}}{1+\eta})]_*^\eta \\
&= \Phi([P_l \bullet_\eta \frac{\lambda P_l}{1+\eta}, \frac{A_{kk}}{1+\eta}]_*^\eta) + \Phi([P_l \bullet_\eta \frac{\lambda P_l}{1+\eta}, \frac{B_{kk}}{1+\eta}]_*^\eta) \\
&= 0.
\end{aligned}$$

Since Φ is injective, we have $\frac{(\lambda-\eta\overline{\lambda})X_{ll}+\lambda X_{lk}-\eta\overline{\lambda}X_{kl}}{1+\eta} = 0$ and $(\lambda - \eta\overline{\lambda})X_{ll} + \lambda X_{lk} - \eta\overline{\lambda}X_{kl} = 0$. Thus, $X_{ll} = 0, X_{lk} = 0$ and $X_{kl} = 0$. For every $C_{kl} \in \mathcal{M}_{kl}, k \neq l$, it follows from Lemma 2.6 that

$$\begin{aligned}
&\Phi(X_{kk}C_{kl}) \\
&= \Phi([I \bullet_\eta \frac{X}{1+\eta}, C_{kl}]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(\frac{X}{1+\eta}), \Phi(C_{kl})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{A_{kk}}{1+\eta}) + \Phi(\frac{B_{kk}}{1+\eta}), \Phi(C_{kl})]_*^\eta \\
&= [\Phi(I) \bullet_\eta \Phi(\frac{A_{kk}}{1+\eta}), \Phi(C_{kl})]_*^\eta + [\Phi(I) \bullet_\eta \Phi(\frac{B_{kk}}{1+\eta}), \Phi(C_{kl})]_*^\eta \\
&= \Phi([I \bullet_\eta \frac{A_{kk}}{1+\eta}, C_{kl}]_*^\eta) + \Phi([I \bullet_\eta \frac{B_{kk}}{1+\eta}, C_{kl}]_*^\eta) \\
&= \Phi(A_{kk}C_{kl}) + \Phi(B_{kk}C_{kl}) \\
&= \Phi(A_{kk}C_{kl} + B_{kk}C_{kl}).
\end{aligned}$$

This implies that $(X_{kk} - A_{kk} - B_{kk})C_{kl} = 0$. For every $C \in \mathcal{M}$, then $(X_{kk} - A_{kk} - B_{kk})CP_l = 0$. It follows from Lemma 1.2 that $X_{kk} = A_{kk} + B_{kk}$. Thus, $\Phi(\frac{A_{kk}+B_{kk}}{1+\eta}) = \Phi(\frac{X}{1+\eta}) = \Phi(\frac{A_{kk}}{1+\eta}) + \Phi(\frac{B_{kk}}{1+\eta})$ and $\Phi(A_{kk}+B_{kk}) = \Phi(A_{kk}) + \Phi(B_{kk})$.

Now we come to the position to show Theorem 2.1.

Proof of Theorem 2.1 Let A and B be in $\mathcal{M}$. Write $A = \sum_{k,l=1}^{2} A_{kl}$ and $B = \sum_{k,l=1}^{2} B_{kl}$, where $A_{kl}, B_{kl} \in \mathcal{M}_{kl}, k, l = 1, 2$. It follows from Lemmas 2.5, 2.6 and 2.7 that

$$\begin{aligned}
\Phi(A+B) &= \Phi(\sum_{k,l=1}^{2}(A_{kl} + B_{kl})) = \sum_{k,l=1}^{2} \Phi(A_{kl} + B_{kl}) \\
&= \sum_{k,l=1}^{2}(\Phi(A_{kl}) + \Phi(B_{kl})) = \sum_{k,l=1}^{2} \Phi(A_{kl}) + \sum_{k,l=1}^{2} \Phi(B_{kl}) \\
&= \Phi(\sum_{k,l=1}^{2} A_{kl}) + \Phi(\sum_{k,l=1}^{2} B_{kl}) \\
&= \Phi(A) + \Phi(B).
\end{aligned}$$

Thus Φ is additive.

3. Linearity

Our main result in this section reads as follows.

