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Several topological properties of H-spaces

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Abstract. An H-space, denoted as $(\mathbb{R}, \tau_A)$, has $\mathbb{R}$ as its point set and a basis consisting of usual open interval neighborhoods at points of A while taking Sorgenfrey neighborhoods at points of $\mathbb{R} \setminus A$. In this paper, we mainly discuss some topological properties of H-spaces. In particular, we prove that, for any subset $A \subset \mathbb{R}$,

(1) $(\mathbb{R}, \tau_A)$ is zero-dimensional iff $\mathbb{R} \setminus A$ is dense in $(\mathbb{R}, \tau_E)$, where τ_E is the natural topology on $\mathbb{R}$;

(2) $(\mathbb{R}, \tau_A)$ is locally compact iff $(\mathbb{R}, \tau_A)$ is a k_ω-space;

(3) if $(\mathbb{R}, \tau_A)$ is σ-compact, then $\mathbb{R} \setminus A$ is countable and nowhere dense; if $\mathbb{R} \setminus A$ is countable and scattered in the real line, then $(\mathbb{R}, \tau_A)$ is σ-compact;

(4) $\prod_{i=1}^{\infty}(\mathbb{R}, \tau_{A_i})$ is perfectly subparacompact, where each A_i is a subset of $\mathbb{R}$;

(5) there exists a subset $A \subset \mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is not quasi-metrizable;

(6) $(\mathbb{R}, \tau_A)$ is metrizable if and only if $(\mathbb{R}, \tau_A)$ is a β-space.

1. Introduction

The usual topology on $\mathbb{R}$, induced by the standard absolute value, is coarser than the topology of the Sorgenfrey line which has been studied intensively. It is well known that Sorgenfrey line has a basis consisting of all half-open intervals of the form $[a, b)$, where $a < b$. The topology of an H-space, mentioned in [14], is between the usual topology of the set of real numbers and the topology of the Sorgenfrey line $\mathbb{S}$, was described by Hattori in [10]. The H-space, denoted as $(\mathbb{R}, \tau_A)$, has $\mathbb{R}$ as its point set and a basis consisting of usual open interval neighborhoods at points of A while taking Sorgenfrey neighborhoods at points of $\mathbb{R} \setminus A$, that is, the topology τ_A is defined as follows:

(1) For each $x \in A$, $\{(x - \varepsilon, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at x, and

(2) for each $x \in \mathbb{R} \setminus A$, $\{[x, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at x.

Chatyrko and Hattori were first who began to study the properties of such spaces, many interesting results were obtained, see [5] and [6]. In particular, for any $A \subset \mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is a regular,

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hereditarily Lindelöf, hereditarily separable and Baire space. Moreover, for any closed subset A of $\mathbb{R}$, they proved that $(\mathbb{R}, \tau_A)$ is homeomorphic to the Sorgenfrey line $\mathbb{S}$ if and only if A is countable. In [14], Kulesza made an improvement and a summary on the basis of Chatyrko and Hattori's work. He called spaces of this kind H-spaces and demonstrated properties of H-spaces with respect to homeomorphisms, functions, completeness and reversibility. In particular, Kulesza proved that $(\mathbb{R}, \tau_A)$ is homeomorphic to $\mathbb{S}$ if and only if A is scattered, and $(\mathbb{R}, \tau_A)$ is complete if and only if $\mathbb{R} \setminus A$ is countable, which implies that $(\mathbb{R}, \tau_{\mathbb{P}})$ is complete, where $\mathbb{P}$ is the set of the irrational numbers. Then Bouziad and Sukhacheva in [1] gave characterizations of some topological properties of $(\mathbb{R}, \tau_A)$, such as total imperfectness, local compactness. In this paper, we continue the work of Chatyrko and Hattori by suggesting additional information about the spaces $(\mathbb{R}, \tau_A)$. The remaining of this paper is organized as follows.

Section 2 is dedicated to outline some concepts and terminologies. In Section 3, we mainly discuss some topological properties of H-spaces, such as zero-dimensionality, σ-compactness, k_ω-property, perfectness, quasi-metrizability. In particular, we give the characterizations of A or $\mathbb{R} \setminus A$ such that $(\mathbb{R}, \tau_A)$ has topological properties of zero-dimensionality, σ-compactness, and k_ω-property respectively. Moreover, we show that $\prod_{i=1}^{\infty}(\mathbb{R}, \tau_{A_i})$ is perfectly subparacompact, where each A_i is a subset of $\mathbb{R}$. Further, we discuss some generalized metric properties of $(\mathbb{R}, \tau_A)$, and prove that there exists a subset $A \subset \mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is not quasi-metrizable. In Section 4, we pose some interesting questions about H-spaces.

