Filomat 39:6 (2025), 1927–1944
https://doi.org/10.2298/FIL2506927S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New discoveries in the history of Euler's equation

Galina I. Sinkevich[a,]

[a]*Saint Petersburg State University of Architecture and Civil Engineering, Vtoraja Krasnoarmejskaja ul. 4, St. Petersburg 190005, Russia*

Abstract. Much has been written about the history of Euler's formula and Euler's equation, but there are many blank spots left in it. The equality $ln(cosx + isinx) = xi$ was first obtained in verbal form forty-two years before it appeared in Euler's work. Was this verbal formula correct? Euler's equation or Euler's identity $e^{i\pi} = -1$ was never written down by Euler himself. Rewritten in the form $e^{i\pi} + 1 = 0$, it connects the five most important numbers in mathematics, but this formula is not in Euler's works. Today this identity attracts wide attention from physicists, philosophers and popularizers of science; it is given an almost mystical meaning. Who received it for the first time? We will look at all these mathematical events in order.

1. Early history of complex numbers

1.1. 1545, Gerolamo Cardano, the first appearance of complex numbers

In 1545, in his book *Artis magnae, sive de regulis algebraicis* [11]), Gerolamo (Hieronymus) Cardano obtained the roots from negative values in studying the general formula of the cubic equation. As the roots of the auxiliary quadratic were mutually conjugate and cancelled each other out in the equation, Cardano regarded them as a useful auxiliary construction.

1.2. 1572, Rafael Bombelli, operations for adding and multiplying complex numbers

In 1572, hydraulic engineer Rafael Bombelli in his book *L'algebra parte maggiore dell'aritmetica divisa in tre libri di Rafael Bombelli da Bologna* [10], showed the possibility of determining the ratio of equality, the sum and production of complex numbers. But the roots of negative values did not yet have physical or geometric meaning. Traditionally, numbers were not perceived as quantities or ratios, so neither negative or imaginary numbers found a place in algebra.

1.3. 1614, John Napier, logarithms

In 1614, John Napier published *Mirifici Logarithmorum Canonis Descriptio: Ejusque usus, in utraque Trigonometria; ut etiam in omni Logistica Mathematica, Amplissimi, Facillimi* [39] (Description of the Marvelous Canon of Logarithms: And its use, in both Trigonometry; as also in all Mathematical Logistics, the most comprehensive, the most easy), containing the concept of the logarithm and tables of logarithms of

2020 *Mathematics Subject Classification*. Primary 01A50
Keywords. Bernoulli, Cotes, Eulerś identity, Fagnano, Argan, Fran[c]ais.
Received: 03 October 2024; Revised: 13 December 2024; Accepted: 15 December 2024
Communicated by Ljubiša D. R. Kočinac
Email address: galina.sinkevich@gmail.com (Galina I. Sinkevich)
ORCID iD: https://orcid.org/0000-0002-8595-1686 (Galina I. Sinkevich)

trigonometric functions for the first quarter of the circle with a step of 1 minute. The need for this new mathematical tool was especially great among astronomists, forced to multiply large numbers. The logarithm and speed of its change were determined by Napier kinematically, and subsequent mathematicians calculated it using the squares of the hyperbole, and in 1688 (Logarithmo-technia, Mercator, N.), a formula also emerged series expansion of a logarithm. A large step in accepting operations on irrational numbers was the introduction of ten-decimal fractions, which Napier used for approximated calculations and assessment of error. In the 1620s, Oughtred and Wingate constructed a logarithmic line. But the logarithm retained the role of a regulating means and was not considered to be a function for a long time to come.

1.4. 1637, Rene Descartes on the status of complex numbers

In 1637, Rene Descartes in his work *Geometry* [20], examined the task of dissecting a circle with a parabola. The case of the separate position of the parabola and the circle was regarded by Descartes as a lack of genuine (real), false (negative) roots, and only the presence of imaginary ones. "Neither the true nor the false roots are always real; sometimes they are imaginary that is, while we can always conceive of as many roots or each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of" ([20], p. 175). Thanks to Descartes, the algebrization of geometry began. Algebraic operations began to be carried out on sections and other geometrical objects, and for us the properties will be important from the algebraic standpoint of the hyperbola, logarithmic spiral and logistic curve, which will be discussed below.

1.5. 1685, John Wallis, an attempt to interpret complex numbers

In 1685, John Wallis published his treatise *A treatise of algebra, both historical and practical* [50], where the first attempt was made to give a geometrical and physical interpretation of negative and imaginary numbers. Wallis was the first to introduce the numerical line, containing positive numbers, zero and negative numbers, and then formed the prototype of a complex surface. Initially he examined an imaginary number as a side of a lost rectangular land plot, then as a medium geometrical between sections divided into positive and negative sides, i.e. as a vertical section in relation to the real line: "As I mean proportional between a positive and a negative quantity may be thus exemplified in geometry" ([50], p. 287). Later, examining the average geometrical in the circle and hyperbola, Wallis found a connection between them using the imaginary support. This idea was subsequently taken up by A. de Moivre, moving from the circumference $x^2 + y^2 = 1$ to the hyperbola $x^2 - y^2 = 1$ by replacing y with $y\sqrt{-1}$.

1.6. 1702, works by I. Newton, A. de Moivre and Joh. Bernoulli to develop methods of integrating rational functions

In the late 17^{th} – early 18^{th} century, with the creation of differential and integral calculation in the works of I. Newton, G.W. Leibniz, J. Bernoulli, A. de Moivre and R. Cotes, intensive work proceeded on devising a method for integrating rational functions. Far from all the area or length of a curve may be expressed in algebraic form, it was said (for example by Newton), that the curve is squared geometrically or in final form. In a different way, there was an attempt to reduce the task to squaring conic sections. By 1711 Newton had obtained series expansion of the binomial, sine, cosine, indicator and several other functions.

Primarily, the properties of functions were investigated which were applied to obtaining the quadrature of the circle and the hyperbola, and also the simplest transcendental functions. It was noted that integration of rational expressions using imaginary substitutions could be brought to final algebraic expressions of squares of indefinite sections of a circle and hyperbola. We may especially single out J. Bernoulli's work of 1702, *Solving the task on integral calculation with certain reductions in relation to this calculation* [7]. In the supplement to this work, *A reduced method of transforming complex differentials into simple and back; and conversely, even simply imaginary differentials into complex real* ([8], p. 289–297), Bernoulli writes of the possibility of transforming an imaginary logarithmic differential into a differential of the real circular sector using an imaginary substitution.

Cubic equations and those of higher powers solved not only algebraically, but also by the trigonometrical method, using the sinuses of whole integer arcs. There was an episode when François Viète in 1594 used this method to solve an algebraic equation of the 45^{th} power. Using known trigonometric rations and moving from arcs of the circle to arcs of the hyperbola using imaginary substitution, Moivre reached the formula of raising to a power and extracting the root of a natural (to the 7^{th}) power from a complex number [36], [37].

Giulio Fagnano used the method of imaginary substitution following Johann Bernoulli [30].

Newton and his colleague, the astronomer R. Cotes developed a table of differentials corresponding to squares. In Newton's time, the concept of an indefinite integral as an antiderivative did not exit, and the Newton-Leibniz formula had not been formalized. Quadratures sought as geometric problems: to express the length or are for a given curve or the length or area of an already known circular (or hyperbolic) arc or sector respectively. The indefinite integral, as an antiderivative, acquired an independent meaning only for Euler.

1.7. 1712, Johann Bernoulli, Gottfried Leibniz, Jean le Rond d'Alembert on the significance of the logarithm of a negative number

Before 1702, imaginary numbers were regarded merely as roots of negative values. For a long time it was unclear whether operations on complex numbers led to numbers of the same kind[1)]. In 1749, Euler showed in the article *Investigations on the imaginary roots of equations* that operations on complex numbers led to number of the same kind [24]. The concept of the logarithm was also unclear: it was either an exponent of an element of some geometric progression, or the quadrature of a hyperbola, or a power series, but not a function with a clear domain of definition. The profound connections between these manifestations had not yet been studied. In 1702, Johann Bernoulli encountered the problem of calculating a logarithm of a negative and complex number. In 1712, Bernoulli and Leibniz argued in their correspondence about what the logarithm of a negative number was [49]. Leibniz placed $x = -2$ in the logarithm expansion formula and concluded that the logarithm from -1 could not be zero. For the positive number a, $ln\sqrt{a} = \frac{1}{2}lna$ is correct. Continuing the argument, it may be concluded that $lni = ln\sqrt{-1} = \frac{1}{2}ln(-1)$. But what is $ln(-1)$ equal to? Leibniz suggested that it should be complex (imaginary), but this term did not have a precise definition for him. Bernoulli, and then d'Alembert [19], believed that logarithms of a negative number should be positive. Bernoulli's arguments were based on the integration of the logarithm of $(-x)$ as an interegral from $dz : z$, taken between the limits of 1 and $(-x)$, but he conducted the integration through the pole $z = 0$ and obtained log(-x)=log(x). Later Euler proved that the logarithm of a negative number would be complex, adding that the logarithm was multivalued.