Theorem 3.1 Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras, one of which has no center abelian projections. Let $\eta \neq \pm 1$ be a non-zero scalar, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a not necessarily linear bijection. Suppose that Φ preserves the second mixed triple $\eta - *-$ product. Then the following statements hold:
(1) When $|\eta| = 1$, then Φ is a linear $*-$ isomorphism;
(2) When $|\eta| \neq 1$, then Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$isomorphism.
In what follows, without loss of generality, we assume that $\mathcal{M}$ has no central abelian projections.

Proof We distinguish two cases.

Case 1 $|\eta| = 1$.

Claim 1.1 For every $A \in \mathcal{M}$, $\Phi(A)^* = \Phi(A)$ if and only if $A^* = A$.

Proof Let $A \in \mathcal{M}$ and $A^* = A$. Since $|\eta| = 1$ and Φ preserves the second mixed triple $\eta - *-$ product, we have

$$0 = \Phi((1+\eta)(A - A^*)) = \Phi([I \bullet_\eta A, I]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A), \Phi(I)]_*^\eta) = (1+\eta)(\Phi(A) - \Phi(A)^*).$$

It follows from $\eta \neq -1$ that $\Phi(A)^* = \Phi(A)$.

Let $A \in \mathcal{M}$ and $\Phi(A)^* = \Phi(A)$. Since Φ^{-1} preserves the second mixed triple $\eta - *-$product, we have

$$0 = \Phi^{-1}([I \bullet_\eta \Phi(A), I]_*^\eta) = \Phi^{-1}([\Phi(I) \bullet_\eta \Phi(A), \Phi(I)]_*^\eta) = [I \bullet_\eta A, I]_*^\eta = (1+\eta)(A - A^*).$$

By $\eta \neq -1$, we get $A^* = A$.

Claim 1.2 $\Phi(\mathcal{Z}(\mathcal{M})) = \mathcal{Z}(\mathcal{N})$.

Proof For every $B \in \mathcal{N}^a$, since Φ is surjective, there exists $A \in \mathcal{M}$ such that $\Phi(A) = B$. It follows from $\Phi(A)^* = B^* = B = \Phi(A)$ and Claim 1.1 that $A^* = A$.

For every $C \in \mathcal{Z}(\mathcal{M})$, we have $AC = CA$ and

$$0 = \Phi([I \bullet_\eta A, C]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A), \Phi(C)]_*^\eta = [I \bullet_\eta B, \Phi(C)]_*^\eta = (1+\eta)(B\Phi(C) - \Phi(C)B).$$

It follows from $\eta \neq -1$ that $B\Phi(C) = \Phi(C)B$. For every $B \in \mathcal{N}$, by the Cartesian decomposition, it can be concluded that $B\Phi(C) = \Phi(C)B$. By the arbitrariness of B, we have $\Phi(C) \in \mathcal{Z}(\mathcal{N})$. By the arbitrariness of C, then we have $\Phi(\mathcal{Z}(\mathcal{M})) \subseteq \mathcal{Z}(\mathcal{N})$.

Similarly, we have $\Phi^{-1}(\mathcal{Z}(\mathcal{N})) \subseteq \mathcal{Z}(\mathcal{M})$, that is, $\mathcal{Z}(\mathcal{N}) \subseteq \Phi(\mathcal{Z}(\mathcal{M}))$. Thus, $\Phi(\mathcal{Z}(\mathcal{M})) = \mathcal{Z}(\mathcal{N})$.

Claim 1.3 $\Phi(iI)^2 = -I$.

Proof On the one hand, it follows from $\Phi(iI)^* = -\Phi(iI)$ and $|\eta| = 1$ that

$$-2\Phi((1+\eta)I) = \Phi([I \bullet_\eta iI, iI]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(iI), \Phi(iI)]_*^\eta = [I \bullet_\eta \Phi(iI), \Phi(iI)]_*^\eta = 2(1+\eta)\Phi(iI)^2. \quad (1)$$

On the other hand,

$$-2\Phi((1-\eta)I) = \Phi([iI \bullet_\eta iI, I]_*^\eta) = [\Phi(iI) \bullet_\eta \Phi(iI), I]_*^\eta = [(1-\eta)\Phi(iI)^2, I]_*^\eta = 2(1-\eta)\Phi(iI)^2. \quad (2)$$

By comparing equations (1) and (2), we get $\Phi(iI)^2 = -I$.