2. Preliminaries

In this section, we introduce the necessary notations and terminologies. First of all, let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all positive integers, all integers and all real numbers, respectively. Throughout this paper, all spaces are assumed to be regular and all results are in the standard set theory ZFC. For undefined terminologies, the reader refer to [7] and [9].

Definition 2.1. ([7]) Let X be a topological space.

(1) X is called *zero-dimensional* if it has a base of open sets that are at the same time open and closed in it (that is, $\operatorname{ind} X = 0$).

(2) X is called *strongly zero-dimensional* if, for each non-empty closed subset A and open subset U with $A \subset U$, there exists a clopen subset V such that $A \subset V \subset U$ (that is, $\operatorname{Ind} X = 0$).

(3) X is called a *Baire space* if every intersection of a countable collection of open dense sets in X is also dense in X.

(4) X is called *locally compact*, if every point x of X has a compact neighbourhood, i.e., there exist an open set U and a compact set K, such that $x \in U \subseteq K$.

(5) X is called a *k_ω-space* if there exists a family of countably many compact subsets $\{K_n : n \in \mathbb{N}\}$ of X such that each subset F of X is closed in X provided that $F \cap K_n$ is closed in K_n for each $n \in \mathbb{N}$.

(6) X is *σ-compact* if it is the union of countably many compact subsets of X.

(7) X is *Lindelöf* if every open cover of X has a countable subcover.

Clearly, each k_ω-space is σ-compact and each σ-compact is Lindelöf.

Definition 2.2. ([7, 9]) (1) A space X is *subparacompact* if each open cover of X has a σ-locally finite closed refinement.

(2) A space X is *perfect* if each closed subset of X is a G_δ in X.

(3) A space X is *perfectly subparacompact* if it is perfect and subparacompact.

(4) A space X is *weakly θ-refinable* if for each open cover $\mathscr{U}$ of X, there exists an open cover $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}(n)$ of X which refines $\mathscr{U}$ and which has the property that if $x \in X$, then there exists an $n \in \mathbb{N}$ such that x belongs to exactly k members of $\mathscr{V}(n)$ for some $k \in \mathbb{N}$.

(5) A family $\mathscr{U}$ of open sets in X is called *interior-preserving* if for $\mathscr{F} \subset \mathscr{U}$ and $y \in \bigcap \mathscr{F}$, $\bigcap \mathscr{F}$ is an open neighborhood of y.

Definition 2.3. ([7]) A family $\mathscr{P}$ of subsets of a space X is a *network* for X if for each point $x \in X$ and any neighbhorhood U of x there is an $P \in \mathscr{P}$ such that $x \in P \subset U$. The *network weight* of a space X is defined as the smallest cardinal number of the form $|\mathscr{P}|$, where $\mathscr{P}$ is a network for X, this cardinal number is denoted by $nw(X)$.

Definition 2.4. ([9]) Recall that (X, τ) is a *β-space* if there exists a function $g : \omega \times X \to \tau$ such that if $x \in g(n, x_n)$ for every $n \in \omega$ then the sequence $\{x_n\}$ has a cluster point in X.

Definition 2.5. ([6]) Let A be a subset of $\mathbb{R}$ of the real number. Define the topology τ_A on $\mathbb{R}$ as follows:

(1) For each $x \in A$, $\{(x - \epsilon, x + \epsilon) : \epsilon > 0\}$ is the neighborhood base at x,

(2) For each $x \in \mathbb{R} - A$, $\{[x, x + \epsilon) : \epsilon > 0\}$ is the neighborhood base at x.

Then $(\mathbb{R}, \tau_A)$ is called an *H-space*. The point x is called an $\mathbb{R}$-*point*, if $x \in A$, otherwise, x is called an $\mathbb{S}$-*point*.

Let τ_E and τ_S denote the usual (Euclidean) topology of $\mathbb{R}$ and the topology of the Sorgenfrey line $\mathbb{S}$ respectively. It is clear that $\tau_A = \tau_E$ if $A = \mathbb{R}$ and $\tau_A = \tau_S$ if $A = \emptyset$. And it is also obvious that $\tau_E \subset \tau_A \subset \tau_S$.

Some topological properties of $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$ are shown in the Table 1 below.