1.8. The history of the logarithmic spiral

Navigators know that it is easier and more convenient to go at a constant angle to the North Star. This curve is loxodroma or a rumb line. Its flat projection is a logarithmic spiral. It was known from 16^{th} century as a spiral at each point of type $\phi = \phi_0 + 2k\pi$ has the same tangent slope (isogonal spiral). We know it as $\rho = ae^{b\varphi}$. It was first described by A. Dürer in 1525 [21]. It was studied by R. Descartes (1638), E. Torricelli (1644) [46], I. Newton (1687) [41], Jacob I Bernoulli (1691, 1692) ([5], [6]), P. Varignon (1706) [48]. Descartes established that for the logarithmic (isogonal) spiral, when the angle is changed in arithmetic progression, the radius vector changes in geometric progression, and showed the equivalence that polar angles for the points are proportional to the logarithms of radius vectors.

In 1714, the English astronomer and mathematician, editor and publisher of Newton, Roger Cotes examined the logarithmic spiral which he called the reciprocal spiral, *Spiralem Aequaianguam*, expanded on the studies of Varignon. Cotes, based on the study its properties, developed his own calculation method for solving analysis problems [28].

2. 1714, Roger Cotes, the logarithm of a complex number. The connection between arc and logarithmic functions

Roger Cotes (1682–1716) made a significant contribution to calculation methods of astronomy and mathematics. Sometimes he is called the English Euler. In his lifetime he only published one article,

1)In 1702, Leibniz made a mistake in calculations and came to the conclusion that there are imaginaries of a different type [35]

Logometria. Measurements of ratios (1714) [17]. The continuation, the second and third parts of Logometry, along with works on calculation methods and the theory of errors in astronomy and geodesy were published posthumously under the title *Harmony of Measurements* (1722) [18].

I. Newton examined a large number of integrals containing the root from the quadratic trinomial, and drew up tables of simple curves comparable by quadrature with the ellipsis and hyperbola, Cotes continued to develop his methods.

In *Harmony of Measurements*, for the first time graphs of the tangent and secant were published, tables of differentials, table of integrals for a large number of algebraic functions, and also the theorem on the expansion into multipliers of the first and second degree of the binomial $a^n \pm b^n$, proved subsequently by Moivre. Cotes provided the first printed calculation of the numbers e and $1/e$ with 12 decimal places, using continued fractions (Leibniz had eight decimal places, later Euler had 14 and then 23). This work contains a more detailed clarification of the method of *Logometria*.

We are obliged to Cotes for introducing the radian, formulas of derivatives of trigonometric functions, methods of approximation of quadratures, including the Newton-Cotes formulas. On the basis of these formulas, Cotes calculated for a large number of curves the length of arcs, areas, volumes and surfaces of bodies of revolution. Cotes applied the results obtained to problems of mechanics, physics and navigation.

Cotes' method is autonomous in relation to series and fluxes. It is based on proportions, the sue of the properties of the hyperbola, the logarithmic spiral and the logistic (logarithmic) curve, and also the correlation of arithmetic and geometric progressions.

The logarithmic (or logistic) curve, which was not yet related to the coordinate axes, was used for any correlation between arithmetic and geometric progressions, for example compound interest (Jacob Bernoulli, *Some issues of advantage, with a solution of the problem of games of changes*, 1690, [5]).

Cotes gets the same results that could be obtained by the fluxion method, but sometimes his method is simpler and faster, and often his way of the only one possible for his time[2)]. Although the expansion of functions into a series and the flux method were already known, the actual concept of the function was unclear, and so even Newton in *Mathematical Principles of Natural Philosophy* made very moderate use of expansions into series using derivatives (for example he obtained the expansion for the sine by reversing the series of the arcsine, not by differentiation), and made very limited use of the fluxion method. Operations on series: addition, multiplication by number, differentiation, determination of series convergence, were yet to be grounded. At the same time, the ratio method, although very complex, was traditionally tested in *Harmonia mensurarum*, Cotes compiled tables of complex proportions for different geometric objects. Cotes made a masterly application of the method of proportions (ratio) for different geometric objects. Cotes made a mastery application of the ratio method, and due his understanding of the logarithm as a measure of ratio, to geometric objects.

Curves, apart from polar ones, did not fit the system of coordinates. The concept of a geometric place of points has only begun to form in the works of J. Bernoulli and G.F. de L'Hôpital. For the values of the argument and for the values of the unction (as we understand them today), they built straight lines, as a rule, parallel, on which plotted the values of the argument were placed on one, corresponding to the value of the function on the other. Then, comparing these values, the curve was constructed outside the coordinate system. If the argument changed according to the law of geometric progression, a scale was drawn for it, on which according sections were place. For example, R. Cotes in the problem on the length of the meridian placed sections with the common origin: $1, (1+x), (1+x)^2, (1+x)^3, ...$, where x is small [29]. On another scale, the values of the according arithmetic progression are placed from the indicators 0, 1, 2, 3, . . . E. Halley went on to construct a logarithmic spiral. With the same conditions Jakob Bernoulli (1690) [5] demonstrated the law of the compound interest and constructed a logistic curve (now we know it as the exponent, i.e. it could also look like a logarithmic curve[3)].

[2)]For example, the length of the subtangent is determined from triangles. Cotes also calculated the perimeter of the ellipsis, which did not become accessible using integration until the 19th century.

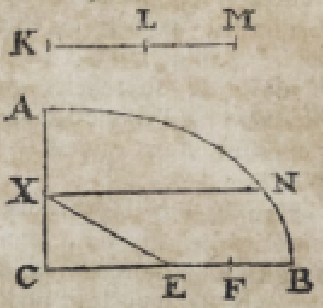
[3)]Euler was the first to determine the exponential function as reversable to the logarithm (1743, Euler, *The Logarithm of a Negative Number*) [26].

The third of the curves used, the hyperbola, was constructed relative to its asymptote, and the area was determined for the hyperbola. It was found that the area proportionate to the logarithm of the relation of limit abscissae. Therefore, natural logarithms were also called hyperbolic.

Cotes uses the method of proportions when calculated the surface area of a layer of an oblate and prolate spheroid (geoid, ellipsoid of revolution), and in the second case obtains a formula which in our notation has the form $ln(cosx + isinx) = xi$, connecting circular and logarithmic functions, in future called Euler's formula (it was 1714, when Euler was 7-years-old). Cotes, whose method allowed him to discover the connection between circular and logarithmic canons (functions), called this connection the miraculous harmony of nature: "Here, of course, the field of applications is very wide, where the power of the method may be tested, especially if *Trigonometry* is added to *Logometry*, in which I once observed the strange kinship between methods coming one after another" ([17], p. 30).

We may note that the length of the arc of the ellipsis could not be calculated before the theory of elliptic functions appeared. The first approximations are found in I. Newton (1669, 1711) in *De analysi per aequationes numero terminorum infinitas* (On Analysis by Equations with an infinite number of terms) [40], then in Fagnano [30], (1750), in James Ivory [34], in 1798, and in F.V. Bessel in 1825 [9]. Euler addressed the problem of straightening curves several times, including in the calculation of the arc of the ellipsis, while Euler used a method that combined geometry, trigonometry and kinematics.

converſione circum axem *CX*, erit ad Circulum ſemidiametro *CB* deſcriptum, ut linearum *KL* & *LM* aggregatum *KM*, ad ſemidiametrum illam *CB*. Ut hæc ultima conſtructio locum habeat, oportet ſemiaxem *CA* circa quem converſio facta eſt, minorem eſſe altero ſemiaxe *CB*; aliter enim Moduli *CE* quantitas $\frac{CAq}{\sqrt{CBq - CAq}}$ evadet impoſſibilis, & conſtructio illa Logometrica (quod in hujuſmodi caſibus fieri ſolet) convertet ſe in Trigonometricam, qualis illa eſt quæ jam ſequitur.



