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On some spectral problems for higher order differential operator equation, I

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Abstract. The aim of the paper is first to study domains of definitions in terms of boundary conditions of minimal and maximal operators, as well as selfadjoint extensions of the minimal operator associated with a fourth-order differential operator equation. Further, we give necessary and sufficient conditions for those operators to have a purely discrete or continuous spectrum, to exist extension with resolvent from σ_p, study asymptotics of spectrum in case of pure discrete spectrum.

1. Introduction

Our aim is first to study domains of definition of minimal and maximal operators generated by a differential operator expression in the space which is larger than one where the differential expression is considered, [3]. Such operators arise upon consideration of boundary value problems for differential equations when boundary conditions contain an eigenvalue parameter. Secondly, to give boundary conditions for defining self-adjoint extensions, extensions with a discrete or continuous spectrum. Thirdly, to derive an asymptotic formula for a spectrum in the case of purely discrete spectrum, finally, and most importantly in our opinion to give a new method for finding regularized trace of the operator associated with the corresponding boundary value problem in one special case. We show a new treat for deriving the trace formula, which is more general in comparison with one applied in our previous works and may be applied in future studies.

Consider in $L_2(H,(0,1))$, where H is an abstract separable Hilbert space, the following differential expression with operator coefficients

$$ly \equiv y^{IV}(t) + Ay(t) + q(t)y(t) \tag{1}$$

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$L_2(H,(0,1))$ is a class of functions $y(t)$ with values from H, where a scalar product is defined as $(y,z)_{L_2(H,(0,1))} = \int_0^1 (y(t),z(t))_H dt$. Further, we will omit index H in integrand $(y(t),z(t))_H$ to denote scalar product in H.

Here A and $q(t)$ are operator coefficients. Our assumptions about them are the followings (later, when deriving a trace formula we will put some additional requirements on A and $q(t)$) :

1. A is a selfadjoint operator in H, moreover $A > I$, where I is an identity operator in H, and $A^{-1} \in \sigma_\infty$. The last condition provides discreteness of the spectrum of A.
2. $q(t)$ is a weakly measurable, selfadjoint, bounded operator-valued function in H for each $t \in [0,1]$.

So, $q(t)$ is bounded in H, while the operator A is bounded only from below. Under that conditions $q(t)$ is bounded also in $L_2(H,(0,1))$.

Consider the direct sum space $\mathcal{H} = L_2(H,(0,1)) \oplus H^2$ with the elements $Y = (y(t), y_1, y_2)$, $Z = (z(t), z_1, z_2)$, where $y_1, y_2, z_1, z_2 \in H$. A scalar product in $\mathcal{H}$ is denoted by $(\cdot,\cdot)_{\mathcal{H}}$ and defined by,

$$(Y,Z)_{\mathcal{H}} = (y(t), z(t))_{L_2(H,(0,1))} + \left(Q_1^{-\frac{1}{2}} y_1, Q_1^{-\frac{1}{2}} z_1\right) + \left(Q_2^{-\frac{1}{2}} y_2, Q_2^{-\frac{1}{2}} z_2\right), \tag{2}$$

$(\cdot,\cdot)$ is a scalar product in H.

Define in $\mathcal{H}$ an operator L'_0 in the following way:

$$D\left(L'_0\right) = \left(Y/ \text{ such that } Y = \left\{y(t), Q_1 y(1), Q_2 y'(1)\right\}, y(t) \in C_0^\infty(H_\infty,(0,1]),\right.$$

$$\left. y(1) \in D(Q_1),\ y'(1) \in D(Q_2), y'''(1), y''(1) \in H,\ L'_0 Y = \{ly, -y'''(1), y'(1)\},\ q(t) \equiv 0\right)$$

where, Q_1 and Q_2 are self-adjoint positive-definite operators in H, $C_0^\infty(H_\infty,\ (0,1])$ is a class of vector functions with the values from $H_\infty \equiv \bigcap_{j=1}^\infty D(A^j)$ and finite in the vicinity of zero. By integrating by parts it may be easily verified that L'_0 is symmetric in $\mathcal{H}$. Denote its closure by L_0 and call it a minimal operator. Adjoint of L_0 is denoted by L_0^* and called a maximal operator. For boundary value problems for differential operator equations of second order see monograph [15]

In [19] the following boundary value problem is considered:

$$l[y] \equiv y^{IV}(t) + Ay(t) = \lambda y(t) \tag{3}$$

$$y'''(0) = \lambda Q_1 y(0), -y''(0) = \lambda Q_2 y'(0) \tag{4}$$

$$\cos C Y'_b - \sin C Y_b = 0 \tag{5}$$

where Q_1, Q_2 are defined as given above, $Y_b = \left(y_b, y'_b\right)$, $Y'_b = (y'''_b, y''_b)$, and y_b, y'_b, y'''_b, y''_b are regularized values at $t = b$ of $y(t)$ and its derivatives to third order according to [13]. In [19] the questions of selfadjointness and compactness of the operator corresponding to that problem with exit to a larger space than $L_2(H,(0,1))$ are studied. However, we have the following notes regarding the statements in the indicated work:

1) The author defines a minimal symmetric operator associated with problem (3)-(5) in space $L_2(H,(0,b))$ as a closure of the symmetric operator L'_0 with domain $D\left(L'_0\right) = C_0^\infty(H_\infty,[0,b))$, which is a set of infinitely many times differentiable vector functions with the values from H_∞, finite in the vicinity of b and $L'_0 y(t) \equiv ly$. It is stated in the work that the domain of closure of L'_0 is given by (5), which obviously is not true, since the closure in $L_2(H,(0,b))$ are the functions satisfying $y(b) = y'(b) = y''(b) = y'''(b) = 0$ which are just a part of the set of functions satisfying (5).

   The adjoint operator is denoted by L_0^*.

2) In Theorem 1 from [19] it is stated that if $Q_1 = Q_2$, then the closure of the operator $L_B^{'}$ with domain of definition $\{Y = \{y(t), \ y_1, y_2\} \in \mathcal{H}, \ y(t) \in C^{\infty}(H_{\infty}(0,b)\}$, where $y_1 = Q_1 y(0), y_2 = Q_2 y^{'}(0)$ and $L_B^{'} = \left\{L_0^* y(t), y^{'''}(0), -y^{''}(0)\right\}$ gives an operator of which domain is a class of functions satisfying conditions (5) and which is selfadjoint (in $\mathcal{H}$. Author defines scalar product by (2).

But this is obviously not true, since the indicated closure consists of the vectors $Y = \{y(t), y_1, y_2\}$, $y(t) \in W_4(0,b), ly \in L_2(H,(0,b))$ and $y(t) \in D(A)$. $W_4(0,b)$ is a closure of $C^4(H_1, [0,b]$ in the norm

$$\left\|y(t)\right\|_{W_4(0,b)}^2 = \left\|y(t)\right\|_{L_2(H_1,(0,b))}^2 + \left\|y^{IV}(t)\right\|_{L_2(H,(0,b))}^2$$

(see [15]).

But, it is obvious that self- adjoint extensions of L_0 will be given by (5).