Claim 1.4 For every $A_1, A_2 \in \mathcal{M}^a$, we have

$$\Phi(A_1 + iA_2) = \Phi(A_1) + \Phi(iI)\Phi(A_2).$$

Proof Since Φ is surjective, there exist operators $B_1, B_2 \in \mathcal{M}^a$ such that $\Phi(A_1 + iA_2) = \Phi(B_1) + i\Phi(B_2)$.

Let $A \in \mathcal{M}$. It follows from $[iI \bullet_\eta iI, A]_*^\eta = 2(\eta - 1)A$ and Theorem 2.1 that

$$2\Phi((\eta-1)A) = \Phi(2(\eta-1)A) = \Phi([iI \bullet_\eta iI, A]_*^\eta) = [\Phi(iI) \bullet_\eta \Phi(iI), \Phi(A)]_*^\eta = 2(\eta-1)\Phi(A).$$

Thus, $\Phi((\eta-1)A) = (\eta-1)\Phi(A)$. By Theorem 2.1, $\Phi(\eta A) = \eta\Phi(A)$.

Let $A \in \mathcal{M}^a$. It follows from $[I \bullet_\eta A, iI]_*^\eta = 0$ that

$$0 = \Phi([I \bullet_\eta A, iI]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A), \Phi(iI)]_*^\eta = [I \bullet_\eta \Phi(A), \Phi(iI)]_*^\eta = (1+\eta)(\Phi(A)\Phi(iI) - \Phi(iI)\Phi(A)).$$

Thus $\Phi(A)\Phi(iI) = \Phi(iI)\Phi(A)$. So $\Phi(B_1)\Phi(iI) = \Phi(iI)\Phi(B_1)$ and $\Phi(B_2)\Phi(iI) = \Phi(iI)\Phi(B_2)$.

It follows from $[iI \bullet_\eta (A_1 + iA_2), iI]_*^\eta = 2(\eta - 1)iA_2$ that

$$\begin{aligned}
&\Phi(2(\eta - 1)iA_2)\\
&= \Phi([iI \bullet_\eta (A_1 + iA_2), iI]_*^\eta) = [\Phi(iI) \bullet_\eta \Phi(A_1 + iA_2), \Phi(iI)]_*^\eta\\
&= [\Phi(iI) \bullet_\eta (\Phi(B_1) + i\Phi(B_2)), \Phi(iI)]_*^\eta\\
&= 2(\eta - 1)i\Phi(B_2).
\end{aligned}$$

By $\Phi((\eta - 1)A) = (\eta - 1)\Phi(A)$ and $\eta \neq 1$, we have $\Phi(iA_2) = i\Phi(B_2)$.

By $\Phi(A_1) + \Phi(iA_2) = \Phi(A_1 + iA_2) = \Phi(B_1) + i\Phi(B_2)$, we have $\Phi(A_1) = \Phi(B_1)$. It follows from Theorem 2.1 and $[iI \bullet_\eta (A_1 + iA_2), I]_*^\eta = 2(\eta - 1)A_2$ that

$$\begin{aligned}
&2\Phi((\eta - 1)A_2) = \Phi(2(\eta - 1)A_2)\\
&= \Phi([iI \bullet_\eta (A_1 + iA_2), I]_*^\eta) = [\Phi(iI) \bullet_\eta \Phi(A_1 + iA_2), \Phi(I)]_*^\eta\\
&= [(1 - \eta)\Phi(iI)(\Phi(B_1) + i\Phi(B_2)), I]_*^\eta\\
&= -2i(\eta - 1)\Phi(iI)\Phi(B_2).
\end{aligned}$$

It follows from $\Phi((\eta-1)A) = (\eta-1)\Phi(A)$ that $\Phi(A_2) = -i\Phi(iI)\Phi(B_2)$. By $\Phi(iI)^2 = -I$, so $i\Phi(B_2) = \Phi(iI)\Phi(A_2)$. Therefore,

$$\Phi(A_1 + iA_2) = \Phi(A_1) + \Phi(iA_2) = \Phi(A_1) + i\Phi(B_2) = \Phi(A_1) + \Phi(iI)\Phi(A_2).$$

Claim 1.5 For every $A, B \in \mathcal{M}$, we obtain $\Phi(A)^* = \Phi(A)^*$ and $\Phi(AB) = \Phi(A)\Phi(B)$.