Sit *ANB* Ellipſis deſcripta centro *C*, verticibus *A* & *B*, foco *F*, ſemiaxe principali *CA*, ſemiaxe conjugato *CB*; & ad axis *CA* punctum quodvis *X* ſit *XN* ordinatim applicata, quæ Ellipſi occurrat ad *N*. Angulo *CXN* inſcribatur recta *CE*, quæ ſit ad *CA* ut *CA* ad *CF*. Tum ſumatur *KL* quæ ſit ad *XC* ut *XE* ad *CE*, & *LM* quæ anguli *XEC* menſura ſit ad Modulum *CE*, hoc eſt, quæ ſit æqualis arcui cujus ſinus eſt *XC* ad radium *CE*: & ſuperficies genita ex arcus *BN* converſione circum axem *CX*, erit ad Circulum ſemidiametro *CB* deſcriptum, ut linearum *KL* & *LM* aggregatum *KM*, ad ſemidiametrum illam *CB*. Poſſet hujus etiam ſuperficiei dimenſio per Logometriam deſignari, ſed modo inexplicabili. Nam ſi quadrantis circuli quilibet arcus, radio *CE* deſcriptus, ſinum habeat *CX* ſinumque complementi ad quadrantem *XE*: ſumendo radium *CE* pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ & *CE* menſura ducta in $\sqrt{-1}$. Verum iſthæc aliis, quibus operæ pretium videbitur, diligentius excutienda relinquo. Ceterum ex præcedentibus intelligi poteſt, quanta ſit cognatio inter angulorum atque rationum menſu-

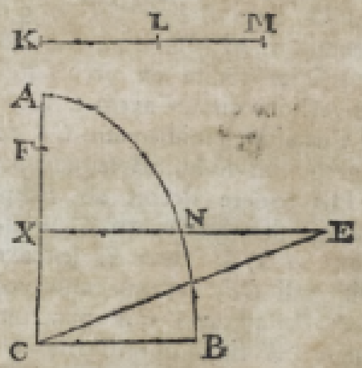


Figure 1 ([17], p. 31)

We will cite Cotes' argument in full, in which Euler's formula appears. Unfortunately, a detailed description of his method would require too much space and time, we'll be brief. The main task of the method was to express the sought quantity (here the surface of a spheroid) through the surface of a spherical layer of a known radius, which he finds with the method of proportions. After a translation of Cotes' text we will provide our own commentary. Here is Gowing's English translation ([28], p. 169–170): It will be allowable to add here the surfaces generated by an ellipse. Let *ANB* be the ellipse described with centre *C*, verticles *A* and *B*, focus *F*, principal semi-axis *CB*, conjugate semi-axis *CA*, and at some point *X* in the axis *CA* let the ordinate *XN* be drawn, which meets the ellipse in *N* [See Fig. 1, top picture]. In the axis *CB* let *CE* be taken to *CA* as *CA* is to *CF*, and let *EX* be joined. Then let *KL* be taken which is to *XC* as *XE* to *CE*, and LM which is the measure of ratio between $EX + XC$ and CE, to modulus CE; and the surface generated by the arc BN rotated about the axis CX will be to the circle on CB as the sum KM of the lines KL and LM to the semi-diameter CB. In order for this last construction to exist, the semi-axis CA about which rotation is made, must be less than the other semi-axis CB, for otherwise the quantity of the modulus CE, $\frac{CAq}{\sqrt{CBq-CAq}}$ (*in modern notation* $\frac{CA^2}{\sqrt{CB^2-CA^2}}$ *- G.S.*) will become impossible, and the logometric construction (which generally arises in this sort of case) becomes trigonometrical, such as that which now follows.

Let ANB [See Fig. 1, bottom picture] be the ellipse described with centre C, vertices A and B, focus F, principal semi-axis CA, conjugate semi-axis CB; and at some point X in the axis CA the ordinate XN is drawn, which meets the ellipse in N. With angle CXN let the straight line CE be drawn which is to CA as CA is to CF. Then KL is taken which is to XC as XE to CE, and LM which is the measure of the angle XEC to modulus CE, i.e., which is equal to the arc whose sine is XC with radius CE; and the surface generated by the arc BN rotated about the axis CX, will be to the circle described on CB as semi-diameter, as KM, the sum of the lines KL and LM, to the semi-diameter CB. It would have been possible to define this surface by logarithms, but by an impractical method. **For if some arc of a quadrant of a circle described with radius CE has sine CF [See Fig. 1, bottom picture] and the sine of the complement of the quadrant XE, taking radius CE as modulus, the arc will be the measure of the ratio between $EX + XC\sqrt{-1}$ and CE, the measure having been multiplied by $\sqrt{-1}$, but I leave this to be examined in more detail by others who will think it worthwhile.** [*Emphasized by me - G.S.*]. Moreover, from the foregoing can be understood the extent of the relationship between the measures of angles and of ratios, further, by simple exchange among themselves, they are easily converted for different cases of the same problem"([28], p. 169-171).

Here is another translation of the last paragraph: "If any arc of the quarter of the circle, described by the radius CE, has the sine CX and the sine complement to the quarter of XE, and if the radius CE is accepted as the modulus, then the arc will be the measure of ratio $EX + XC\sqrt{-1}\&CE$, multiplied by $\sqrt{-1}$ ([47], p. 61).

The explanation of the last paragraph is as follows. CE is a radius, The angle $XEC = \theta$, $\frac{CX}{CE} = sin\theta$, $\frac{EX}{CE} = cos\theta$; $\frac{EX}{CE} + \frac{CX}{CE}\sqrt{-1}$. The logarithm is a measure of a ratio, i.e. $ln(\frac{EX}{CE} + \frac{CX}{CE}\sqrt{-1}) = ln(cos\theta + sin\theta\sqrt{-1})$. The arc corresponding to angle XEC and radius CE, measured in radians, is $\theta\sqrt{-1}$. So $ln(cos\theta + sin\theta\sqrt{-1}) = \theta\sqrt{-1}$.

Owing to differences in translation, the meaning changes: is the left or right side of the equation to be multiplied by $\sqrt{-1}$? We may note that in *Harmonia mensuarum* of 1722 this phrase is repeated word for word, but the difference is a comma, namely: "Nam si quadrantis circuli quilibet arcus, radio CE descriptus, sinum habeat CX sinumque complementi ad quadrantem XE : sumendo radium CE pro Modulo, arcus erit rationis inter $EX + CX\sqrt{-1}$ mensura, ducta in $\sqrt{-1}$"[4)] ([18]).

Some sources mention an error by Cotes: indeed, if the multiplier $\sqrt{-1}$ is placed not in the right, but in the left side of the equation, a superfluous minus appears. Cotes' English researcher Ronald Gowing believes that the minus side justifiably arises because of Cotes' ambiguous interpretation of the direction of the movement of the arc and its complex nature ([28], p. 38). But, as we have now noticed, the misreading was due to the absence of a comma in the 1714 Cotes publication, restored in the 1722 publication. Thus, the Cotes wording was correct.

We should note that Cotes was the first to introduce the angle to mathematics which has a measure always equal to the radius, i.e. providing a relation whose measure is always equal to the module. This modular ratio is equal to 2.71828. . . , the according angle is 57.295 degrees, which is one radian. It was thanks to Cotes that the radian measure of angles appeared, without which Euler's identity could not have appeared.

Cotes calls this connection the "marvelous harmony" ([17], p.33).

Cotes never returned to this topic.

Cotes determines the area of the surface of an oblate and prolate spheroid, and rotates the ellipsis around the vertical axis. Initially, this is an ellipsis with the semiaxes $a > b$, which gives a oblate spheroid. Here

[4)]Note that starting with the works of François Vieta, the word *ducere*, *ductio* acquired a broader meaning: not just multiplication, but product: "the product of A and B is denoted by the word in."

the element of the arc contains the multiplier $\frac{CX^2}{\sqrt{CB^2-CX^2}}$, which finally leads to the statement that the surface formed by the arc BN, rotating around the axis CX, will relate to the circle described by the radius CB, as the sum of the sections KL and LM, comprising KM, to this radius CB, and namely with S as the area of the upper dome of the oblate spheroid, $\frac{S}{\pi CR^2} = \frac{KL+LM}{CB}$, where $KL = \frac{XC \cdot XE}{CE}$, LM is the measure of the relation, i.e. the logarithm of the relation (indicated by l), and namely $LM = l(\frac{CB}{CX} + \frac{\sqrt{CB^2-CX^2}}{CX})$, taking into account that the point X slides along the vertical axis from zero to b, and moving to the natural logarithm, we finally receive for the upper half of the oblate spheroid $S = \pi a^2 + \frac{\pi ab^2}{\sqrt{a^2-b^2}} ln(\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b})$.

Cotes then examines the ellipsis with the semi-axis $a < b$, the focus on the vertical axis, rotation around the vertical axis, and examines the arc of the upper dome of the prolate spheroid. As it still has the point X on the vertical axis, the polar radius is greater than the equatorial radius, and relations are arranged the same way, a root of a negative value arises, and the measure of the arc XEC, multiplied by $\sqrt{-1}$, is determined as $l(\frac{CB}{CX} + \frac{\sqrt{CB^2-CX^2}}{CX}\sqrt{-1})$. As we know, this area will be expressed through the arcsine[5)].