For that reason, we decide not to refer to that but give definitions of the minimal and maximal operators and self-adjoint extensions, then treat some spectral questions for the operators generated by $l[y]$ in the direct sum space. Hence:

1. Define a minimal symmetric operator corresponding to differential expression (1) with exit to direct sum space and give boundary conditions defining selfadjoint extensions of that operator.

2. Give conditions for that extension to have purely discrete spectrum or to have spectrum filling any preset interval from the real axis. Also, define selfadjoint extensions resolvents of which are from σ_p which is a Schatten von Neumann class of functions. Refer here to [19] where the domains of minimal and maximal operators generated by $2n$-th order differential operator expression without exit to larger space are studied. Note here also [16],[21], where selfadjoint extensions and eigenvalue asymptotics for the Sturm-Liouville operator equation by exiting to a larger space are searched and [13], where selfadjoint extensions of operators generated by $2n$-th order differential operator expressions (ones having unbounded operator coefficients) without exit to a larger space are studied.

3. Consider the eigenvalue problem

$$ly = \lambda y \tag{6}$$

$$y(0) = y''(0) = 0 \tag{7}$$

$$-y'''(1) = \lambda Q_1 y(1), \ \ y''(1) = \lambda Q_2 y'(1). \tag{8}$$

The operator corresponding to this problem is one of selfadjoint extensions of the minimal operator corresponding to (1) with an exit to a larger space. We study its eigenvalue distribution.

4. Give a regularized trace formula for the operator associated with (6)-(8). For regularized traces, more general method than one used in our previous works will be suggested. The suggested method will let to treat these problems from unique point of view.

Results for such problems are applicable to boundary value problems for some classes of partial differential equations. At the end of paper we give application of obtained results to partial differential equations.

Boundary value problems for ordinary differential equations with an unbounded operator coefficient cover boundary value problems for partial differential equations. Problems with eigenvalue-dependent boundary conditions arise upon separation of variables in partial differential equations, when boundary conditions and differential equation both contain a partial derivative with respect to time, hence, for problems with dynamic boundary conditions. Problems with dynamic boundary conditions are very natural in many mathematical models as heat transfer in solid in contact with moving fluid, thermoalasticity, diffusion phenomena, the heat transfer in two-phase medium, thermal energy storage devices, problems in fluid mechanics, diffusion in porous media, electronics and semiconductors, long cables. We refer here to Langer [18], also to Fulton [10], [22], [6] and references given there . For elliptic type partial differential equations see, for example, [9], [8]. In the scalar case, for spectral problems for Sturm-Liouville equations with rational Herglotz –Nevanlinna function of eigenvalue parameter in boundary conditions we refer to [7], [17] and references there and for fourth-order differential equation to [2].

At the end we give an example of a boundary value problem for a partial differential equation which after separation of variables is reduced to spectral problem for differential operator with unbounded operator coefficient.

2. Domains of definition of adjoint operator, selfadjoint extensions, selfadjoint extensions with compact resolvents

Recall that since $q(t)$ is bounded in $L_2(H,(0,1))$, the existence of $q(t)$ in $l[y]$ is not essential for the domain of definitions of minimal, maximal operators and self-adjoint extensions, that is why when studying these questions we will take $q(t) \equiv 0$.

We will use the following notations :H_j (a scale of Hilbert spaces generated by A)as always denotes $(j > 0)$ a completion of $D(A^j)$ with respect to the scalar product $(f,g)_j = \left(A^j f, A^j g\right)$ (see [2]) for $j > k$, $H_j \subseteq H_k \subseteq H$. H_{-j} is a space with a negative norm constructed with respect to H, H_j. H_{-j} is the completion of H in the norm $\left\|A^{-j} f\right\|$. H_{-j} is usually considered as an adjoint to H_j with respect to the scalar product $(\cdot,\cdot)$, so that for $g \in H_{-j}$, $f \in H_j$, $g(f)$ will be written as (f,g). The operator A is an isometric operator from H_1 to H. The adjoint of A denoted by $\tilde{A}$ acts from H to H_{-1} and is the extension of A.

Theorem 2.1. *The domain $D(L_0^*)$ of L_0 consists of those elements $Y = \left(y(t), Q_1 y(1),\quad Q_2 y'(1)\right)$ of the space $\mathcal{H} = L_2(H,(0,1)) \oplus H^2$, where*

$$y(t) = e^{\alpha_1 \sqrt[4]{\tilde{A}} t} f_1 + e^{\alpha_2 \sqrt[4]{\tilde{A}} t} f_2 + e^{-\alpha_1 \sqrt[4]{A}(t-1)} g_1 +$$

$$+ e^{-\alpha_2 \sqrt[4]{A}(t-1)} g_2 + \int_0^1 G(t,s)\, h(s) ds, \tag{9}$$

$$G(t,s) = \left[\frac{e^{\alpha_1 \sqrt[4]{A}|t-s|}}{4\alpha_1^3} + \frac{e^{\alpha_2 \sqrt[4]{A}|t-s|}}{4\alpha_2^3}\right] A^{-\frac{3}{4}}, \tag{10}$$

$$f_1,\ f_2 \epsilon H_{-\frac{1}{8}}, g_1,\ g_2 \epsilon H_{\frac{3}{4}}\ (or\ A^{\frac{3}{4}} g_i \in H,\ i = 1,2), \qquad Q_1 g_i \in H,\ A^{\frac{1}{4}} Q_2\, g_i \epsilon H,\ for\ i = 1,2, \tag{11}$$

α_1, α_2 are the roots of the equation $\alpha^4 = -1$ with negative real parts, so, $\alpha_1 = e^{\frac{3\pi i}{4}}, \alpha_2 = e^{\frac{5\pi i}{4}}$ and

$$L_0^* Y = \left(ly, -y'''(1), y''(1)\right). \tag{12}$$

Since $g_1,\ g_2 \in H$ and $f(A)g = f\left(\tilde{A}\right) g$ for a bounded function f on H, then in the (9) in third and fourth terms one may take A but not $\tilde{A}$.

Proof. In [13], $ly = (-1)^n y^{(2n)} + Ay$ in $L_2(H,(0,1))$ is considered, and there it was shown that the values of $y(t)$ at endpoints of the interval are from a larger space than H, namely, $f_1,\ f_2,\ g_1,\ g_2 \in H_{-\frac{1}{8}}$. But we take $f_1,\ f_2 \in H_{-\frac{1}{8}}$, and $g_1,\ g_2 \in H_{\frac{3}{4}}$,$Q_1 g_i \in H$, $A^{\frac{1}{4}} Q_2 g_i \in H$, for $i = 1,2$, because we define an operator in $\mathcal{H} = L_2(H,(0,1)) \oplus H^2$ and for that reason $y'(1), y'''(1), Q_1 y(1)$, $Q_2 y''(1)$ must be from H. □

As it follows from Theorem 2.1 (relations (9), (11)) values of $y(t)$ and its derivatives at zero are distributions.