Proof There exist operators $A_1, A_2 \in \mathcal{M}^a$ such that $A = A_1 + iA_2$. By Claims 1.1 and 1.4, we have $\Phi(A^*) = \Phi(A_1 - iA_2) = \Phi(A_1) - \Phi(iA_2) = \Phi(A_1) - \Phi(iI)\Phi(A_2) = (\Phi(A_1) + \Phi(iI)\Phi(A_2))^* = \Phi(A)^*$.

It follows from Theorem 2.1, Claims 1.4 and 1.3 that

$\Phi(iA) = \Phi(iA_1 - A_2) = \Phi(iI)\Phi(A_1) - \Phi(A_2) = \Phi(iI)(\Phi(A_1) + \Phi(iI)\Phi(A_2)) = \Phi(iI)\Phi(A)$.

It follows from $[I \bullet_\eta A, B]_*^\eta = (1 + \eta)(AB - BA^*)$ that

$$\Phi((1 + \eta)(AB - BA^*)) = \Phi([I \bullet_\eta A, B]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A), \Phi(B)]_*^\eta = (1 + \eta)(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*).$$

By the proceeding results, we get

$$\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^* = \Phi(A)\Phi(B) - \Phi(B)\Phi(A^*). \tag{3}$$

Replacing A with iA in equation (3), we have $\Phi((iA)B - B(iA)^*) = \Phi(iA)\Phi(B) - \Phi(B)\Phi(iA)^*$. It follows from $\Phi(iA) = \Phi(iI)\Phi(A)$ that

$$\begin{aligned}
&\Phi(iI)\Phi(AB + BA^*)\\
&= \Phi(i(AB + BA^*)) = \Phi((iA)B - B(iA)^*) = \Phi(iA)\Phi(B) - \Phi(B)\Phi(iA)^*\\
&= \Phi(iI)\Phi(A)\Phi(B) - \Phi(B)(\Phi(iI)\Phi(A))^* = \Phi(iI)\Phi(A)\Phi(B) + \Phi(iI)\Phi(B)\Phi(A)^*\\
&= \Phi(iI)(\Phi(A)\Phi(B) + \Phi(B)\Phi(A^*)).
\end{aligned}$$

By Claim 1.3, we get

$$\Phi(AB + BA^*) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A^*). \tag{4}$$

By combining equations (3) and (4), we obtain $\Phi(AB) = \Phi(A)\Phi(B)$.

Claim 1.6 For every $\lambda \in \mathbb{R}$ and $A \in \mathcal{M}$, we have $\Phi(\lambda A) = \lambda\Phi(A)$ and $\Phi(iA) = i\Phi(A)$.

Proof For every rational number q, by Theorem 2.1, we have $\Phi(qI) = qI$. Let E be a positive element in $\mathcal{M}$. Then there exists an operator $B \in \mathcal{M}^a$ such that $E = B^2$. By Claim 1.5, $\Phi(B)$ is self adjoint and $\Phi(E) = \Phi(B)^2$. It follows that $\Phi(E)$ is a positive element. So Φ preserves positive elements.

There exist two sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers with $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. By $a_n \leq \lambda \leq b_n$, we get $a_nI \leq \lambda I \leq b_nI$. Taking the limit of the above equation, we have $\Phi(\lambda I) = \lambda I$ and $\Phi(\lambda A) = \Phi((\lambda I)A) = \Phi(\lambda I)\Phi(A) = \lambda\Phi(A)$. So Φ is real linear.

Suppose that $\eta = a + bi$ with $a, b \in \mathbb{R}$. It follows from Theorem 2.1 and the above result that

$$a\Phi(A) + b\Phi(iA) = \Phi((a + bi)A) = \Phi(\eta A) = \eta\Phi(A) = (a + bi)\Phi(A) = a\Phi(A) + bi\Phi(A).$$

If $|\eta| = 1$ and $\eta \neq \pm 1$ are used, we have $b \neq 0$ and $\Phi(iA) = i\Phi(A)$. By Theorem 2.1, Claims 1.4, 1.5 and 1.6, we obtain Φ is linear $*-$isomorphism.

Case 2 $|\eta| \neq 1$.

Claim 2.1 Φ preserves projections.