Euler (1747), studying Cotes's work, found a simpler path: he compared the primitives of the two integrals $\int \frac{dx}{\sqrt{1-x^2}}$ and $\int \frac{dx}{\sqrt{1+x^2}}$, and noted that one integral was obtained from the other through imaginary substitution, after which he compared the antiderivatives[6)].

Thanks to Cotes, an algebraic formula connecting logarithmic and circular functions first appeared in geometry and analysis.

3. L. Euler

.

3.1. Laying the Basics of Complex analysis

From 1730, Leonhard Euler developed the theory of elementary functions of a complex variable. In 1734–1735 Euler obtained a condition $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}; \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$ for the function $f(z) = P(x, y) + iQ(x, y)$. After him in 1752, D'Alembert obtained this condition, then in the 19th century A. Cauchy (1814) and B. Riemann (1851). It is now known as the Cauchy–Riemann equations. Euler introduces the concept of a complex variable and its functions, formulates a theorem on the decomposition of a polynomial into factors of the first and second order, expands functions of a complex variable into generalized power series and infinite products ("Introduction to an analysis of infinitesimals", 1748), [23]. He studies conformal transformation conditions (1777); he applies complex functions to evaluate integrals (1776 and later); he introduces the concept of the gamma function (the name was given later by Legendre) and several other special functions.

From 1743 Euler goes from point coordinates (x, y) to a complex number $p = x \pm \sqrt{-1}y$ representing it in polar coordinates $p = s(cos\omega \pm \sqrt{-1}sin\omega)$. In 1743, Euler created a method for solving linear differential equations of higher orders, in which imaginary numbers arise when solving characteristic algebraic equations. In this case, the general solution of the equation is real. Euler used more convenient designations, namely the letter e as the basis of the natural logarithm from 1728, the letter πas the relation of the circumference to the diameter[7)], from 1736, the letter i as the designation of the imaginary unit from 1777 [27] and gave it its modern definition ($i^2 = -1$ or $\frac{1}{i} = -i$). However, for a long time mathematicians continued to use $\sqrt{-1}$ instead of the letter i.

---

5)In Cotes's times there were no symbols or definition of the inverse functions of the trigonometric functions.

6)Note that at the time of Cotes the concept of an indefinite integral as an antiderivative did not yet exist, this is the merit of Euler.

7)We may note that the number as the ratio of the length of the circumference to the diameter first appeared in 1706 in the work by W. Jones [33] as an abbreviation of the words "perimeter" or "periphery" of the circle, but entered mathematical usage thanks to Euler.

3.2. 1743, Euler's formulae

By 1711, Newton had obtained series expansions of the binomial, sine, cosine, exponential and some other functions. Using the series expansion, Euler obtained expressions of the sine and cosine through the exponent of a complex number (Euler's formulae). He wrote about this in letter to Christian Goldbach (9/12/1741 and 8/05/1742), and to Nicholas II Bernoulli (16/01/1742 and 10/11/1742) (the last letter also contains formulas for exponentiation in trigonometric form), then published it in 1743.

Euler wrote: "After examination logarithms and exponents of quantities, we should examine the arc of the circle and their sines and cosines, as they form a new type of transcendent quantities, and also because they form the actual logarithmic and exponential quantities, when these latter are imaginary numbers" ([23], p. 103–103). And further: "Here the arc z is infinitesimal, n takes the value of an infinite number i (NB! Here i this is not an imaginary unit, but a real number), so iz acquires the final value v. So it will be $nz = v$ and $z = \frac{v}{i}$. This implies $sinz = \frac{v}{i}$ and $cosz = 1$; substituting this, we get $cosv = \frac{(1+\frac{v\sqrt{-1}}{i})^i+(1-\frac{v\sqrt{-1}}{i})^i}{2}$ and $sinv = \frac{(1+\frac{v\sqrt{-1}}{i})^i-(1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In the previous chapter we saw that $(1+\frac{z}{i})^i = e^z$, where e denotes the base of hyperbolic logarithms; if instead of z we write $+v\sqrt{-1}$ in one case, and $-v\sqrt{-1}$ in another, we get $cosv = \frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2}$ and $sinv = \frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}$. From this it is clear how imaginary exponential quantities lead to the sines and cosines of real arcs. Exactly, $e^{+v\sqrt{-1}} = cosv + \sqrt{-1}sinv$, $e^{-v\sqrt{-1}} = cosv - \sqrt{-1}sinv$". You can see it below:

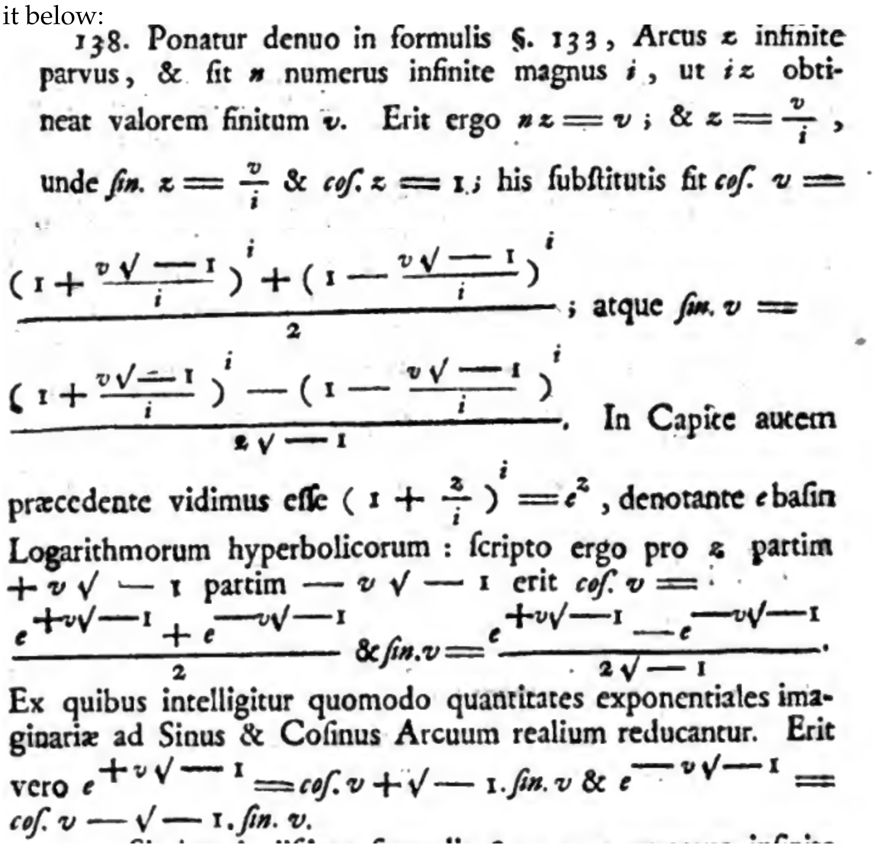

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & fit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde *fin.* $z = \frac{v}{i}$ & *cof.* $z = 1$; his fubftitutis fit *cof.* $v = \frac{(1+\frac{v\sqrt{-1}}{i})^i+(1-\frac{v\sqrt{-1}}{i})^i}{2}$; atque *fin.* $v = \frac{(1+\frac{v\sqrt{-1}}{i})^i-(1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In Capite autem præcedente vidimus effe $(1+\frac{z}{i})^i = e^z$, denotante e bafin Logarithmorum hyperbolicorum: fcripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit *cof.* $v = \frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2}$ & *fin.* $v = \frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}$. Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cofinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} =$ *cof.* $v + \sqrt{-1}$. *fin.* v & $e^{-v\sqrt{-1}} =$ *cof.* $v - \sqrt{-1}$. *fin.* v.

Figure 2, 3. Euler's representation of the sine and cosine through an exponent ([23], p. 103–103).

Euler also derived the arc formula $z = \frac{1}{2\sqrt{-1}ln\frac{1+\sqrt{-1}tgz}{1-\sqrt{-1}tgz}}$, which which is equivalent to Euler's formula. This concept was used after Euler by Lagrange and other mathematicians in two-dimensional tasks of hydrodynamics.

3.3. 1749, Euler on logarithms of negative and imaginary numbers

In 1747 (published in 1749), after the death of his teacher J. Bernoulli, Euler delivered a paper at the Berlin Academy of Sciences *On logarithms of negative and imaginary numbers* [26], where he gave the formula $ln(-1) = (\pi \pm 2\pi n)\sqrt{-1}$ and and its special cases for $\pm\pi\sqrt{-1}$, $\pm 3\pi\sqrt{-1}$, , etc., and also the formula for $ln(1 = ln(-1)^2$, giving the values $\pm 2\pi\sqrt{-1}$, $\pm 6\pi\sqrt{-1}$, etc.