Table 1: Some topological properties of $(\mathbb{R}, \tau_E)$ **and** $(\mathbb{R}, \tau_S)$

Number	Property	$(\mathbb{R}, \tau_E)$	$(\mathbb{R}, \tau_S)$
1	metrizable	Yes	No
2	Hereditarily Separable	Yes	Yes
3	Normality	Yes	Yes
4	Lindelöff	Yes	Yes
5	Baire Space	Yes	Yes
6	Zero-dimension	No	Yes
7	Compactness	No	No
8	Countably Compact	No	No
9	Local Compactness	Yes	No
10	Sequential Compactness	No	No
11	Paracompactness	Yes	Yes
12	σ-Compactness	Yes	No
13	Connectedness	Yes	No
14	Path Connectedness	Yes	No
15	Local Connectedness	Yes	No
16	Every compact subset is countable	No	Yes
17	First countability	Yes	Yes

By Table 1, it is easy to see that, for every subset A of real numbers, an H-space is always a hereditarily separable, paracompact, Lindelöff, normal and first-countable space. And we also know that, an H-space is always not a compact, countably compact or sequentially compact space for any subset A of $\mathbb{R}$. According to [6, Proposition 2.3], H-space $(\mathbb{R}, \tau_A)$ is second-countable if and only if $\mathbb{R} - A$ is countable.

3. The main results

In this section, we mainly discuss some topological properties of H-spaces, such as zero-dimensionality, σ-compactness, k_ω-property, perfectness, quasi-metrizability. First, we give an obvious lemma.

Lemma 3.1. *Let D be a dense subset of* $(\mathbb{R}, \tau_A)$. *Then D is dense in* $(\mathbb{R}, \tau_E)$ *and* $(\mathbb{R}, \tau_S)$.

Proof. Obviously, D is a dense subset of $(\mathbb{R}, \tau_E)$ since τ_A is finer than τ_E. In order to prove D being dense in $(\mathbb{R}, \tau_S)$, take an arbitrary non-empty open subset U in τ_S, then there exists a non-empty open subset $V \subset U$ in τ_E, hence $\emptyset \neq V \cap D \subset U \cap D$. Therefore, D is also dense in $(\mathbb{R}, \tau_S)$. □

By Lemma 3.1, we have the following corollary.

Corollary 3.2. *For an arbitrary subset A of $\mathbb{R}$, we have $d(\mathbb{R}, \tau_A) = d(\mathbb{R}, \tau_E) = d(\mathbb{R}, \tau_S) = \omega$.*

Since $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$ are all Baire, we have the following corollary.

Corollary 3.3. ([6, Proposition 3.1]) *For an arbitrary subset A of $\mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is a Baire space.*

Proposition 3.4. *For an arbitrary subset A of $\mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is homeomorphic to $(\mathbb{R}, \tau_E)$ if and only if $A = \mathbb{R}$.*

Proof. Assume that $(\mathbb{R}, \tau_A)$ is homeomorphic to $(\mathbb{R}, \tau_E)$ and $A \neq \mathbb{R}$. Hence $\mathbb{R} \setminus A \neq \emptyset$. Take an arbitrary point $a \in \mathbb{R} \setminus A$. Then $(-\infty, a)$ is an open and closed subset in $(\mathbb{R}, \tau_A)$. Hence $(\mathbb{R}, \tau_A)$ is not connected. However, $(\mathbb{R}, \tau_E)$ is connected. Hence $A = \mathbb{R}$. □

Now, we can prove one of the main results of this paper, which gives a characterization of subset $\mathbb{R} \setminus A$ such that $(\mathbb{R}, \tau_A)$ is zero-dimensional.

The following lemma is evident.

Lemma 3.5. *Let $A \subset \mathbb{R}$ and $x, y \in \mathbb{R}$ such that $x < y$.*

(1) *If $(x, y) \subset A$ then $ind(\mathbb{R}, \tau_A) = 1$.*

(2) *If $\{x, y\} \subset \mathbb{R} \setminus A$ then $[x, y)$ is a clopen subset of $(\mathbb{R}, \tau_A)$.*

Theorem 3.6. *For an arbitrary subset A of $\mathbb{R}$, the following are equivalent:*

(1) *$(\mathbb{R}, \tau_A)$ is zero-dimensional;*

(2) *$(\mathbb{R}, \tau_A)$ is strongly zero-dimensional;*

(3) *$\mathbb{R} \setminus A$ is dense in $(\mathbb{R}, \tau_E)$.*

Proof. (2) ⇔ (1) by [7, Theorem 7.1.11]. (1) ⇔ (3) by Lemma 3.5. □

Clearly, if A is a subset of $\mathbb{R}$ such that $\mathbb{R} \setminus A$ is dense in $(\mathbb{R}, \tau_E)$, then arbitrary product of $(\mathbb{R}, \tau_A)$ is zero-dimensional.

Remark 3.7. *Let $A \subset \mathbb{R}$. (1) If $(\mathbb{R}, \tau_A)$ is homeomorphic to $\mathbb{P}$, then the product $\prod_1 = (\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is also homeomorphic to $\mathbb{P}$. In particular, $Ind(\prod_1) = 0$. (2) If $(\mathbb{R}, \tau_A)$ is homeomorphic to $\mathbb{S}$, then the product $\prod_2 = (\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is also homeomorphic to $\mathbb{S} \times \mathbb{S}$, which is not normal. In particular, $Ind(\prod_2) = \infty$.*

However, the following question is still open for us.