Proof For $A \in \mathcal{M}$, by $\Phi(I) = I$, we have

$$\Phi((1 - |\eta|^2)A) = \Phi([I \bullet_\eta I, A]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(I), \Phi(A)]_*^\eta) = [I \bullet_\eta I, \Phi(A)]_*^\eta) = (1 - |\eta|^2)\Phi(A)$$

and

$$\Phi(|\eta|^2 A) = |\eta|^2\Phi(A). \tag{5}$$

For $A \in \mathcal{M}^a$, we get $\Phi((1 - |\eta|^2)A^2) = \Phi([I \bullet_\eta A, A]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A), \Phi(A)]_*^\eta = (1 - |\eta|^2)\Phi(A)^2$. By Theorem 2.1, $\Phi(A^2) - \Phi(|\eta|^2 A^2) = \Phi(A)^2 - |\eta|^2\Phi(A)^2$. By equation (5), we have $\Phi(|\eta|^2 A^2) = |\eta|^2\Phi(A)^2$ and $\Phi(A^2) = \Phi(A)^2$.

For $A \in \mathcal{M}^a$, by $[I \bullet_\eta A, I]_*^\eta = (1 - |\eta|^2)A$, we have

$$\Phi((1 - |\eta|^2)A) = \Phi([I \bullet_\eta A, I]_*^\eta) = [I \bullet_\eta \Phi(A), I]_*^\eta = \Phi(A) + \eta\Phi(A) - \eta\Phi(A)^* - |\eta|^2\Phi(A)^*.$$

So $(\eta + |\eta|^2)(\Phi(A) - \Phi(A)^*) = 0$. By $\eta \neq 0$ and $\eta \neq -1$, we have $\eta + |\eta|^2 \neq 0$ and $\Phi(A)^* = \Phi(A)$.

For every projection $P \in \mathcal{M}$, we have $P^2 = P = P^*$. From the proceeding results, $\Phi(P)^2 = \Phi(P) = \Phi(P)^*$. Thus $\Phi(P)$ is a projection in $\mathcal{N}$. Therefore, Φ preserves projections.

Let $Q_k = \Phi(P_k), k = 1, 2$. Then Q_k is a projection in $\mathcal{N}$. Let $\mathcal{N}_{kl} = Q_k\mathcal{N}Q_l, k, l = 1, 2$. So $\mathcal{N} = \sum_{k,l=1}^2 \mathcal{N}_{kl}$. For every $A \in \mathcal{N}$, we can write $A = \sum_{k,l=1}^2 A_{kl}$ with $A_{kl} \in \mathcal{N}_{kl}$. It follows from $\underline{P_1} = 0$ and $\overline{P_1} = I$ that $\underline{Q_1} = 0$ and $\overline{Q_1} = I$. Furthermore, $\underline{Q_2} = 0$ and $\overline{Q_2} = I$.

Claim 2.2 $\Phi(\mathcal{M}_{kl}) = \mathcal{N}_{kl}, k, l = 1, 2$ and $k \neq l$.

Proof For every $A_{kl} \in \mathcal{M}_{kl}$, it follows from $[I \bullet_\eta P_k, \frac{1}{1+\eta}A_{kl}]_*^\eta = A_{kl}$ that

$$\begin{aligned}\Phi(A_{kl}) &= \Phi([I \bullet_\eta P_k, \frac{1}{1+\eta}A_{kl}]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(P_k), \Phi(\frac{1}{1+\eta}A_{kl})]_*^\eta = [I \bullet_\eta Q_k, \Phi(\frac{1}{1+\eta}A_{kl})]_*^\eta \\ &= (1 + \eta)Q_k\Phi(\frac{1}{1+\eta}A_{kl}) - \eta(1 + \overline{\eta})\Phi(\frac{1}{1+\eta}A_{kl})Q_k.\end{aligned}$$

Multiplying the above equation by Q_l from left side and Q_l from right side yields, $Q_l\Phi(A_{kl})Q_l = 0$.

Similarly, we can prove $Q_k\Phi(A_{kl})Q_k = 0$.