We now have this formula in the form $Lnz = ln|z| + i\phi + qk\pi i$. This is Euler's description (ibidem, p. 269): "§1. In the correspondence between Leibniz and Johann Bernoulli there was a major debate about the logarithms of negative and imaginary numbers, both sides stubbornly insisted on their own opinions, while maintaining full agreement on other issues of analysis".

As the starting point of the argument, Euler takes the same integrals and their antiderivative, which Cotes obtained in the problem of the surface of the spheroid, namely the arc sine and the "long" logarithm.

Euler then examines arcs on a single circle, their sines and cosines taking into account periodicity $\pm 2\pi n + \phi$ for the values of the argument $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ etc. He designates $x = sin\phi$, $y = cos\phi$, $y = \sqrt{1-x^2}$, , the letter l designates the natural logarithm. As $d\phi = \frac{dx}{y} = \frac{dx}{\sqrt{1-x^2}}$, Euler introduces the designation $x = z\sqrt{-1}$ hence $d\phi = \frac{dz\sqrt{-1}}{\sqrt{1+z^2}}$. Accordingly, $\int \frac{dz\sqrt{-1}}{\sqrt{1+z^2}} = l(\sqrt{1+z^2} + z) + C$. Accordingly, $\overline{\phi} = \sqrt{-1}l(\sqrt{1-x^2} + \frac{x}{\sqrt{-1}}) + C$. So, Euler writes, the constant $C = 0$.

Then $\overline{\phi} = \frac{1}{\sqrt{-1}}l(\sqrt{1-x^2} - x\sqrt{-1})$, hence $\phi = \frac{1}{\sqrt{-1}}l(\sqrt{1-x^2} + x\sqrt{-1})$, or $\phi = \frac{1}{\sqrt{-1}}l(y + x\sqrt{-1})$. Then $\phi \pm 2\pi n = \frac{1}{\sqrt{-1}}l(y + x\sqrt{-1})$ and $l(y + x\sqrt{-1}) = (\phi \pm 2\pi n)\sqrt{-1}$. Returning to the previous designations, $l(cos\phi + sin\phi\sqrt{-1}) = (\phi \pm 2\pi n)\sqrt{-1}$ (ibidem, p. 277).

And finally Euler calculates $ln(-1)$.

§ 29. Soit maintenant l'arc proposé φ de 180°, ou soit $\varphi = \pi$, et nous aurons $\sin\varphi = 0$ et $\cos\varphi = -1$. Cette supposition faite, l'équation générale trouvée se changera en cette forme

$$l(-1) = (\pi \pm 2n\pi)\sqrt{-1}\ (= 1 \pm 2n)\pi\sqrt{-1},$$

d'où nous tirons toute l'infinité des logarithmes du nombre négatif -1, car nous aurons

$$l(-1) = \pm\pi\sqrt{-1},\ \pm 3\pi\sqrt{-1},\ \pm 5\pi\sqrt{-1},\ \pm 7\pi\sqrt{-1},\ \text{etc.}$$

et de là nous voyons clairement, que tous les logarithmes de -1 sont imaginaires et tous différents des logarithmes de $+1$. Cela non obstant, les logarithmes de $(-1)^2$ qui seront

$$\pm 2\pi\sqrt{-1},\ \pm 6\pi\sqrt{-1},\ \pm 10\pi\sqrt{-1},\ \text{etc.}$$

sont visiblement contenus dans les logarithmes de $+1$; ce qui suffit pour sauver les contradictions apparentes dont j'ai fait mention là-haut, quoiqu'il n'en suive pas réciproquement que les moitiés de tous les logarithmes de $+1$ soient logarithmes de -1: ce que la nature même des quantités ne permet pas, puisque -1 n'est pas la seule racine carrée de $+1$.

Figure 4. Euler calculates $ln(-1)$. ([26], p. 277).

As we can see, Euler already has the equation $ln(-1) = \pm\pi i$. But he does not have the desired expression through the exponent.

On page 279 Euler notes: "Hence the problem that worried Leibniz and Bernoulli, the problem of whether logarithms of negative numbers are real or imaginary, is solved in favor of the former, who insisted they were imaginary, and all of Bernoulli's objections and protests no longer have any influence on this conclusion". We may note that Euler did not allow himself to publish this malicious comment until one year after the death of his teacher Johann Bernoulli.

From the property of logarithms, Euler expresses the logarithm of a negative number as the sum of the logarithm of −1 and the logarithm of the module of the respective number; he expresses the logarithm of the imaginary unit as $(\pm 2n + \frac{1}{2})\pi\sqrt{-1}$. On the basis of logarithms obtained he proves the exponentiation formula $(cos\phi + sin\phi\sqrt{-1})^{\mu} = cos\mu\phi + sin\mu\phi\sqrt{-1}$ and the root extraction formula.

As we can see, it is just a small step to the identity named at the start of our thesis, but Euler does not do this. Perhaps he did not attach significance to this form as an isolated case.

It was not until after Euler's death, in the 19th century, that the geometrical interpretation of the complex number was discovered by Wessel, Argand, and developed in the works of Gauss, Grassman, Hamilton and other scientists.

4. 1750, Giulio Carlo Fagnano

Marchese Giulio Carlo d'e Toschi di Fagnano, (1682–1766), after receiving a fine education in the humanities, lived almost entirely on his estate on the coast of the Adriatic Sea. He only became interested in mathematics at the age of 24. He never spoke with any major mathematician of his time, but after studying the works of Descartes, Newton, Leibniz; Johann, Jacob and Nicolas I Bernoulli, L'Hospital and many other, he embarked on mathematical research himself[8)] and corresponded with many mathematicians. Fagnano's first works studied lemniscates and attempted to find their length. The difficult of intergation in the problem posed by Jacob and Johann Bernoulli, because the integral was not expressed elementarily. Fagnano began to look for other methods and discovered rather simple relations between certain arcs of leminscates. Later Euler, studying Fagnano's works, noted the structural properties of corresponding integrals. Fagnano published his works in Italian in the Venetian journal *Giornale dei letterati l'Italia*, and in 1750 published a two-volume edition of his works [30], which he sent to the Berlin Academy of Sciences for review by Euler. Euler was delighted and assisted Fagnano's election as a foreign member of the Berlin Academy, and in many ways continued Fagnano's studies. This applies to Fagnano's works which were written around 1719 and covered the development of methods of inegration using imaginary variables, and also their application to calculating the areas and arcs of several curves, including the elipsis ([30], p. 469). Fagnano, not encumbered by collegial relations and professional applied tasks, worked on the purely mathematical aspects and developed several original methods, both in solving algebraic equations, and on methods of inegration, in particular calculating the arcs of curves without using series, only by comparing their differences. In solving quadratures, Fagnano first seeks a way to find, by approximation, a circular sector whose arc is equal to a given hyperbola interval, and then a hyperbolic area equal to the corresponding circular sector.

Solving the posed problem: to find by approximation, but without using series inversion, a sector of a circle equal to a given interval between an equilateral hyperbola, an asymptote and two ordinates of the same asymptote, Fagnano obtains and studies an equation in which Euler later became interested (1) $\frac{dt\sqrt{-1}}{1+tt} = \frac{dx\sqrt{-1}}{1+x}$ ([30], p. 480).

Fagnano obtains the expression of the arctangent through the logarithm, in our notation $\int \frac{dt}{1+t^2} =$

[8)] We may note that from the Catholic church's standpoint, Leibniz and Bernoulli had only one shortcoming: they were Protestants. Their works, in particular several issues of Acta eruditorum, were on the index of prohibited books, L'Index librorum prohibitorum (decrees of 29 March 1690, 4 March 1709, 15 January 1714), and Fagnano was regularly obliged to request permission from Abbot Guido Grandi to read these books (the books themselves, as Fagnano wrote, were obtained with certain difficulties) ([43], p. 5).

$ln((1-it)^{\frac{i}{2}} \times (1+it)^{-\frac{i}{2}})$ and also its equivalent forms:

DIvidendo per $(1-t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}}$ il numeratore, e il denominatore di quella frazione, il logaritmo della quale fa il ſecondo membro dell' equazione (6), notata nella I. ſoluzione del II. problema, e dividendo per $(1+t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}}$ il numeratore, e il denominatore di quella frazione, il di cui logaritmo fa il ſecondo membro dell' equazione (7), ſi à ſempre

$$S.\frac{dt}{1+tt} = \log.(1-t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1+t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

e in virtù del corollario precedente, ſi à ancora

$$(8)\ S.\frac{dt}{1+tt} = \log.(AA-BB)$$

Queſte due ultime equazioni manifeſtano una nuova, e belliſſima proprietà del cerchio, ciaſcun arco del di cui quadrante à per ſuo elemento $\frac{dt}{1+tt}$, quando la t denota la tangente dell' arco medeſimo.