Let Y_0 and $Y_0^{'}$ be defined in H^2 by

$$Y_0 = \left\{y_0, y_0^{'}\right\}, \quad Y_0^{'} = \{y_0^{'''}, y_0^{''}\} \tag{13}$$

where $y_0, y_0^{'}, y_0^{'''}, y_0^{''}$ are regularized values of $y(t)$ and its derivatives at zero which are obtained from [13],by taking $n = 2$

$$\begin{aligned} y_0 = A^{-\frac{1}{8}} y(0), \; y_0^{'} = A^{-\frac{3}{8}} y^{'}(0), \; y_0^{''} = A^{\frac{3}{8}} \left(y^{''}(0) - \sqrt{2} A^{\frac{1}{4}} y'(0) + A^{\frac{1}{2}} y(0)\right) \\ y_0^{'''} = A^{\frac{1}{8}} \left(-y^{'''}(0) + A^{\frac{1}{2}} y^{'}(0) + \sqrt{2} A^{\frac{3}{4}} y(0)\right) \end{aligned} \tag{14}$$

Now let $\tilde{y} = \left\{y_0, y'_0\right\}, \; \widetilde{y'} = \left\{y'''_0, y''_0\right\}$ be arbitrary vectors from H^2. By the similar way done in [12], [16] the following lemma might be easily proved

Lemma 2.2. *For each* $\{\tilde{y}, \; \tilde{y}'\} \epsilon H^4$ *there exists* $Y = \left\{y(t), Q_1 y(1), Q_2 y^{'}(1)\right\} \in D(L_0^*)$ *so that* $y_0, y_0^{'}, y_0^{''}, y_0^{'''}$ *are defined by* (14)

By the methods of the work [20] (where condition for binary relations to be hermitian is given), and [12], [13] and [14] (where Sturm-Liouville operator with an unbounded operator coefficient and with exit to larger space is defined) the following theorem may be easily verified.

Theorem 2.3. *The domain of self-adjoint extensions* L_0^s *of the operator* L_0 *in* $\mathcal{H}$ *consists of those* $Y \in D\left(L_0^*\right)$ *which satisfy also*

$$\cos C Y_0^{'} - \sin C Y_0 = 0 \tag{15}$$

with a selfadjoint operator C *on* $H^2 : C = (C_1, C_2)$, C_i *act in* H *for* $i = 1, 2$, *and* $Y_0, Y_0^{'}$ *are defined by* (13), (14). *For simplifying the notations, we will take* $C = (C, C)$.

Note 2.1. Since $q(t)$ is a selfadjoint and bounded operator in $\mathcal{H}$ the statement of the theorem remains true also for $L = L_0 + Q$, where $Q = \{q(t), \; 0, 0\}$

Denote selfadjoint extension of L by L_s.

Theorem 2.4. *The spectrum of selfadloint extensions* L_0^s *of the minimal operator* L_0 *is discrete if and only if* $\cos C$ *is compact and* $Q_1 A^{-\frac{3}{4}}, Q_2 A^{-\frac{1}{2}}$ *are bounded.*

Proof. Let λ be non-real, then for selfadjoint extension L_0^s and $\tilde{h} = (h(t), \; h_1, h_2) \in \mathcal{H}$, $Y \in D\left(L_0^s\right)$, we consider the equation ,

$$L_0^s Y - \lambda Y = \tilde{h} \tag{16}$$

or in the equivalent form

$$y^{IV} + Ay - \lambda y = h(t), \tag{17}$$

$$-y^{'''}(1) - \lambda Q_1 y(1) = h_1, \tag{18}$$

$$y^{''}(1) - \lambda Q_2 y'(1) = h_2 \tag{19}$$

in addition to (17)-(19) Y as a vector from the domain of L_0^s satisfies the condition (15). From (16)-(19) the resolvent $R_\lambda(L_0^s)$ of L_0^s is

$$R_\lambda\left(L_0^s\right)\tilde{h} = Y = \begin{pmatrix} y(t,\lambda) \\ Q_1 y(1) \\ Q_2 y'(1) \end{pmatrix} \tag{20}$$

where $y(t,\lambda)$ is the solution of (17) defined by

$$y(t,\lambda) = e^{\alpha_1 \sqrt[4]{A-\lambda I}t} A^{\frac{1}{8}} f_1 + e^{\alpha_2 \sqrt[4]{A-\lambda I}t} A^{\frac{1}{8}} f_2 + e^{-\alpha_1 \sqrt[4]{A-\lambda I}\,(t-1)} A^{-\frac{3}{4}} g_1 +$$

$$+ e^{-\alpha_2 \sqrt[4]{A-\lambda I}(t-1)} A^{-\frac{3}{4}} g_2 + \int_0^1 G(t,s,\lambda)\, h(s) ds, \tag{21}$$

where

$$f_1,\ f_2 \epsilon H, g_1,\ g_2 \epsilon H, Q_1 A^{-\frac{3}{4}} g_i \in H,\ Q_2\, A^{-\frac{1}{2}} g_i \epsilon H, \text{ for } i = 1,2. \tag{22}$$

Introduce the notations:

$$\omega_j(t,\lambda) = \begin{cases} e^{\alpha_i \sqrt[4]{A-\lambda I}t} A^{\frac{1}{8}},\ i = 1,2,\ j = 1,2 \\ e^{-\alpha_i \sqrt[4]{A-\lambda I}(t-1)} A^{-\frac{3}{4}},\ i = 1,2,\ j = 3,4 \end{cases} \tag{23}$$

where, $\omega_j(t,\lambda) f_i, (j = 1,2,\ i = 1,2)$ and $\omega_j(t,\lambda) g_i\ (j = 3,4, i = 1,2)$ form a fundamental system of solutions of the homogenous equation corresponding to (**??**)

Let also $Y_0 = \left\{y_0, y_0'\right\},\ Y_0' = \{y_0''', y_0''\}$, elements of which are defined by (14). With (21)-(23) in mind we can write:

$$R_\lambda\left(L_0^s\right)\tilde{h} = \begin{pmatrix} y(t) \\ Q_1 y(1) \\ Q_2 y'(1) \end{pmatrix} =$$

$$= \begin{pmatrix} \omega_1(t,\lambda) f_1 + \omega_2(t,\lambda) f_2 + \omega_3(t,\lambda) g_1 + \omega_4(t,\lambda) g_2 + \int_0^1 G(t,s,\lambda) h(s) ds \\ Q_1\omega_1(1,\lambda) f_1 + Q_1\omega_2(1,\lambda) f_2 + Q_1 A^{-\frac{3}{4}} g_1 + Q_1 A^{-\frac{3}{4}} g_2 + Q_1 \int_0^1 G(1,s,\lambda) h(s) ds \\ Q_2 k_1 \omega_1(1,\lambda) f_1 + Q_2 k_2 \omega_2(1,\lambda) f_2 - Q_2 k_1 A^{-\frac{3}{4}} g_1 - Q_2 k_2 A^{-\frac{3}{4}} g_2 + Q_2 \int_0^1 G_t'(1,s,\lambda) h(s) ds \end{pmatrix}$$

$$k_i = \alpha_i \sqrt[4]{A-\lambda I}, i = 1,2.$$

Rewrite the last relation in the following matrix form:

$$R_\lambda\left(L_0^s\right)\tilde{h} = \begin{pmatrix} \omega_1(t,\lambda) & \omega_2(t,\lambda) & \omega_3(t,\lambda) & \omega_4(t,\lambda) \\ Q_1\omega_1(1,\lambda) & Q_1\omega_2(1,\lambda) & Q_1 A^{-\frac{3}{4}} & Q_1 A^{-\frac{3}{4}} \\ Q_2 k_1 \omega_1(1,\lambda) & Q_2 k_2 \omega_2(1,\lambda) & -Q_2 k_1 A^{-\frac{3}{4}} & -Q_2 k_2 A^{-\frac{3}{4}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^1 G(t,s,\lambda) h(s) ds \\ Q_1 \int_0^1 G(1,s,\lambda) h(s) ds \\ Q_2 \int_0^1 G_t'(1,s,\lambda) h(s) ds \end{pmatrix} \tag{24}$$