Question 3.8. *Let A be a subset of $\mathbb{R}$ such that $\mathbb{R} \setminus A$ is dense in $(\mathbb{R}, \tau_E)$, and let κ be a cardinal. Is the Tychonoff product $(\mathbb{R}, \tau_A)^\kappa$ strongly zero-dimensional?*

Moreover, the following question is in a more general form.

Question 3.9. *Let I be a non-empty index set. For each $\alpha \in I$, let A_α be a subset of $\mathbb{R}$ such that $\mathbb{R} \setminus A_\alpha$ is dense in $(\mathbb{R}, \tau_E)$. Is the Tychonoff product $\prod_{\alpha \in I}(\mathbb{R}, \tau_{A_\alpha})$ strongly zero-dimensional?*

Next we prove the second main result of this paper, which shows that the local compactness is equivalent to the k_ω-property in $(\mathbb{R}, \tau_A)$. Indeed, A. Bouziad and E. Sukhacheva in [1] have proved that, for an arbitrary subset A of $\mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is locally compact if and only if $\mathbb{R} \setminus A$ is closed in $\mathbb{R}$ and discrete in $\mathbb{S}$.

Theorem 3.10. *For an arbitrary subset A of $\mathbb{R}$, the following conditions are equivalent:*

(1) $(\mathbb{R}, \tau_A)$ *is locally compact;*

(2) $(\mathbb{R}, \tau_A)$ *is a k_ω-space;*

(3) $\mathbb{R} \setminus A$ *is discrete and closed in* $(\mathbb{R}, \tau_A)$.

Proof. From [1], we have (1) ⇔ (3). The implications (1) ⇒ (2) and (2) ⇒ (1) are followed by [8, Theorem 10] and [8, Theorem 20] respectively. □

From Theorem 3.10, it is natural to pose the following question.

Question 3.11. *For what subsets A of $\mathbb{R}$ are $(\mathbb{R}, \tau_A)$ σ-compact?*

Next we give a partial answer to this question. First, we give some lemmas.

Proposition 3.12. *For an arbitrary subspace B of $(\mathbb{R}, \tau_S)$, we have $nw(B) \geq |B|$. In particular, $w(B) \geq |B|$.*

Proof. Let $\mathscr{P}$ be an arbitrary network of the subspace B with $|\mathscr{P}| = nw(B)$. For each $x \in B$, let

$$\mathscr{B}_x = \{P \in \mathscr{P} : x \in P \text{ and } P \subset [x, x + \frac{1}{n}) \cap B \text{ for some } n \in \mathbb{N}\}.$$

Then $\bigcup_{x \in B} \mathscr{B}_x$ is contained in $\mathscr{P}$ and is a network of the subspace B. However, for any $x, y \in B$ with $x \neq y$, we have $\mathscr{B}_x \cap \mathscr{B}_y = \emptyset$. Indeed, assume $\mathscr{B}_x \cap \mathscr{B}_y \neq \emptyset$, then take any $P \in \mathscr{B}_x \cap \mathscr{B}_y$, hence $x \in P$, $y \in P$ and there exist $n, m \in \mathbb{N}$ such that $P \subset [x, x + \frac{1}{n})$ and $P \subset [y, y + \frac{1}{m})$. Thus, $x = y$, which is a contradiction. Therefore, $nw(B) \geq |B|$. □

By Proposition 3.12 and the hereditary separability of $(\mathbb{R}, \tau_S)$, we have the following corollary.

Corollary 3.13. ([6, Proposition 2.3]) *For any subspace X of $(\mathbb{R}, \tau_S)$, X is metrizable if and only if it is countable.*

A topological space X is called *submetrizable* if X has a weaker metrizable topology.

Lemma 3.14. *For an arbitrary subset A of $\mathbb{R}$, $(\mathbb{R}, \tau_A)$ is submetrizable.*

Proof. Since $(\mathbb{R}, \tau_A)$ and $\tau_E \subset \tau_A$, it follows that $(\mathbb{R}, \tau_A)$ is submetrizable. □

Lemma 3.15. *If K is a compact subset of $(\mathbb{R}, \tau_A)$, then $K \cap (\mathbb{R} \setminus A)$ is countable.*

Proof. By Lemma 3.14, K is metrizable. Put $X = K \cap (\mathbb{R} \setminus A)$. Then X is metrizable. Moreover, X is subspace of $(\mathbb{R}, \tau_S)$. By Corollary 3.13, X is countable. Therefore, $K \cap (\mathbb{R} \setminus A)$ is countable. □