Figure 5. Fagnano, [30], p. 490

Finally, in the last chapter Fagnano raises the issue of calculating the ellipsis arc (or more precisely, finding this circular or hyperbolic arc, which is equal to the ellipsis arc). Using his method of studying leminscates, Fagnano noticed the ratio between the arcs on the ellipsis. In solving this problem, an integral appeared that was later called ellipitical. Fagnano was proud to be the first to use imgainary degrees. His work served as a stimulus for Euler's thinking.

5. 1797, Caspar Wessel, the complex number as a directed segment, geometric interpretation of operations

In 1797, geodesist and cartographer Caspar Wessel submitted to the Danish Academy of Sciences the work *An essay on the analytical presentation of direction and its applications, primarily to solving flat and spherical polygons* [51], published in 1799. Wessel introduced the concept of a complex number as a directed segment, defined addition as a parallel displacement of the plane, and multiplication as rotation of the plane with stretching. Operations on complex numbers were applied to operations on geometric objects. The work was intended for cartographers, published in Danish, and remained unknown to the mathematical community for over a century.

6. 1806, 1813/14, Argand. Geometric diagrams

Argand's biography is the subject of debate, and there is even doubt over his real name. It is believed that his name was Jean-Robert Argand and that he lived from 1768 to 1822. We recommend that readers consult the study by G. Schubring [44], although his interpretation also contains contradictions.

In 1806 in France, an anonymous brochure was published with the title *An essay on a certain method of presenting imaginary values in geometric constructions* [2] , which developed the geometric theory of the complex number. In particular, it states that when multiplying complex numbers, their arguments add up, and the modules extend. The so-called Argand diagrams are introduced in the work, depicting operations of multiplication on the circumference, raising to a power and extracting root from a complex number. As it was subsequently confirmed, the author of the brochure was the amateur mathematician Argand. It is known that A.-M. Legendre highly valued the copy sent to him by the author [45].

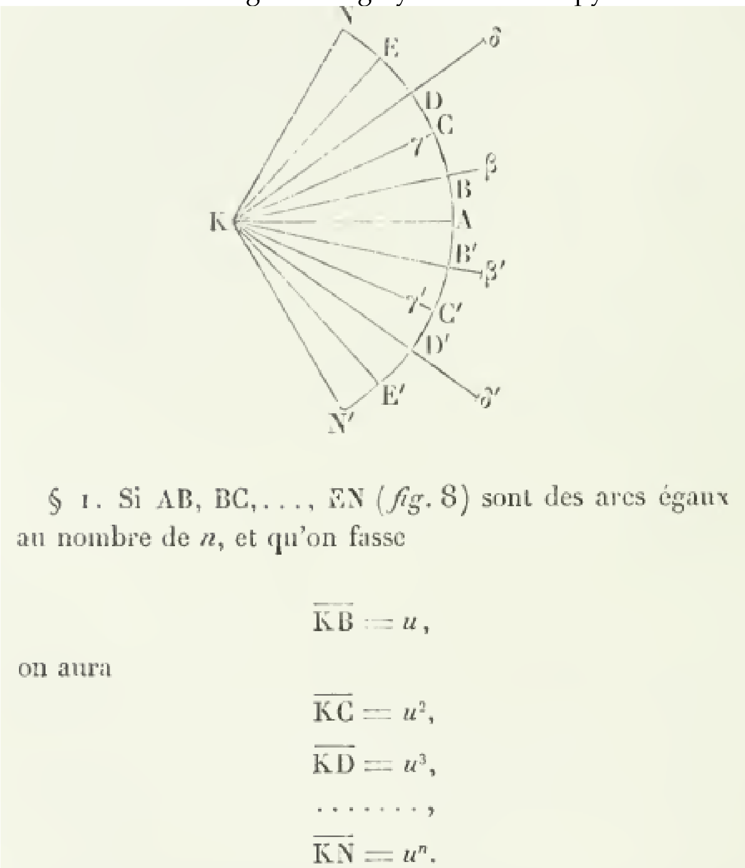


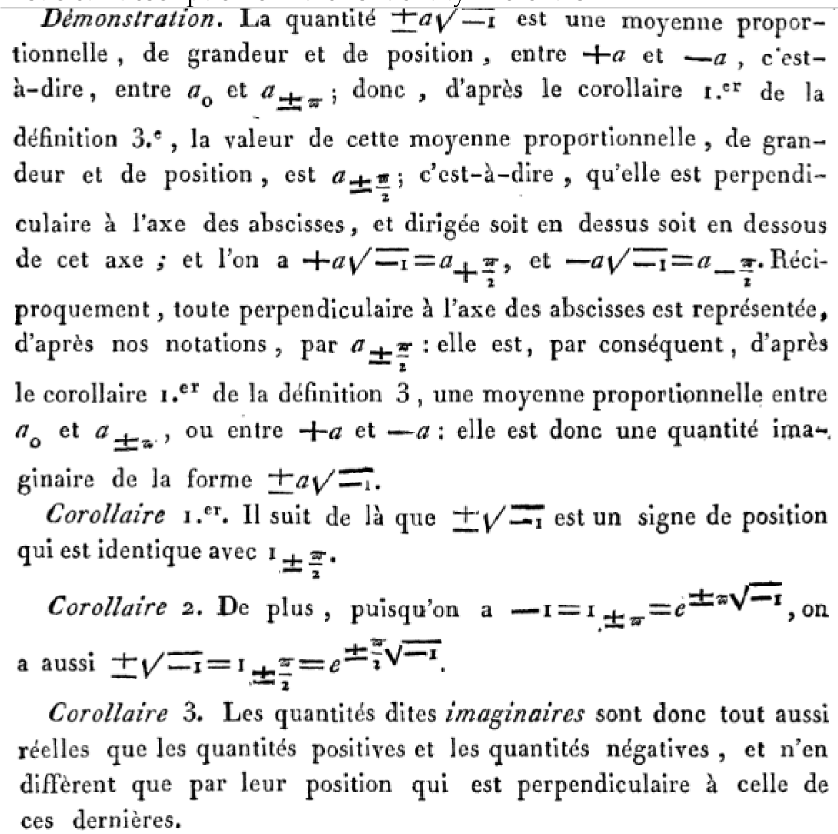
Figure 6. Argand's diagram on raising a complex number to a power [4]

7. 1813, Jacques Français. The first appearance of the famous formula

At this time there were two brothers living in France, army officers who became teachers of mathematics after they retired from service. The elder brother, François Joseph Français (1768–1810), taught mathematics at civil and military academies. He was friendly with L.-F. Arbogast, wrote several unpublished memoirs about differential calculation and its application in artillery, which was highly valued by A.-M. Legendre, J.L. Lagrange, S.F. Lacroix and J.B. Biot. Legendre gave him a copy of Argand's anonymous brochure of 1806. The younger brother, Jacques Frédéric Français (1775–1833) served in the engineers' corps, and then was a professor of military science in Metz. He wrote works on the integration of partial differential equations, and on analytical geometry, including on the transforming oblique coordinates and their application to solving the problem of finding a sphere tangent to four given spheres. After the death of his elder brother in 1810, Jacques Français studied his mathematical archive and continued his mathematical researches. He published four memoirs on his brother's ideas, adding to them in such a way that it difficult to tell the contributions of each brother apart. In September 1813 in the second issue of the 4th volume of Gergonne's journal *New principles of positional geometry and interpretation of imaginary symbols* [32], in which he gave a geometric presentation of complex numbers with interesting applications. This publication was preceded by a furious discussion between J. Francais, Argand and F. Servois. Français and Argand argued for the correctness of geometric presentation, while Servois claimed that complex numbers should only be interpreted using pure algebra.

In his work, J. Français referred to Argand's anonymous brochure of 1806: "In fairness, I must state that the basis of these new ideas does not belong to me. I found this in a letter by Legendre to my late brother, which the great geometer told him about. Accordingly, my contribution comes down to the method of explaining and demonstrating these principles, to the designation and the idea of indicating the position . . . I publish the results I have obtained in the hope that the first author of these ideas will make himself known".