Define the vector $\begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}$ from equalities (15), (18), (19) by substituting $y(t,\ \lambda)$ from (21) into them. Introduce some notations.

Firstly, let

$$B_j(t,\lambda) = \begin{cases} \frac{e^{-\alpha_i \sqrt[4]{A-\lambda I}t}}{4\alpha_i^3} A^{-\frac{3}{4}}, i = 1,2,\ j = 1,2 \\ \frac{e^{\alpha_i \sqrt[4]{A-\lambda I}t}}{4\alpha_i^3} A^{-\frac{3}{4}}, i = 1,2,\ j = 3,4 \end{cases}.$$

Note that with this notation

$$G(t,s,\lambda)=\begin{cases}\left[B_1(t,\lambda)e^{\alpha_1\sqrt[4]{A-\lambda I}s}+B_2(t,\lambda)e^{\alpha_2\sqrt[4]{A-\lambda I}s}\right], & t\le s\\ \left[B_3(t,\lambda)e^{\alpha_1\sqrt[4]{A-\lambda I}(-s)}+B_4(t,\lambda)e^{\alpha_2\sqrt[4]{A-\lambda I}(-s)}\right], & s\le t\end{cases} \tag{25}$$

Now we will write boundary conditions (15), (18), (19) in the matrix form with $y(t)$ defined from (21) and with (25) in mind. For simplifying $G^{(n)}(0,s,\lambda), n=1,4$, (since $t=0\le s$ we have to use the second row expressions from (25)) introduce the following notations, obtained by taking $B_j(t,\lambda)$ in (14) instead of $y(t)$:

$$B'''_{1,0}=A^{\frac18}\left[-\frac{1}{4\alpha_1^3}\left(\alpha_1\sqrt[4]{A-\lambda I}\right)^3A^{-\frac34}+\frac{1}{4\alpha_1^2}\sqrt[4]{A-\lambda I}A^{-\frac14}+\frac{\sqrt2}{4\alpha_1^3}\right]$$

$$B'''_{2,0}=A^{\frac18}\left[-\frac{1}{4\alpha_2^3}\left(\alpha_2\sqrt[4]{A-\lambda I}\right)^3A^{-\frac34}+\frac{1}{4\alpha_2^2}\sqrt[4]{A-\lambda I}A^{-\frac14}+\frac{\sqrt2}{4\alpha_2^3}\right]$$

$$B''_{1,0}=A^{\frac38}\left[\frac{1}{4\alpha_1^3}\left(\alpha_1\sqrt[4]{A-\lambda I}\right)^2A^{-\frac34}-\frac{\sqrt2}{4\alpha_1^2}\sqrt[4]{A-\lambda I}A^{-\frac12}+\frac{\sqrt2}{4\alpha_1^3}A^{-\frac14}\right]$$

$$B''_{2,0}=A^{\frac38}\left[\frac{1}{4\alpha_2^3}\left(\alpha_2\sqrt[4]{A-\lambda I}\right)^2A^{-\frac34}-\frac{\sqrt2}{4\alpha_2^2}\sqrt[4]{A-\lambda I}A^{-\frac12}+\frac{\sqrt2}{4\alpha_2^3}A^{-\frac14}\right]$$

$$B_{1,0}=\frac{1}{4\alpha_1^3}A^{-\frac18}A^{-\frac34}, B_{2,0}=\frac{1}{4\alpha_2^3}A^{-\frac18}A^{-\frac34}, B'_{1,0}=-A^{-\frac38}\frac{\alpha_1\sqrt[4]{A-\lambda I}}{4\alpha_1^3}A^{-\frac34}$$

$$B'_{2,0}=-A^{-\frac38}\frac{\alpha_2\sqrt[4]{A-\lambda I}}{4\alpha_2^3}A^{-\frac34} \tag{26}$$

Regularized values of $\omega_j(t,\lambda)$ and its derivatives at zero defined by (14) denote by $\omega_{j,0}, \omega'_{j,0}, \omega''_{j,0}, \omega'''_{j,0}$ and for a shortcut of notations denote the values of $\omega_j(t,\lambda)$ and its derivatives at 1 by $\omega_j(1), \omega'_j(1), \omega''_j(1), \omega'''_j(1)$, respectively. Hence,

$$\omega_j(1,\lambda)\equiv\omega_j(1)=\begin{cases}e^{\alpha_i\sqrt[4]{A-\lambda I}}A^{\frac18}, & i=1,2,\ j=1,2\\ A^{-\frac34}, & i=1,2,\ j=3,4\end{cases},$$

$$\omega'_j(1,\lambda)\equiv\omega'_j(1)=\begin{cases}\alpha_i\sqrt[4]{A-\lambda I}e^{\alpha_i\sqrt[4]{A-\lambda I}}A^{\frac18}, & i=1,2,\ j=1,2\\ -\alpha_i\sqrt[4]{A-\lambda I}\,A^{-\frac34}, & i=1,2,\ j=3,4\end{cases}$$

$$\omega''_j(1,\lambda)\equiv\omega''_j(1)=\begin{cases}\left(\alpha_i\sqrt[4]{A-\lambda I}\right)^2e^{\alpha_i\sqrt[4]{A-\lambda I}}A^{\frac18}, & i=1,2,\ j=1,2\\ \left(\alpha_i\sqrt[4]{A-\lambda I}\right)^2A^{-\frac34}, & i=1,2,\ j=3,4\end{cases}$$

$$\omega'''_j(1,\lambda)\equiv\omega'''_j(1)=\begin{cases}\left(\alpha_i\sqrt[4]{A-\lambda I}\right)^3e^{\alpha_i\sqrt[4]{A-\lambda I}}A^{\frac18}, & i=1,2,\ j=1,2\\ \left(-\alpha_i\sqrt[4]{A-\lambda I}\right)^3A^{-\frac34}, & i=1,2,\ j=3,4\end{cases}$$