Lemma 3.16. *For an arbitrary subset A of $\mathbb{R}$, if the closure of $\mathbb{R} \setminus A$ under the topology $(\mathbb{R}, \tau_A)$ is countable then $(\mathbb{R}, \tau_A)$ is σ-compact.*

Proof. Put $U = \mathbb{R} \setminus \overline{\mathbb{R} \setminus A}$, where $\overline{\mathbb{R} \setminus A}$ denote the closure of $\mathbb{R} \setminus A$ under the topology $(\mathbb{R}, \tau_A)$. Then U is open in $(\mathbb{R}, \tau_A)$ and $U \subset A$, hence U is open in $(\mathbb{R}, \tau_E)$, which implies that U is σ-compact. From $U \subset A$, it follows that U is σ-compact in $(\mathbb{R}, \tau_A)$. By the countability of $\overline{\mathbb{R} \setminus A}$, $(\mathbb{R}, \tau_A)$ is σ-compact. □

Now we have the following two results.

Theorem 3.17. *For an arbitrary subset A of $\mathbb{R}$, if $(\mathbb{R}, \tau_A)$ is σ-compact, then $\mathbb{R} \setminus A$ is countable and nowhere dense in $(\mathbb{R}, \tau_A)$.*

Proof. By Lemma 3.15, it is easy to see that $\mathbb{R} \setminus A$ is countable. It suffices to prove that $\mathbb{R} \setminus A$ is nowhere dense.

Assume that $\mathbb{R} \setminus A$ is not nowhere dense. Then there exists an open subset V being contained in the closure of $\mathbb{R} \setminus A$ under the topology $(\mathbb{R}, \tau_A)$. Hence there exist $a, b \in \mathbb{R} \setminus A$ such that $[a, b) \subset V$. Then $[a, b)$ is σ-compact since $[a, b)$ is open and closed in $(\mathbb{R}, \tau_A)$, hence there exists a sequence of compact subsets $\{K_n\}$ of $(\mathbb{R}, \tau_A)$ such that $[a, b) = \bigcup_{n\in\mathbb{N}} K_n$. By Corollary 3.3, one can easily note that $[a, b)$ is a Baire space, then there exists $n \in \mathbb{N}$ such that K_n contains a non-empty open subset W in $(\mathbb{R}, \tau_A)$. Since $W \subset [a, b) \subset V$, there exist $c, d \in \mathbb{R} \setminus A$ such that $[c, d) \subset [a, b)$. Since $[c, d)$ is closed and $[c, d) \subset K_n$, it follows that $[c, d)$ is compact, which is a contradiction. Therefore, $\mathbb{R} \setminus A$ is nowhere dense. □

Theorem 3.18. *For an arbitrary subset A of $\mathbb{R}$, if $\mathbb{R} \setminus A$ is countable and scattered in $(\mathbb{R}, \tau_E)$, then $(\mathbb{R}, \tau_A)$ is σ-compact.*

Proof. Assume that $\mathbb{R} \setminus A$ is countable and scattered in the real line. Then it follows from [12, Corollary 3] that $\mathbb{R} \setminus A$ is homeomorphic to a subspace of $[0, \omega_1)$. Hence one can easily check that the closure of $\mathbb{R} \setminus A$ under the topology $(\mathbb{R}, \tau_A)$ is countable. By Lemma 3.16, $(\mathbb{R}, \tau_A)$ is σ-compact. □

Remark 3.19. The condition for $\mathbb{R} \setminus A$, in Theorem 3.18, to be scattered in $(\mathbb{R}, \tau_E)$ is essential. By [4, Proposition 2.11] there exists a space $(\mathbb{R}, \tau_A)$ such that $\mathbb{R} \setminus A$ is countable, scattered in $(\mathbb{R}, \tau_A)$ and $(\mathbb{R}, \tau_A)$ is not σ-compact.

The following example shows that the property of σ-compact in $(\mathbb{R}, \tau_A)$ does not imply local compactness.

Example 3.20. There exists a subset A such that $(\mathbb{R}, \tau_A)$ is σ-compact but not locally compact.

Proof. Let $A = \mathbb{R} \setminus (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\})$. Then $\mathbb{R} \setminus A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is closed, countable scattered and non-discrete. By Theorem 3.18, $(\mathbb{R}, \tau_A)$ is σ-compact. However, it follows from Theorem 3.10 that $(\mathbb{R}, \tau_A)$ is not locally compact and not a k_ω-space. □

Next we prove that $\prod_{i=1}^{\infty}(\mathbb{R}, \tau_{A_i})$ is perfectly subparacompact for arbitrary sequence $\{A_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{R}$. Indeed, Lemma 2.3 of [11] asserts that, for every $n \in \mathbb{N}$, the product $(\mathbb{R}, \tau_S)^n$ is perfect. Theorem 3.21 is a natural generalization of [11, Lemma 2.3], and the proof of Theorem 3.21 is a very minor modification of the proof of [11, Lemma 2.3]. However, for the convenience of readers, we give the proof.