Let $\Phi(A_{kl}) = B_{kl} + B_{lk}$ with $B_{kl} \in \mathcal{N}_{kl}, B_{lk} \in \mathcal{N}_{lk}$. It follows from $[I \bullet_\eta A_{kl}, P_k]_*^\eta = 0$ that

$$\begin{aligned}0 &= \Phi([I \bullet_\eta A_{kl}, P_k]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(A_{kl}), \Phi(P_k)]_*^\eta \\ &= [I \bullet_\eta \Phi(A_{kl}), Q_k]_*^\eta = (1 + \eta)\Phi(A_{kl})Q_k - \eta(1 + \overline{\eta})Q_k\Phi(A_{kl})^* \\ &= (1 + \eta)(B_{kl} + B_{lk})Q_k - \eta((1 + \overline{\eta})Q_k(B_{kl} + B_{lk})^* = (1 + \eta)B_{lk} - \eta(1 + \overline{\eta})B_{lk}^*.\end{aligned}$$

This implies that $(1 + \eta)B_{lk} = \eta(1 + \overline{\eta})B_{lk}^*$ and $B_{lk} = 0$. Thus $\Phi(A_{kl}) = B_{kl} \in \mathcal{N}_{kl}$. Due to the arbitrariness of A_{kl}, we have $\Phi(\mathcal{M}_{kl}) \subseteq \mathcal{N}_{kl}$.

By considering Φ^{-1}, we can get $\Phi^{-1}(\mathcal{N}_{kl}) \subseteq \mathcal{M}_{kl}$. Hence $\Phi(\mathcal{M}_{kl}) = \mathcal{N}_{kl}$.

Claim 2.3 $\Phi(\mathcal{M}_{kk}) = \mathcal{N}_{kk}, k = 1, 2$.

Proof For every $A_{kk} \in \mathcal{M}_{kk}$, suppose $l = 1, 2$ and $l \neq k$. Then

$$0 = \Phi([I \bullet_\eta P_l, A_{kk}]_*^\eta) = [\Phi(I) \bullet_\eta \Phi(P_l), \Phi(A_{kk})]_*^\eta = [I \bullet_\eta Q_l, \Phi(A_{kk})]_*^\eta = (1 + \eta)Q_l\Phi(A_{kk}) - \eta(1 + \overline{\eta})\Phi(A_{kk})Q_l.$$

So $Q_l\Phi(A_{kk})Q_k = Q_k\Phi(A_{kk})Q_l = Q_l\Phi(A_{kk})Q_l = 0$ and $\Phi(A_{kk}) = Q_k\Phi(A_{kk})Q_k \in \mathcal{N}_{kk}$. By the arbitrariness of A_{kk}, we have $\Phi(\mathcal{M}_{kk}) \subseteq \mathcal{N}_{kk}$.

By considering Φ^{-1}, we can get $\Phi^{-1}(\mathcal{N}_{kk}) \subseteq \mathcal{M}_{kk}$. Hence $\Phi(\mathcal{M}_{kk}) = \mathcal{N}_{kk}$.

Claim 2.4 For every $A, B \in \mathcal{M}$, we have $\Phi(AB) = \Phi(A)\Phi(B)$.

Proof By Theorem 2.1, just need to prove

$$\Phi(A_{kl}B_{gh}) = \Phi(A_{kl})\Phi(B_{gh}), k, l, g, h = 1, 2.$$

If $l \neq g$ is used, it follows from Lemma 2.1, Claims 2.2 and 2.3 that $\Phi(A_{kl}B_{gh}) = 0 = \Phi(A_{kl})\Phi(B_{gh})$.

By $\Phi(B_{kl})\Phi(A_{kk})^* = 0$, we have

$$\Phi(A_{kk}B_{kl}) - \Phi(\eta B_{kl}^* A_{kk}^*) = \Phi([A_{kk} \bullet_\eta B_{kl}, I]_*^\eta) = \Phi(A_{kk})\Phi(B_{kl}) - \eta\Phi(B_{kl})^*\Phi(A_{kk})^*.$$

Thus $\Phi(A_{kk}B_{kl}) - \Phi(A_{kk})\Phi(B_{kl}) = \Phi(\eta B_{kl}^* A_{kk}^*) - \eta\Phi(B_{kl})^*\Phi(A_{kk})^*$.