With all that notations substituting $y(t,\lambda)$ into (15), (18), (19) and then writing them in the matrix form, we have :

$$\left[\begin{pmatrix}\cos C & O & O & O\\ O & \cos C & O & O\\ O & O & I & O\\ O & O & O & I\end{pmatrix}\begin{pmatrix}\omega'''_{1,0} & \omega'''_{2,0} & \omega'''_{3,0} & \omega'''_{4,0}\\ \omega''_{1,0} & \omega''_{2,0} & \omega''_{3,0} & \omega''_{4,0}\\ \omega'''_1(1) & \omega'''_2(1) & \omega'''_3(1) & \omega'''_4(1)\\ \omega''_1(1) & \omega''_2(1) & \omega''_3(1) & \omega''_4(1)\end{pmatrix}-\right.$$

$$-\begin{pmatrix} \sin C & O & O & O \\ O & \sin C & O & O \\ O & O & I & O \\ O & O & O & I \end{pmatrix} \times$$

$$\times \begin{pmatrix} \omega_{1,0} & \omega_{2,0} & \omega_{3,0} & \omega_{4,0} \\ \omega'_{1,0} & \omega'_{2,0} & \omega'_{3,0} & \omega'_{4,0} \\ \lambda Q_1 \omega_1(1) & \lambda Q_1 \omega_2(1) & \lambda Q_1 \omega_3(1) & \lambda Q_1 \omega_4(1) \\ \lambda Q_2 \omega'_1(1) & \lambda Q_2 \omega'_2(1) & \lambda Q_2 \omega'_3(1) & \lambda Q_2 \omega'_4(1) \end{pmatrix} \Bigg] \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix} =$$

$$= \left[\left(\begin{matrix} \cos CA^{-\frac{1}{8}} B'''_{1,0} & \cos CA^{-\frac{1}{8}} B'''_{2,0} & O \\ \cos CA^{-\frac{1}{8}} B''_{1,0} & \cos CA^{-\frac{1}{8}} B''_{2,0} & O \\ O & O & \frac{A^{-\frac{7}{8}}}{4\alpha_1^3}\left[\left(\alpha_1 \sqrt[4]{A-\lambda I}\right)^3 + \lambda Q_1\right] \\ O & O & \frac{A^{-\frac{7}{8}}}{4\alpha_1^3}\left[\left(\alpha_1 \sqrt[4]{A-\lambda I}\right)^2 + \lambda Q_2 \alpha_1 \sqrt[4]{A-\lambda I}\right] \end{matrix} \right. \right.$$

$$\left. \begin{matrix} O & O & O \\ O & O & O \\ \frac{A^{-\frac{7}{8}}}{4\alpha_2^3}\left[\left(\alpha_2 \sqrt[4]{A-\lambda I}\right)^3 + \lambda Q_1\right] & I & O \\ \frac{A^{-\frac{7}{8}}}{4\alpha_2^3}\left[\left(\alpha_2 \sqrt[4]{A-\lambda I}\right)^2 + \lambda Q_2\left(\alpha_2 \sqrt[4]{A-\lambda I}\right)\right] & O & I \end{matrix} \right) -$$

$$\left. - \begin{pmatrix} \sin CA^{-\frac{1}{8}} B_{1,0} & \sin CA^{-\frac{1}{8}} B_{2,0} & O & O & O & O \\ \sin CA^{-\frac{1}{8}} B'_{1,0} & \sin CA^{-\frac{1}{8}} B'_{2,0} & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{pmatrix} \right] \times$$

$$\times \begin{pmatrix} A^{\frac{1}{8}} \int_0^1 e^{\alpha_1 \sqrt[4]{A-\lambda I} s} h(s) ds \\ A^{\frac{1}{8}} \int_0^1 e^{\alpha_2 \sqrt[4]{A-\lambda I} s} h(s) ds \\ A^{\frac{1}{8}} \int_0^1 e^{\alpha_1 \sqrt[4]{A-\lambda I}\,(1-s)} h(s) ds \\ A^{\frac{1}{8}} \int_0^1 e^{\alpha_2 \sqrt[4]{A-\lambda I}\,(1-s)} h(s) ds \\ h_1 \\ h_2 \end{pmatrix} \tag{27}$$

In the last column in (27) the integral terms are from $D(A^{\frac{1}{8}})$ and to get terms from H we put before them the factors $A^{\frac{1}{8}}$, that is why there appear the factors $A^{-\frac{1}{8}}$ in front of $B's$ in matrices within braces in right of (27).

Denote the matrix within the brackets in the left hand side of (27) by D, the difference of matrices in the brackets on the right of (27) by $\widetilde{B}_1 - \widetilde{B}_2$ and column matrix on the right by $\widetilde{H}$, respectively. Each term of $\tilde{H}$ is from H, because linear operators $h(s) \to \int_0^1 e^{\alpha_i \sqrt[4]{A-\lambda I} s} h(s) ds$, $h(s) \to \int_0^1 e^{\alpha_i \sqrt[4]{A-\lambda I}(1-s)} h(s) ds$ continuously act from $L_2(H,(0,1))$ to $H_{\frac{1}{8}}$ as the adjoint to the operator $f \to e^{\alpha_i \sqrt[4]{A-\lambda I} s} f$ which continuously acts from $H_{-\frac{1}{8}}$ to $L_2(H,(0,1))$. Hence,

$$\begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix} = D^{-1}\left(\widetilde{B}_1 - \widetilde{B}_2\right)\tilde{H}. \tag{28}$$

Denote in (24), the matrix in front of the vector $\begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}$ by J and the second term by G

$$R_\lambda\left(L_s^0\right)\tilde{h} = JD^{-1}(\widetilde{B}_1 - \widetilde{B}_2)\widetilde{H} + G, \tag{29}$$

Let $\cos C, Q_1A^{-\frac{3}{4}}, Q_2A^{-\frac{1}{2}}$ be bounded in H. For $Im\lambda \neq 0$ the inverse operator D^{-1} exists in H and it may be easily verified under above conditions is bounded in H. $\widetilde{H}$ is continuous from $\mathcal{H}$ to H^6. Because the first four terms are continuous from $L_2(H,(0,1))$ to H, the last two terms are continuous from H to H.

In the operator matrix $\widetilde{B}_2$ the term $\sin CA^{-\frac{1}{8}}B_{i,0}$ is $\sin CA^{-\frac{1}{8}}B_{i,0} = \frac{\sin CA^{-1}}{4\alpha_i^3}$ and in $\widetilde{B}_1$ the term $\cos CA^{-\frac{1}{8}}B_{i,0}^{'''}$ (in other terms too) is representable as $\cos CA^{-\frac{1}{8}}B_{i,0}^{'''} = \cos C(\beta I + F)$ where β is a number defined by the coefficients of the terms $\widetilde{B}_1$, F is a bounded operator in H. $D^{-1}(\widetilde{B}_1 - \widetilde{B}_2)$ is completely continuous from H^6 to H^4 because of $\cos C$ is compact by our assumption, $Q_1A^{-\frac{7}{8}}$ is compact because of $Q_1A^{-\frac{7}{8}} = Q_1A^{-\frac{3}{4}}A^{-\frac{1}{8}}$, where $Q_1A^{-\frac{3}{4}}$ is bounded and $A^{-\frac{1}{8}}$ is compact, since A^{-1} is compact, and $Q_2A^{-\frac{5}{8}}$ is compact, since by our assumption. Compactness of $Q_2A^{-\frac{4}{8}}$ and $Q_2A^{-\frac{5}{8}}$ a obvious under stated requirements of theorem. J is continuous H^4 to $\mathcal{H}$, G is completely continuous from $\mathcal{H}$ to $\mathcal{H}$ since the first term is compact in $L_2(H,(0,1))$ as an integral operator which kernel $G(t,s,\lambda)$ with values from H is compact for each (t,s,λ) and next two terms are compact in H because of the conditions $\cos C, Q_1A^{-\frac{3}{4}}, Q_2A^{-\frac{1}{2}}$ are compact.