It was in this article, on page 64, that Jacques Français writes: $+1 = e^{0\pi\sqrt{-1}}$, and $-1 = e^{\pm\pi\sqrt{-1}}$. This is the first clear description of Euler's identity. Here it is:



Démonstration. La quantité $\pm a\sqrt{-1}$ est une moyenne proportionnelle, de grandeur et de position, entre $+a$ et $-a$, c'est-à-dire, entre a_0 et $a_{\pm\pi}$; donc, d'après le corollaire 1.er de la définition 3.e, la valeur de cette moyenne proportionnelle, de grandeur et de position, est $a_{\pm\frac{\pi}{2}}$; c'est-à-dire, qu'elle est perpendiculaire à l'axe des abscisses, et dirigée soit en dessus soit en dessous de cet axe; et l'on a $+a\sqrt{-1}=a_{+\frac{\pi}{2}}$, et $-a\sqrt{-1}=a_{-\frac{\pi}{2}}$. Réciproquement, toute perpendiculaire à l'axe des abscisses est représentée, d'après nos notations, par $a_{\pm\frac{\pi}{2}}$: elle est, par conséquent, d'après le corollaire 1.er de la définition 3, une moyenne proportionnelle entre a_0 et $a_{\pm\pi}$, ou entre $+a$ et $-a$: elle est donc une quantité imaginaire de la forme $\pm a\sqrt{-1}$.

Corollaire 1.er. Il suit de là que $\pm\sqrt{-1}$ est un signe de position qui est identique avec $1_{\pm\frac{\pi}{2}}$.

Corollaire 2. De plus, puisqu'on a $-1=1_{\pm\pi}=e^{\pm\pi\sqrt{-1}}$, on a aussi $\pm\sqrt{-1}=1_{\pm\frac{\pi}{2}}=e^{\pm\frac{\pi}{2}\sqrt{-1}}$.

Corollaire 3. Les quantités dites *imaginaires* sont donc tout aussi réelles que les quantités positives et les quantités négatives, et n'en diffèrent que par leur position qui est perpendiculaire à celle de ces dernières.

Figure 7. The first appearance of Euler's famous identity, 1813. Français, J. F. Nouveaux principes ([32], p. 64)

In November 1813 in the fifth issue of the 4th volume of this journal, Argand responded with his article *Thoughts on the new theory of imaginary numbers with subsequent application to providing the theorem of analysis* [3], where he admitted his authorship of the anonymous brochure of 1806, mentioned that Legendre was familiar with it and had a high opinion the brochure, and also set out with several clarifications his new proof of the main theorem of algebra, and introduced the term "module of a complex number" in its modern meaning.

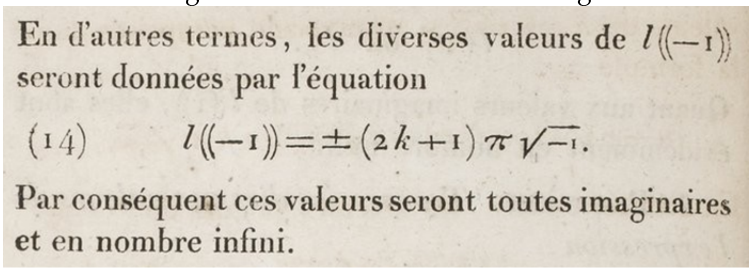
In Gergonne's journal *Annales de mathématiques pures et appliquées* before 1832 there were several articles with examples of calculations of logarithms and exponential functions of a complex variable. Other French and German mathematical journals of the period of 1830-1875 did not devote any attention to Euler's identity.

8. Augustin Cauchy

8.1. 1821, Analyse algébrique

In 1821, Augustin Cauchy gave a course of lectures of analysis at the Polytechnic School. Cauchy expresses the logarithm of -1: $l(-1) = \pm(2k + 1)\pi\sqrt{-1}$ ([12], p. 315), but does not give an explanatory note. He proposes to denote the main value of the logarithm (with $k = 0$) as $l(a)$ or simply la. Cauchy has no geometric interpretation of complex numbers and operations on them. This was a study course, not a scientific research. In his description Cauchy holds to the main points of Euler's article *On the logarithm of the negative number* [26].

Here is a fragment on the calculation of the logarithm of minus one:



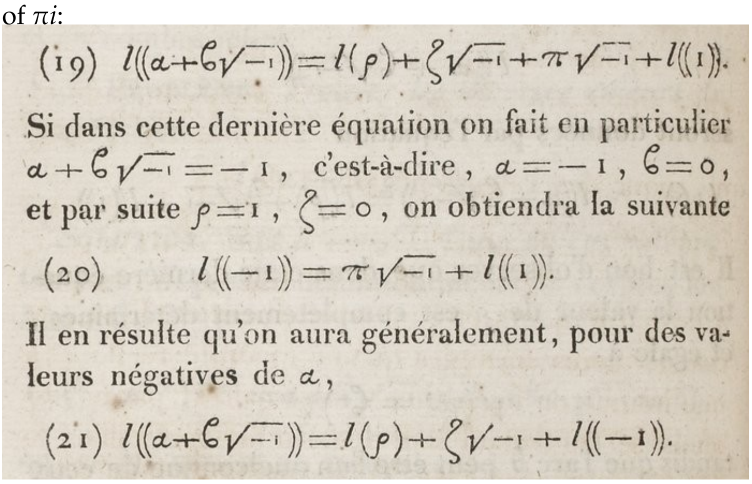
En d'autres termes, les diverses valeurs de $l((-1))$ seront données par l'équation

$$(14) \qquad l((-1)) = \pm(2k+1)\pi\sqrt{-1}.$$

Par conséquent ces valeurs seront toutes imaginaires et en nombre infini.

Figure 8. Cauchy. ([12], p. 316)

On page 318 Cauchy repeats the formula for the main and general value of the logarithm for the value of πi:



$$(19) \quad l((\alpha+\beta\sqrt{-1})) = l(\rho) + \zeta\sqrt{-1} + \pi\sqrt{-1} + l((1)).$$

Si dans cette dernière équation on fait en particulier $\alpha+\beta\sqrt{-1} = -1$, c'est-à-dire, $\alpha = -1$, $\beta = 0$, et par suite $\rho = 1$, $\zeta = 0$, on obtiendra la suivante

$$(20) \qquad l((-1)) = \pi\sqrt{-1} + l((1)).$$

Il en résulte qu'on aura généralement, pour des valeurs négatives de α,

$$(21) \quad l((\alpha+\beta\sqrt{-1})) = l(\rho) + \zeta\sqrt{-1} + l((-1)).$$

Figure 9. Cauchy. ([12], p. 318)

8.2. *Subsequent works*

Later, in 1829–1832 Cauchy made a great contribution to the theory of complex variable functions – he created the theory of residues. In works on this topic Cauchy often used the illustrative form of presentation both for proving continuity (*On expansion of functions into ordered sequences by rising variables*) ([13]), when calculating contour integrals.

He repeatedly addressed the topic of representing numbers and functions in exponential form. In 1831–1832, in emigration in Turin, Cauchy published a lithographed edition in two notebooks *Résumés Analytiques*, which he republished several times after returning to France [14]. This resume contained the results of various fields of his studies, including §XV. *Imaginary Exponents. Expansion through the functions cosx, sinx* [15]. Here on p. 140 Euler's identity is found in the form $e^{\pi\sqrt{-1}} = -1$. In 1846, Cauchy published *A Memoir on functions of the imaginary variable* ([14]), and in 1846 *Memoir on a new theory of imaginary numbers and on symbolic roots of equations and on identities*, also published in the 30th issue of Crelle's journal. Here Cauchy presents his theory of algebraic equivalence, in which imaginary numbers are examined as equivalent classes of polynomials with real coefficients according to the module $(x^2 + 1)$. The identity that he gives here and calls symbolic is $i^2 + 1 = 0$ ([16], p. 319).

In subsequent years in European journals, authors did not display great interest in issues of trigonometric and exponential presentation of complex numbers and their interpretation.

9. William Hamilton

William Hamilton[9)] was the royal astronomer of Ireland, mathematician, mechanic, physicist, and examined algebra not as an art, a language, a science on quantity, but as a science on order in certain series. Hamilton determined the vector as the tranfer. His symbol *i* means firstly the single vector of the *Ox* axis, secondly, an imaginary unit, and thirdly, the rotation operator, the versor. Hamilton wanted to apply the system of complex numbers to three-dimensional space, but discovered difficulties with determining multiplication – either the commutative law or the distributive law was violated. This contradicted Peacock principle of permanence[10)], until in 1843 Hamilton determined operations on quaternions. Subsequently his theory of quaternions served as the basis for creating vectoral analysis.

10. The mid-19th century, USA. B. Peirce

The American astronomer and mathematician Benjamin Peirce[11)] made a contribution to celestial and analytical mechanics, statistics (Peirce's criteria), number theory, linear algebra, geodesy and the philosophy of mathematics. It was thanks to his influence that courses on mathematics began to be read in American universities, and mathematical research was carried out; mathematics became an academic science ([**?**]), [8]. His courses included a study of the classics of mathematics, and expressed great admiration for their mathematical discoveries. One of his students, W. E. Byerly, subsequently a Harvard University professor recalled one episode in 1864: "...In one of his lectures he gave a ratio linking π, e and i, in this form: $e^{\frac{\pi}{2}} = \sqrt[i]{i}$, which evidently dazzled his imagination. He dropped the chalk and eraser, thrust his hands into his pockets, and after contemplating the formula for several minutes turned to the class and said very slowly and impressively: "Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth" [1], [38].