Theorem 3.21. *For each $i \in \mathbb{N}$, let A_i be an arbitrary subset of $\mathbb{R}$. Then $\prod_{i=1}^{n}(\mathbb{R}, \tau_{A_i})$ is perfect for every $n \in \mathbb{N}$.*

Proof. By induction. The theorem is clear for $n = 1$ since $(\mathbb{R}, \tau_{A_1})$ is a Lindelöf space. Therefore let us suppose the theorem for n and let us prove it for $n + 1$.

Let $Z = \prod_{i=1}^{n+1} Z_i$ with $Z_i = (\mathbb{R}, \tau_{A_i})$ for all $i \leq n + 1$. For every $m \leq n + 1$, put $Z(m) = \prod_{i=1}^{n+1} Z_i(m)$, where $Z_m(m) = (\mathbb{R}, \tau_E)$ and $Z_i(m) = (\mathbb{R}, \tau_{A_i})$ if $i \neq m$.

Now it suffices to prove that an arbitrary open subset U of Z is an F_σ in Z. For every $m \leq n + 1$, let $U(m)$ be the interior of U as a subset of $Z(m)$, and let $U^\star = \bigcup_{m=1}^{n+1} U(m)$. It follows from [11, Lemma 2.2] and our assumption that each $Z(m)$ is perfect, hence $U(m)$ is an F_σ in $Z(m)$ and thus also in Z. Therefore, $U^\star$ is also an F_σ in Z. Put $U' = U \setminus U^\star$. Thus it only remains to prove that U' is an F_σ in Z. Clearly, for each $x = (x_1, \ldots, x_{n+1}) \in U'$, it has $x_i \in \mathbb{R} \setminus A_i$ for each $i \leq n + 1$.

For each $z \in Z$, let $\{W_j(z) : j \in \mathbb{N}\}$ denote the base of neighborhoods of z in Z defined by

$$W_j(z) = \{y \in Z : y_i \in [z_i, z_i + \frac{1}{j}) \text{ for each } i \leq n + 1\}.$$

For every $j \in \mathbb{N}$, let

$$U'_j = \{z \in U' : W_j(z) \subset U\}.$$

One can easily check that $U' = \bigcup_{j=1}^{\infty} U'_j$. Next we shall prove that each U'_j is closed in Z. Taking an arbitrary $j \in \mathbb{N}$ and assuming $z \notin U'_j$, it suffices to prove that z is not in the closure of U'_j in Z.

For each $F \subset \{1, \cdots, n+1\}$, let

$$U'_{j,F}(z) = \{y \in U'_j : z_i = y_i \text{ iff } i \in F\}.$$

Clearly, $U'_j = \bigcup\{U'_{j,F}(z) : F \subset \{1, \cdots, n+1\}\}$. Then it suffices to prove that for each $F \subset \{1, \cdots, n+1\}$ there exists a neighborhood of z in Z disjoint from $U'_{j,F}(z)$. Indeed, suppose that $W_j(z) \cap U'_{j,F}(z) \neq \emptyset$. Then we can choose a point $x \in W_j(z) \cap U'_{j,F}(z)$. Then the set

$$V = W_j(z) \cap \{y \in Z : y_i < x_i \text{ if } i \notin F\}$$

is a neighborhood of z in Z, and it will suffice to prove that $V \cap U'_{j,F}(z) = \emptyset$.

Suppose not, then there exists some $y \in V \cap U'_{j,F}(z)$. Clearly, $y \in W_j(z)$ and $y \neq z$, thus there is an $m \leq n+1$ such that $y_m > z_m$. Then $m \notin F$. Put

$$W = W_j(y) \cap \{u \in Z : y_m < u_m\}.$$

Clearly, W is open in $Z(m)$ and $W \subset W_j(y) \subset U$. It follows from the definition of $U(m)$ that $W \subset U(m) \subset U^\star$. Moreover, it easily check that $x \in W$. Therefore, $x \in U^\star$, which is a contradiction. That completes the proof. □

For every $n \in \mathbb{N}$, Proposition 3.1 of [15] asserts that the product $(\mathbb{R}, \tau_S)^n$ is perfectly subparacompact. Our Theorem 3.22 is a simple generalization of [15, Proposition 3.1]. By mimicking and lightly modifying the proof of [15, Proposition 3.1], one can obtain the proof of Theorem 3.22.

Theorem 3.22. *For each $i \in \mathbb{N}$, let A_i be an arbitrary subset of $\mathbb{R}$. Then $\prod_{i=1}^{n}(\mathbb{R}, \tau_{A_i})$ is perfectly subparacompact for each $n \in \mathbb{N}$.*

Proof. By Theorem 3.21, $\prod_{i=1}^{n}(\mathbb{R}, \tau_{A_i})$ is perfect. It suffices to prove that $\prod_{i=1}^{n}(\mathbb{R}, \tau_{A_i})$ is subparacompact.