So, it follows that $R_\lambda\left(L_s^0\right)$ is compact if and only if $\cos C$, $Q_1A^{-\frac{3}{4}}, Q_2A^{-\frac{1}{2}}$ are compact. □

Note 2.2. Since $q(t)$ is bounded in $L_2(H,(0,1))$, then in virtue of relation

$$R_\lambda(L_s) = R_\lambda\left(L_s^0\right) - R_\lambda(L_s)\,QR_\lambda\left(L_s^0\right), \tag{30}$$

where $L_s = L_0^s + Q$,

$$QY = \{q(t)\,y(t),\ 0, 0\}, \tag{31}$$

statement of Theorem 2.4 holds also for selfadjoint extensions L_s of the minimal operator $L = L_0 + Q$.

3. Asymptotics of eigenvalue distribution of a class of selfadjoint extensions, definition of the domain of selfadjoint extensions with resolvents from σ_p and extensions with continuous spectrum

Take in boundary conditions (15) $C = (C_1, C_2)$, where C_1 and C_2 are operators on H, moreover $C_1 = \frac{\pi}{2}I$ (I is an identity operator in H), $C_2 = arctg(-\sqrt{2}A)$, then the corresponding selfadjoint extension will be given by the boundary conditions $y(0) = y''(0) = 0$. The eigenvalue problem corresponding to that operator is:

$$ly = \lambda y \tag{32}$$

$$y(0) = y''(0) = 0. \tag{33}$$

$$-y'''(1) = \lambda Q_1 y(1),\ \ y''(1) = \lambda Q_2 y'(1) \tag{34}$$

Note here that boundary conditions (33),(34) are obtained from (4),(5) with indicated above choice of C and by the setting $b = 1$ and making a change of variable $1 - t = x$.

The operator corresponding to that problem for $q(t) \equiv 0$ denote by L_1^0 which in virtue of Theorems 2.3 and 2.4 is self-adjoint and has purely discrete spectrum.

Now we study the asymptotics of eigenvalues of that operator.

The solution of (32) is

$$y(t,\lambda) = e^{\alpha_1\sqrt[4]{A-\lambda I}t}A^{\frac{1}{8}}f_1 + e^{\alpha_2\sqrt[4]{A-\lambda I}t}A^{\frac{1}{8}}f_2 +$$

$$+e^{-\alpha_1\sqrt[4]{A-\lambda I}(t-1)}A^{-\frac{3}{4}}g_1+e^{-\alpha_2\sqrt[4]{A-\lambda I}(t-1)}A^{-\frac{3}{4}}g_2 \quad (35)$$

with $f_1,\ f_2, g_1, g_2 \in H$ defined as in (22).

Substituting the function (35) in the the boundary conditions (33),

$$A^{\frac{1}{8}}f_1+A^{\frac{1}{8}}f_2+e^{\alpha_1\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_1+e^{\alpha_2\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_2=0 \quad (36)$$

$$\left(\alpha_1\sqrt[4]{A-\lambda I}\right)^2A^{\frac{1}{8}}f_1+\left(\alpha_2\sqrt[4]{A-\lambda I}\right)^2A^{\frac{1}{8}}f_2+\left(\alpha_1\sqrt[4]{A-\lambda I}\right)^2e^{\alpha_1\sqrt[4]{A-\lambda I}}\times$$

$$\times A^{-\frac{3}{4}}g_1+\left(\alpha_2\sqrt[4]{A-\lambda I}\right)^2e^{\alpha_2\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_2=0 \quad (37)$$

rewrite (36), (37) as

$$\left(A^{\frac{1}{8}}f_1+e^{\alpha_1\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_1\right)+\left(A^{\frac{1}{8}}f_2+e^{\alpha_2\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_2\right)=0$$

$$\left(\alpha_1\sqrt[4]{A-\lambda I}\right)^2\left(A^{\frac{1}{8}}f_1+e^{\alpha_1\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_1\right)+$$

$$+\left(\alpha_2\sqrt[4]{A-\lambda I}\right)^2\left(A^{\frac{1}{8}}f_2+e^{\alpha_2\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_2\right)=0$$

hence

$$f_1=-A^{-\frac{1}{8}}e^{\alpha_1\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_1=-A^{-\frac{7}{8}}e^{\alpha_1\sqrt[4]{A-\lambda I}}g_1=-A^{-\frac{7}{8}}e^{\sqrt[4]{\lambda I-A}}g_1, \quad (38)$$

$$f_2=-A^{-\frac{1}{8}}e^{\alpha_2\sqrt[4]{A-\lambda I}}A^{-\frac{3}{4}}g_2=-A^{-\frac{7}{8}}e^{\alpha_2\sqrt[4]{A-\lambda I}}g_2=$$

$$=-A^{-\frac{7}{8}}e^{i\alpha_1\sqrt[4]{A-\lambda I}}g_2=-A^{-\frac{7}{8}}e^{i\sqrt[4]{\lambda I-A}}g_2 \quad (39)$$

Taking in (35) f_1 and f_2 as in (38), (39) yields

$$y(t,\lambda)=-e^{\alpha_1\sqrt[4]{A-\lambda I}t}A^{\frac{1}{8}}A^{-\frac{7}{8}}e^{\sqrt[4]{\lambda I-A}}g_1-e^{\alpha_2\sqrt[4]{A-\lambda I}t}A^{\frac{1}{8}}A^{-\frac{7}{8}}e^{i\sqrt[4]{\lambda I-A}}g_2+$$

$$+e^{-\sqrt[4]{\lambda I-A}(t-1)}A^{-\frac{3}{4}}g_1+e^{-\sqrt[4]{\lambda I-A}(t-1)}A^{-\frac{3}{4}}g_2=-e^{\sqrt[4]{\lambda I-A}t}A^{-\frac{3}{4}}e^{\sqrt[4]{\lambda I-A}}g_1+$$

$$+e^{-\sqrt[4]{\lambda I-A}(t-1)}A^{-\frac{3}{4}}g_1-e^{i\sqrt[4]{\lambda I-A}t}A^{-\frac{3}{4}}e^{i\sqrt[4]{\lambda I-A}}g_2+$$

$$+e^{-i\sqrt[4]{\lambda I-A}(t-1)}A^{-\frac{3}{4}}g_2=-2sh\sqrt[4]{\lambda I-A}tA^{-\frac{3}{4}}e^{\sqrt[4]{\lambda I-A}}g_1-2i\sin\sqrt[4]{\lambda I-A}tA^{-\frac{3}{4}}e^{i\sqrt[4]{\lambda I-A}}g_2.$$