By $\Phi(A_{kk}B_{kl}) - \Phi(A_{kk})\Phi(B_{kl}) \in \mathcal{N}_{kl}$ and $\Phi(\eta B_{kl}^* A_{kk}^*) - \eta\Phi(B_{kl})^*\Phi(A_{kk})^* \in \mathcal{N}_{lk}$, so $\Phi(A_{kk}B_{kl}) - \Phi(A_{kk})\Phi(B_{kl}) = 0$. That is, $\Phi(A_{kk}B_{kl}) = \Phi(A_{kk})\Phi(B_{kl})$.

For every $T_{kl} \in \mathcal{N}_{kl}, k \neq l$, there exists an operator $C_{kl} \in \mathcal{M}_{kl}$ such that $T_{kl} = \Phi(C_{kl})$. Thus $\Phi(A_{kk}B_{kk})T_{kl} = \Phi(A_{kk}B_{kk})\Phi(C_{kl}) = \Phi(A_{kk}B_{kk}C_{kl}) = \Phi(A_{kk})\Phi(B_{kk}C_{kl}) = \Phi(A_{kk})\Phi(B_{kk})\Phi(C_{kl}) = \Phi(A_{kk})\Phi(B_{kk})T_{kl}$. For every $T \in \mathcal{N}$, then $(\Phi(A_{kk}B_{kk}) - \Phi(A_{kk})\Phi(B_{kk}))TQ_l = 0$. It follows from Lemma 1.2 that $\Phi(A_{kk}B_{kk}) = \Phi(A_{kk})\Phi(B_{kk})$.

By $\Phi(B_{lk})\Phi(A_{kl}^*) = 0$ and $\Phi(A_{kl}^*)\Phi(B_{lk}) = 0$, we have

$$\begin{aligned}\Phi(A_{kl}B_{lk}) - \Phi(|\eta|^2 B_{lk}A_{kl}) &= \Phi([A_{kl} \bullet_\eta I, B_{lk}]_*^\eta) = [\Phi(A_{kl}) \bullet_\eta \Phi(I), \Phi(B_{lk})]_*^\eta \\ &= [\Phi(A_{kl}) \bullet_\eta I, \Phi(B_{lk})]_*^\eta = \Phi(A_{kl})\Phi(B_{lk}) - |\eta|^2\Phi(B_{lk})\Phi(A_{kl}).\end{aligned}$$

That is, $\Phi(A_{kl}B_{lk}) - \Phi(|\eta|^2 B_{lk}A_{kl}) = \Phi(A_{kl})\Phi(B_{lk}) - |\eta|^2\Phi(B_{lk})\Phi(A_{kl})$.

Combining equation (5), it can be concluded that

$$\begin{aligned}\Phi(A_{kl}B_{lk}) - \Phi(A_{kl})\Phi(B_{kl}) &= \Phi(|\eta|^2 B_{lk}A_{kl}) - |\eta|^2\Phi(B_{lk})\Phi(A_{kl}) \\ &= |\eta|^2\Phi(B_{lk}A_{kl}) - |\eta|^2\Phi(B_{lk})\Phi(A_{kl}) = |\eta|^2(\Phi(B_{lk}A_{kl}) - \Phi(B_{lk})\Phi(A_{kl})).\end{aligned}$$

It follows from Claims 2.2 and 2.4 that $\Phi(A_{kl}B_{lk}) - \Phi(A_{kl})\Phi(B_{kl}) \in \mathcal{N}_{kk}$ and $\Phi(B_{lk}A_{kl}) - \Phi(B_{lk})\Phi(A_{kl}) \in \mathcal{N}_{ll}$. Thus $\Phi(A_{kl}B_{lk}) - \Phi(A_{kl})\Phi(B_{lk}) = 0$. That is, $\Phi(A_{kl}B_{lk}) = \Phi(A_{kl})\Phi(B_{lk})$.

For every $T_{lk} \in \mathcal{N}_{lk}, k \neq l$, there exists an operator $S_{lk} \in \mathcal{M}_{lk}$ such that $T_{lk} = \Phi(S_{lk})$. Hence

$$\Phi(A_{kl}B_{ll})T_{lk} = \Phi(A_{kl}B_{ll})\Phi(T_{lk}) = \Phi(A_{kl})\Phi(B_{ll})\Phi(S_{lk}) = \Phi(A_{kl}B_{ll}S_{lk}) = \Phi(A_{kl})\Phi(B_{ll})T_{lk}.$$

For every $T \in \mathcal{N}$, then $(\Phi(A_{kl}B_{ll}) - \Phi(A_{kl})\Phi(B_{ll}))TQ_k = 0$. It follows from Lemma 1.2 that $\Phi(A_{kl}B_{ll}) = \Phi(A_{kl})\Phi(B_{ll})$.