By induction. The result is certainly true for $n = 1$. Let us assume that $\prod_{i=1}^{n}(\mathbb{R}, \tau_{A_i})$ is subparacompact. Next, we will prove that $\prod_{i=1}^{n+1}(\mathbb{R}, \tau_{A_i})$ is subparacompact. By [15, Proposition 2.9], it suffices to prove that $\prod_{i=1}^{n+1}(\mathbb{R}, \tau_{A_i})$ is weakly θ-refinable. Put $Z = \prod_{i=1}^{n+1}(\mathbb{R}, \tau_{A_i})$. Let

$$\mathscr{W} = \{W(\alpha_1, \ldots, \alpha_{n+1}) : \alpha_i \in A_i, i \leq n+1\}$$

be a basic open cover of the space Z, where $W(\alpha_1, \ldots, \alpha_{n+1}) = U(1, \alpha_1) \times \cdots \times U(n+1, \alpha_{n+1})$ such that $U(k, \alpha_k) = (a(k, \alpha_k), b(k, \alpha_k))$ if $a(k, \alpha_k) \in A_k$ and $U(k, \alpha_k) = [a(k, \alpha_k), b(k, \alpha_k))$ if $a(k, \alpha_k) \notin A_k$. By the same notations in the proof of Theorem 3.21, let $W(\alpha_1, \ldots, \alpha_{n+1}, m)$ be the interior of the set $W(\alpha_1, \ldots, \alpha_{n+1})$ in $Z(m)$ for each $m \leq n+1$, thus $W(\alpha_1, \ldots, \alpha_{n+1}, m)$ is open in $Z(m)$, hence also open in Z. For each $m \leq n+1$, put

$$\mathscr{G}(m) = \{W(\alpha_1, \ldots, \alpha_{n+1}, m) : \alpha_i \in A_i, i \leq n+1\}.$$

By [15, Corollary 2.6] and our assumption, each $Z(m)$ is perfect subparacompact. Then it follows from [15, Porposition 2.9] that $\mathscr{G}(m)$ has a weakly θ-refinement $\mathscr{H}(m)$ which covers $\bigcup \mathscr{H}(m)$ and which consists of open subsets of $Z(m)$. Clearly, $\mathscr{H}(m)$ is also a collection of open subsets of Z. Let

$$Y = Z \setminus \bigcup\{\bigcup \mathscr{H}(m) : 1 \leq m \leq n+1\}.$$

Then for each $y \in Y$ it has $y_i \in \mathbb{R} \setminus A_i$ for each $i \leq n+1$, hence there exists $\alpha_i \in A$ such that $y_i = a(i, \alpha_i)$ for each $i \leq n+1$. For each $y \in Y$, pick $\alpha_i(y) \in A_i$ for each $i \leq n+1$ such that $y \in W(\alpha_1(y), \ldots, \alpha_{n+1}(y))$. Put

$$\mathscr{H}(0) = \{W(\alpha_1(y), \ldots, \alpha_{n+1}(y)) : y \in Y\}.$$

One can easily check that if x and y are distinct elements of Y, then $y \notin W(\alpha_1(x), \ldots, \alpha_{n+1}(x))$. Therefore, $\mathscr{H}(0)$ is a collection of open subsets of Z which covers Y in such a way that each point of Y belongs to exactly one member of $\mathscr{H}(0)$. Hence $\mathscr{H} = \{\mathscr{H}(m) : m \in \omega\}$ is a weak θ-refinement of $\mathscr{W}$. Therefore, Z is weakly θ-refinable. □

By Theorem 3.22 and [15, Proposition 2.7], we have the following theorem.

Theorem 3.23. *For each $i \in \mathbb{N}$, let A_i be an arbitrary subset of $\mathbb{R}$. Then $\prod_{i=1}^{\infty}(\mathbb{R}, \tau_{A_i})$ is perfectly subparacompact.*

Corollary 3.24. ([15, Theorem 3.2]) *The space $(\mathbb{R}, \tau_S)^{\aleph_0}$ is perfectly subparacompact.*

Finally, we consider the quasi-metrizability of *H*-spaces. It is well-known that $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$ are all quasi-metrizable, it natural to pose the following question.

Question 3.25. *For an arbitrary $A \subset \mathbb{R}$, is $(\mathbb{R}, \tau_A)$ quasi-metrizable?*

We give a negative answer to Question 3.25 in Example 3.28. Indeed, from the definition of generalized ordered space, we have the following proposition.