Denoting $F_1=-2e^{\sqrt[4]{\lambda I-A}}A^{-\frac{3}{4}}g_1$, $\quad F_2=2ie^{\sqrt[4]{\lambda I-A}}A^{-\frac{3}{4}}g_2$, $F_1,\ F_2\epsilon H$ we have

$$y(t)=sh\sqrt[4]{\lambda I-A}tF_1+\sin\sqrt[4]{\lambda I-A}tF_2 \quad (40)$$

Writing that solution in boundary conditions (34), from expansion of a selfadjoint operator with discrete spectrum $A=\sum\limits_{k=1}^{\infty}\gamma_k(\cdot,\varphi_k)\varphi_k$, where γ_k are eigenvalues and φ_k are orthonormal basis formed by the eigenvectors of A, we have

$$-\sqrt[4]{\lambda-\gamma_k}^3ch\sqrt[4]{\lambda-\gamma_k}(F_1,\ \varphi_k)+\sqrt[4]{\lambda-\gamma_k}^3\cos\sqrt[4]{\lambda-\gamma_k}(F_2,\ \varphi_k)=$$

$$=\lambda sh\sqrt[4]{\lambda-\gamma_k}\left(F_1,\ Q_1\varphi_k\right)+\lambda\sin\sqrt[4]{\lambda-\gamma_k}\left(F_2,\ Q_1\varphi_k\right) \quad (41)$$

and

$$\sqrt{\lambda-\gamma_k}sh\sqrt[4]{\lambda-\gamma_k}(F_1,\ \varphi_k)-\sqrt{\lambda-\gamma_k}\sin\sqrt[4]{\lambda-\gamma_k}(F_2,\ \varphi_k)=$$

$$\lambda \sqrt[4]{\lambda-\gamma_k} ch \sqrt[4]{\lambda-\gamma_k}\left(F_1,\ Q_2\varphi_k\right)+\lambda \sqrt[4]{\lambda-\gamma_k}\cos\sqrt[4]{\lambda-\gamma_k}\left(F_2,\ Q_2\varphi_k\right).$$

For simplicity of calculations take here $Q_1 = Q_2 = A^{\alpha}$, and $0 < \alpha < \frac{1}{2}$, which is important for holding the hypothesis of Theorem 2.4.

By replacing

$$\sqrt[4]{\lambda-\gamma_k} = z, (F_1,\ \varphi_k) = c_{1k},\ (F_2,\ \varphi_k) = c_{2k} \tag{42}$$

in virtue of $Q_i\varphi_k = {\gamma_k}^{\alpha}\varphi_k,\ i = 1,2$ we have

$$-z^3 chz c_{1k} + z^3\cos z c_{2k} = \left(z^4+\gamma_k\right) shz \gamma_k^{\alpha} c_{1k} + \left(z^4+\gamma_k\right)\sin z \gamma_k^{\alpha} c_{2k} \tag{43}$$

$$zshz c_{1k} - z\sin z c_{2k} = \left(z^4+\gamma_k\right) chz \gamma_k^{\alpha} c_{1k} + \left(z^4+\gamma_k\right)\cos z \gamma_k^{\alpha} c_{2k} \tag{44}$$

which is a system of linear algebraic equations in c_{1k}, c_{2k} and has nonzero roots if and only if the characteristic determinant $\Delta(z)$ of boundary value problem (32)-(34) the (determinant formed by coefficients of (43), (44)) is zero:

$$\Delta(z) = \begin{vmatrix} -z^3 chz - \left(z^4+\gamma_k\right) shz\ \gamma_k^{\alpha} & z^3\cos z - \left(z^4+\gamma_k\right)\sin z\ \gamma_k^{\alpha} \\ zshz - \left(z^4+\gamma_k\right) chz\ \gamma_k^{\alpha} & -zsinz - \left(z^4+\gamma_k\right)\cos z\ \gamma_k^{\alpha} \end{vmatrix} = 0 \tag{45}$$

After simplifications in (45)

$$tgz = \frac{-2z^3\left(z^4+\gamma_k\right)\ \gamma_k^{\alpha} + z^4 thz - \left(z^4+\gamma_k\right)^2 \gamma_k^{2\alpha} thz}{z^4 + 2z\left(z^4+\gamma_k\right)\ \gamma_k^{\alpha} thz - \left(z^4+\gamma_k\right)^2\gamma_k^{2\alpha}} \tag{46}$$

Since λ as the eigenvalue of a selfadjoint operator must be real, feasible values for z are $y, -y, iy, -iy (y > 0)$ or $\pm y \pm iy$.

Thus, (46) may have only real, imaginary roots and the roots of the form $y \pm iy$, where y iare real. It can't have other complex roots with exception of these roots, because complex roots will give complex eigenvalues for a selfadjoint operator, which is impossible.

Hence, setting $\sqrt[4]{\lambda_{k,j}-\gamma_k} = z_{k,j}$ from (46) for real roots as $|z| \to \infty$ we have $z = z_{k,j} \sim \frac{\pi}{4} + \pi j + O\left(\frac{1}{k}\right), k$ is an entire number large in modulus .

Writing in (46) iz in place of z shows that if z is a real root, then iz is also a root of that equation, thus for imaginary roots

$$z = iz_{k,j} \sim \left(\frac{\pi}{4} + \pi j + O\left(\frac{1}{k}\right)\right) i,$$

But in virtue of (42)

$$\lambda = z^4 + \gamma_k$$

which shows that real and imaginary roots of (46) result in the same eigenvalues but in linearly dependent eigenvectors of the operator and that is why geometric multiplicity of each eigenvalue corresponding to those roots is 2.

Writing in (46) $y \pm iy$ in place of z, after simplifications we get

$$-4y^4\left[\frac{sin2y}{2} - \frac{sh2y}{2i}\right] + 4iy^2\left(y+iy\right)\left(\gamma_k - 4y^4\right){\gamma_k}^{\alpha}\left[\frac{cos2y}{2i} + \frac{ch2y}{2}\right]$$

$$+2y\left(y+iy\right)\left(\gamma_k - 4y^4\right){\gamma_k}^{\alpha}\left[\frac{cos2y}{2i} - \frac{ch2y}{2i}\right] + 4y^4\left[\frac{sh2y}{2} + \frac{isin2y}{2}\right] +$$

$$+\left(\gamma_k - 4y^4\right)^2 {\gamma_k}^{2\alpha}\left[\frac{sh2y}{2} + \frac{isin2y}{2}\right] - \left(\gamma_k - 4y^4\right)^2 {\gamma_k}^{2\alpha}\left[\frac{sin2y}{2} + \frac{sh2y}{2i}\right] = 0 \tag{47}$$

It is easy to see that (47) has the form $K = iK$, where K is real and

$$K = -4y^4\frac{sin2y}{2} - 4y^3\left(\gamma_k - 4y^4\right){\gamma_k}^{\alpha}\left[\frac{cos2y}{2} + \frac{ch2y}{2}\right] + 2\left(\gamma_k - 4y^4\right){\gamma_k}^{\alpha} y\left[\frac{cos2y}{2} - \frac{ch2y}{2}\right] +$$

$$+4y^4\frac{sh2y}{2} + \left(\gamma_k - 4y^4\right)^2 {\gamma_k}^{2\alpha}\frac{sh2y}{2} - \left(\gamma_k - 4y^4\right)^2 {\gamma_k}^{2\alpha}\frac{sin2y}{2}$$

Equation (47) has solution if and only if

$$K = 0. \tag{48}$$

But for the large values of y (48) is equivalent to

$$4y^7{\gamma_k}^{\alpha}e^{2y} + y^7{\gamma_k}^{\alpha}e^{2y} + 8y^8{\gamma_k}^{2\alpha}e^{2y} + 4y^7{\gamma_k}^{\alpha}e^{2y} + y^5{\gamma_k}^{\alpha}e^{2y}$$

which has no positive roots. But by applying Descarte's rule of signs we see that (47) has small in modulus roots, that is way they don't change asymptotics of eigenvalues.