Claim 2.5 Φ is a sum of a linear $*-$isomorphism and a conjugate linear $*-$isomorphism.

Proof For every $A \in \mathcal{M}$, we have $A = A_1 + iA_2$ with $A_1, A_2 \in \mathcal{M}^a$. It follows from Claim 2.1 that Φ preserves self adjoint elements. It follows from $\Phi(iI)^* = -\Phi(iI)$, Claims 2.2 and 2.4 that

$$\begin{aligned}\Phi(A^*) &= \Phi(A_1 - iA_2) = \Phi(A_1) - \Phi((iI)A_2) = \Phi(A_1) - \Phi(iI)\Phi(A_2) \\ &= \Phi(A_1)^* + \Phi(iI)^*\Phi(A_2)^* = \Phi(A_1)^* + (\Phi(A_2)\Phi(iI))^* \\ &= \Phi(A_1)^* + \Phi(iA_2)^* = \Phi(A_1 + iA_2)^* \\ &= \Phi(A)^*.\end{aligned}$$

By Claim 2.4 and Theorem 2.1, we have $\Phi(iI)^2 = \Phi((iI)^2) = \Phi(-I) = -\Phi(I) = -I$.

For any rational number q, by Theorem 2.1, we have $\Phi(qI) = qI$. For any positive element A in $\mathcal{M}$, there exists an operator $B \in \mathcal{M}^a$ such that $A = B^2$. By Claim 2.4, $\Phi(A) = \Phi(B)^2$, where $\Phi(B)$ is a self adjoint element. Thus $\Phi(A)$ is a positive element in $\mathcal{N}$.

For any $\lambda \in \mathbb{R}$, there exist two rational sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. By $a_nI \leq \lambda I \leq b_nI$, we have $a_nI \leq \Phi(\lambda I) \leq b_nI$. Taking the limit of the above equation, we get $\Phi(\lambda I) = \lambda I$. For every $A \in \mathcal{M}$, by Claim 2.4, we have

$$\Phi(\lambda A) = \Phi((\lambda I)A) = \Phi(\lambda I)\Phi(A) = \lambda\Phi(A).$$

It follows from Theorem 2.1 that Φ is real linear.

Let $F = \frac{I - i\Phi(iI)}{2}$. Then $F^2 = \frac{I - \Phi(iI)^2 - 2i\Phi(iI)}{4} = F$ and $F^* = \frac{I + i\Phi(iI)^*}{2} = F$. Thus F is a projection in $\mathcal{N}$. For every $B \in \mathcal{N}$, there exists an operator $C \in \mathcal{M}$ such that $B = \Phi(C)$. It follows from Claim 2.4 that

$$\begin{aligned} BF &= \frac{BI - iB\Phi(iI)}{2} = \frac{B - i\Phi(C)\Phi(iI)}{2} = \frac{B - i\Phi(iC)}{2} \\ &= \frac{B - i\Phi((iI)C)}{2} = \frac{B - i\Phi(iI)\Phi(C)}{2} = \frac{IB - i\Phi(iI)B}{2} \\ &= FB. \end{aligned}$$

Then F is a central projection in $\mathcal{N}$.

Let $E = \Phi^{-1}(F)$. It follows from Claim 2.1 that E is a central projection in $\mathcal{M}$.

For every $A \in \mathcal{M}$, by Claim 2.4, we get

$$\Phi(iAE) = \Phi(A)\Phi(E)\Phi(iI) = i\Phi(A)F(2F - I) = i\Phi(A)F = i\Phi(A)\Phi(E) = i\Phi(AE),$$
$$\Phi(iA(I - E)) = \Phi(A)\Phi(I - E)\Phi(iI) = -i\Phi(A)(I - F)(I - 2F) = -i\Phi(A)(I - F) = -i\Phi(A(I - E)).$$

It follows from Claim 2.4 that Φ is a linear $*-$isomorphism restricted to $\mathcal{M}E$ and Φ is a conjugate linear $*-$isomorphism restricted to $\mathcal{M}(I - E)$.

Thus the proof is completed.

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