Proposition 3.26. *For an arbitrary subset $A \subset \mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is a generalized ordered space.*

By [13, Theorem 10], we can easily give a characterization of subset A of $\mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is quasi-metrizable, see Theorem 3.27.

Theorem 3.27. *For any subset $A \subset \mathbb{R}$, the H-space $(\mathbb{R}, \tau_A)$ is quasi-metrizable if and only if $\mathbb{R} \setminus A$ is a F_σ-set in $(\mathbb{R}, \tau_{S^-})$, where $(\mathbb{R}, \tau_{S^-})$ is the set of real numbers with the topology generated by the base $\{(a, b] : a, b \in \mathbb{R}, a < b\}$.*

Now, we can give a negative answer to Question 3.25.

Example 3.28. There exists a subset A of $\mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is not quasi-metrizable.

Proof. Indeed, let $A = \mathbb{Q}$ be the set of all rational numbers. By Theorem 3.27, assume $\mathbb{R} \setminus A$ is a F_σ-set in $(\mathbb{R}, \tau_{S^-})$, then $\mathbb{Q}$ is a G_δ-set in $(\mathbb{R}, \tau_{S^-})$. However, it follows from [3, Theorem 3.4] that $(\mathbb{R}, \tau_{S^-})$ does not have a dense metrizable G_δ-space, which is a contradiction. □

Obviously, if $\mathbb{R} \setminus A$ is an F_σ-set in $(\mathbb{R}, \tau_A)$, then $\mathbb{R} \setminus A$ is an F_σ-set in $(\mathbb{R}, \tau_{S^-})$, hence we have the following corollary.

Corollary 3.29. *If $\mathbb{R} \setminus A$ is an F_σ-set in $(\mathbb{R}, \tau_A)$, then $(\mathbb{R}, \tau_A)$ is quasi-metrizable.*

We now close this section with a result about generalized metric property of *H*-space.

Theorem 3.30. *For an arbitrary $A \subset \mathbb{R}$, then the following statements are equivalent:*

(1) $(\mathbb{R}, \tau_A)$ *is metrizable;*

(2) $(\mathbb{R}, \tau_A)$ *is a β-space;*

(3) $\mathbb{R} \setminus A$ *is countable.*

Proof. Obviously, it suffices to prove (2) ⇒ (3). Assume that $(\mathbb{R}, \tau_A)$ is a β-space. Since $(\mathbb{R}, \tau_A)$ is a paracompact submetrizable space, it follows from [9, Theorem 7.8 (ii)] that $(\mathbb{R}, \tau_A)$ is semi-stratifiable. By Proposition 3.26 and [9, Theorems 5.16 and 5.21], $(\mathbb{R}, \tau_A)$ is a stratifiable space, hence $(\mathbb{R}, \tau_A)$ is a σ-space by [9, Theorem 5.9]. Then $(\mathbb{R}, \tau_A)$ has a countable network since $(\mathbb{R}, \tau_A)$ is separable, hence $\mathbb{R} \setminus A$ has a countable network. Therefore, it follows from Proposition 3.12 that $\mathbb{R} \setminus A$ must be countable. □

4. Open questions

It is well known that $(\mathbb{R}, \tau_E) \times (\mathbb{R}, \tau_E)$ is Lindelöf, and $(\mathbb{R}, \tau_S) \times (\mathbb{R}, \tau_S)$ is not Lindelöf, hence it is natural to pose the following question.

Question 4.1. *For an arbitrary subset A of $\mathbb{R}$, are the following statements equivalent?*

(1) $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ *is Lindelöf;*

(2) $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ *is normal;*

(3) $(\mathbb{R}, \tau_A)$ *is metrizable.*

The following example gives a negative answer to Question 4.1 under the assumption of CH.

Example 4.2. Under the assumption of CH, there exists a subspace $A \subset \mathbb{R}$ such that $\mathbb{R} \setminus A$ is uncountable and $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf.

Proof. By [2, Theorem 3.4], there exists an uncountable subset $Y \subset \mathbb{S}$ such that Y^2 is Lindelöf. Put $A = \mathbb{R} \setminus Y$. Then $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf. Indeed, it is obvious that

$$(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A) = (A \times A) \cup (A \times Y) \cup (Y \times A) \cup (Y \times Y).$$

Since A is a separable metrizabale space (see also [16, Proposition 5]), the subspace $A \times A$, $A \times Y$ and $Y \times A$ are Lindelöf. Therefore, $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf. □

By Theorem 3.18, we have the following question.

Question 4.3. *If $(\mathbb{R}, \tau_A)$ is σ-compact, is A a scattered subspace?*

The following question was posed by Boaz Tsaban.

Question 4.4. *When is the space $(\mathbb{R}, \tau_A)$ Menger (Hurewicz) for any $A \subset \mathbb{R}$?*

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