Hence, we get the next theorem

Theorem 3.1. *The geometric multiplicity of eigenvalues $\lambda_{k,j}$ of the operator L_1^0 is two and the following asymptotic formula is true:*

$$\lambda_{k,j} = \gamma_k + z_{k,j}^4, z_{k,j} \sim \begin{cases} \pi j + \frac{\pi}{4} + O\left(\frac{1}{k}\right), \\ i\left(\pi j + \frac{\pi}{4} + O\left(\frac{1}{k}\right)\right) \end{cases} \tag{49}$$

as $k, j \to \infty$ and

$$\lambda_{k,j} = \gamma_k + \eta_{k,j}^4,$$

where $\eta_{k,j}$ are small in modulus roots of (46).

Let for eigenvalues $\gamma_k, k = 1, \infty$, of the operator A as $k \to \infty$ holds: $\gamma_k \sim ck^{\beta}$, $(\beta > 0)$.

Analogous to [4], [21] the following statements may be justified:

Lemma 3.2. *For the distribution function $N(\lambda) = \sum_{\lambda_{n<\lambda}} 1$ of the eigenvalues of the operator L_1^0 the following relation*

$$N(\lambda) \sim C_1\lambda^{\frac{4+\beta}{4\beta}} \tag{50}$$

is valid for sufficiently large λ .

Lemma 3.3. *For large values of n the following asymptotic formula is true*

$$\lambda_n \sim C_2 n^{\frac{4\beta}{4+\beta}}, n \to \infty. \tag{51}$$

Setting $L_1 = L_1^0 + Q$, one can easily see that the hypotheses of Theorem 2.3 and Theorem 2.4 hold also for L_1. Denote the eigenvalues of L_1 by $\{\mu_n\}$: $\mu_1 < \mu_2 < \dots$.

From (50) it follows that inverse of L_1^0 is from Neumann von Shcetten class σ_p, if and only if $p \cdot \frac{4\beta}{4+\beta} > 1$ or $\beta > \frac{4}{4p-1}$, which means that $A^{-\frac{1}{4}} \in \sigma_{4p-1}$.

Lemma 3.4. *The operator* $\left(L_1^0\right)^{-1}$ *is from* σ_p *if and only if* $A^{-\frac{1}{4}} \in \sigma_{4p-1}$.
Because of relation (30) *that statement holds also for the operator* L_1.

Now we can prove the next theorem.

Theorem 3.5. *Let* $A^{-\frac{1}{4}} \in \sigma_{4p-1}$. *Then* $R_\lambda\left(L_s^0\right)$, $R_\lambda\left(L_s\right) \epsilon \sigma_p\left(H_1\right)$ *if and only if cosC,* $Q_1 A^{-\frac{3}{4}}, Q_2 A^{-\frac{1}{2}}$ *are from* σ_p.

Proof. Setting in formula (29) $C = (C_1, C_2)$, $C_1 = \frac{\pi}{2} I$, $C_2 = arctg(-\sqrt{2}A)$ and subtracting obtained by that way formula from (29) we get

$$R_\lambda\left(L_0^s\right)\tilde{h} = R_\lambda\left(L_0\right)\tilde{h} + JD^{-1}\left(\widetilde{B}_1 - \widetilde{B}_2\right)\tilde{H} - JD_0{}^{-1}\left(\widetilde{B}_{01} - \widetilde{B}_{02}\right)\tilde{H} \tag{52}$$

where D_0 and $\widetilde{B}_{01} - \widetilde{B}_{020}$ are obtained from D and $\widetilde{B}_1 - \widetilde{B}_2$, respectively, by taking there $C = (C_1, C_2)$, $C_1 = \frac{\pi}{2} I$, $C_2 = arctg(-\sqrt{2}A)$. Writing (52) in the open form and with Lemma 3.4 in mind one can easily see that $R_\lambda\left(L_s^0\right)$ is from σ_p if and only if $\cos C$, $Q_1 A^{-\frac{3}{4}}$, $Q_2 A^{-\frac{1}{2}}$ are from σ_p. Since Q is bounded in $\mathcal{H}$, the statement of the theorem is true also for $R_\lambda(L_s)$. □

Theorem 3.6. *If the inverse of the operator A is compact, then for any closed set F of the real axis, there exists a selfadjoint extension of the minimal operator* L_0, *which spectrum coincides with F.*

Proof. Let $C = \begin{pmatrix} \frac{\pi}{2} I & O \\ O & f(A) \end{pmatrix}$, where $f(\mu)$ is any function Borel measurable on $(1, \infty)$. Then boundary conditions (15) take the form

$$y\left(0\right) = 0, \tag{53}$$

$$cosf\left(A\right) A^{\frac{3}{8}}\left(y''(0) - \sqrt{2}A^{\frac{1}{4}} y'(0) + A^{\frac{1}{2}} y(0)\right) - sinf\left(A\right) A^{-\frac{3}{8}} y'\left(0\right) = 0 \tag{54}$$

Let us corresponding to f selfadjoint extension denote by L_f. Obviously, λ is an eigenvalue of L_f , if in addition to (53) and (54) holds (34). After substituting $y(t, \lambda)$ from (35) into those relations and denoting by $K_\lambda^f(A)$ the determinant of the matrix formed by the coefficients of f_1, f_2, g_1, g_2 in relations (53), (54), (34), it is easy to see that λ is an eigenvalue of L_f if and only if the zero is an eigenvalue of $K_\lambda^f(A)$.

Let F be a closed set of real zeros and $\{\lambda_k\}$ is a set dense in F. Construct the function $f_F(\gamma)$ in the following way:

$$f_F(\gamma) = f_k,\ for\ \gamma_{k-1} < \gamma \leq \gamma_k$$

with f_k defined from the equation

$$K_{\lambda_k}^f(\gamma_k) = 0$$

But from the last relation it follows that 0 is the eigenvale of the operator $K_{\lambda_k}^{f_F}$, hence λ_k is an eigenvalue of the operator L_{f_F}.

Therefore, the set $\{\lambda_k\}$ as well as F are contained in the spectrum of L_{f_F}. The Theorem is proved. □

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