Peirce later repeated his ideas about this equation in his —emphLinear Associative Algebra. This treatise, in the words of Peirce himself, was designed to bring the fundamental principles of science to the central profound source, and from there take a short path to the most fruitful forms of research.

11. The 20th century

In 1963, Richard Feynman in his famous lecturers on physics called Euler's equation "This is a fantastic fact, which we must leave to the Mathematics Department to prove. The proofs are very beautiful and very interesting, but certainly not self-evident. [...] The most remarkable formula in mathematics: $e^{i\theta} = cos\theta + isin\theta$. This is our jewel" ([31], Vol. I. Lecture 22. Algebra. 22-4, 22-6).

In 1988, mathematician David Wells conducted a survey among readers of *The Mathematical Intelligencer*, to choose the most beautiful theorem, one of twenty-four [52]. In 1990 he summed up the results of the survey in an article [53]. The majority of readers chose Euler's identity.

12. Conclusion

Thus, we have traced the journey of Euler\'s formulas which arose from observations of the correspondence between arithmetic and geometric progressions, expressed geometrically; the emergence of the algebraic form of the connection between them, the logarithmic and exponential functions and their expression through series; their auxiliary role in integration. The significance of natural sources of these formulas was replaced with a working application in mathematics and applied matters, the attempt to strengthen the algebraic aspect, the wary attitude of logicians and philosophers of mathematics, the enthusiastic attitude of physicists, and finally the recognition of Euler\'s identity as the most beautiful formula in mathematics. In the 20th century, the apology for this formula grew like the apology for the Golden ratio.

---

9) William Rowan Hamilton, 1805–1865.

10) "Whatever form is algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote" ([42], p. 104).

11) Benjamin Peirce, 1809–1880, father of Charles Peirce.

Acknowledgement

The paper has been presented at the International Conference Moderm Problems of Mathematics and Mechanics, dedicated to the memory of the brilliant Azerbaijani scientist and thinker of the XIII century Nasireddin Tusi (Baku, Azerbaijan, July 3–6, 2024). The author is grateful to the organizers of the conference.

References

[1] R. C. Archibald, A. L. Lowell, W. E. Byerly, A.B. Chace, *Benjamin Peirce*, The American Mathematical Monthly. **32** (1925), 1, 1–30.
[2] R. Argand, *Essai sur une manière de représenter des quantités imaginaires dans les constructions géométriques*, (2e éd.), Gauthier Villars, Paris, 1874.
[3] Argand, *Réflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théorème d'analise*, Annales de mathématiques pures et appliquées. **5** (1814–1815), 197–209.
[4] J. R. Argand, *Imaginary quantities; their geometrical interpretation*, D. Van Nostrand, New York, 1881.
[5] Jacob Bernoulli, *Quaestiones nonnullæ de usuris, cum solutione problematis de sorte alearum, propositi in Ephem. Gall. A.*, Acta Eruditorum, **May** (1690), 219–223.
[6] Jacob Bernoulli, *Lineae cycloidales, Evolutae, Ant-Evolutae, Causticae, Anti-Causticae, Peri-Causticae. Earum usus & simplex relation ad se invicem. Spira mirabilis*, Acta Eruditorum, **May** (1692), 207-213.
[7] Johann I. Bernoulli, *Op. LXX, Solution d'un Problême Concernant le calcul intégral, avec quelques abregés par raport à ce calcul. Par M. Bernoulli Professeur à Groningue. Le tout extrait d'une de ses Lettres écrite de Groningue le 5. Aoust 1702.*, Mémoires de l'Académie royale des sciences (1704), 289-297.
[8] Johann I. Bernoulli, *Manière abrégées de transformer les différentielles composées en simples, & reciproquement; Et même les simples imaginaires en réelles composées*, Tomus primus quo continentur ea quae ab anno 1690 ad annum 1713 prodierunt. Lausannae & Genevae, sumptibus Marci-Michaelis Bousquet & sociorum. **6**, XXIV, 563, [1] p., XXIII c. di tav. 399–400.
[9] F. W. Bessel, *Ueber die Berechnung der geo-graphischen Längen und Breiten aus geodätischen Vermessungen*, Astronomische Nachrichten **4** (1825), 241-254.
[10] R. Bombelli, *L'algebra parte maggiore dell'aritmetica divisa in tre libri di Rafael Bombelli da Bologna*, Nella stamperia do Guovanni Rossi, Bologna, 1572.
[11] H. Cardano, *Artis magnae, sive de regulis algebraicis, liber unus*, Papiae, A. Osiandro, 1545.
[12] A.-L. Cauchy, *Cours d'Analyse de L'École Royale Polytechnique. Analyse Algébrique*, Imprimerie royale, Paris, 1821.
[13] A.-L. Cauchy, *Sur les développement des fonctions series ordonnées suivant les puissances ascendan les des variables*, Journal des Mathématiques Pures et Appliquées, (1846), 313–330.
[14] A.-L. Cauchy (1846), *Mémoire sur les fonctions de variables imaginaires*. Oeuvres completes. Série 2. Tome 13. l'Académie des sciences, Paris, 405–435.
[15] A.-L. Cauchy (1847), *Mémoire sur une nouvelle théorie des imaginaires, et sur les racines symboliques des équations et des équivalences*, Oeuvres completes. Série 1. Tome 10, l'Académie des sciences, Paris, 312-323.
[16] A.-L. Cauchy (1847), *Des exponentielles imaginaires. Développements des fonctions cosx, sinx*, Oeuvres completes. Série 1. Tome 10, l'Académie des sciences, Paris, 133-141.
[17] R. Cotes, *Logometria*, Philosophical Transactions of the Royal Society of London, **29 (338)**, 1714, 5–45.
[18] R. Cotes, with R. Smith *Harmonia mensurarum: sive analysis & synthesis per rationum & angulorum mensuras promotæ: accedunt alia opuscula mathematica*, Convolute, separate pagination. Edidit et auxit Robertus Smith, Cambridge, 1722.
[19] J. Le Rond D'Alembert, *Sur les Logarithmes des quantités negatives. Sixiéme Mémoire*, Opuscules mathématiques ou Mémoires sur différens sujets de géométrie, de méchanique, d'optique, d'astronomie. Tome 1. Jombert, David, Briasson. Paris, (1756), 210–230.
[20] R. Descartes, (1737), *The geometry of Renâe Descartes*, ed. D.E. Smith, The Open Court Pub. Co., Chicago, 1925.
[21] A. Dürer, *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*, Hieronymus Andreae, Nüremberg, 1525.
[22] L. Euler, *Da summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum*, Miscellanea Berolinensia ad incrementum scientiarum, **VII** (1743), 172-192. = Opera omnia, Series 1, Volume 14: 138–155.
[23] L. Euler, *Introductio in analysin infinitorum*, Volume 1, Marcum-Michaelem Bousquet, Lausanne, 1748.
[24] L. Euler, *Recherches sur les racines imaginaires des équations*, Mémoires de l'académie des sciences de Berlin, (1749), 1751, 222–288.
[25] L. Euler, *De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrale licet. M.S. Academiae exhibit. Die 5 Maii 1777*, Euler L. Institutiones calculi integralis. Vol. 4. Petropoli: Impensis Academiae Imperialis Scientarum. 1794. 183–194. – p. 184.
[26] L. Euler, (1743) *Sur les logarithmes des nombres négatifs et imaginaires*, Opera postuma mathematica et physica, Vol.1 (1862), 269-281.
[27] L. Euler L. *De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrale licet*, M.S. Academiae exbibit. Die 5 Maii 1777. P. 183–194. in: Euler L. Institutiones calculi integralis. Vol. 4. Petropoli: Impensis Academiae Imperialis Scientarum. 1794.
[28] R. Gowing, *Roger Cotes – Natural Philosopher*, Cambridge University Press. 1983.
[29] R. Gowing, *Halley, Cotes, and the Nautical Meridian*, Historia Mathematica, **22** (1995), 119–32. – p. 27.
[30] G. C. Fagnano, *Produzioni matematiche del conte Giulio Carlo di Fagnano... Tomo primo*, In Pesaro, nella stamperia Gavelliana, 1750.
[31] R. P. Feynman, *The Feynman Lectures on Physics, Vol. I. Lecture 22. Algebra. No 22-6. Imaginary exponents*, Basic Books; 50th New Millennium ed. edition, 2011.
[32] J. F. Français, *Nouveaux principes de géométrie de position, et interprétation géométrique des symboles imaginaires*, Annales de Mathématiques pures et appliquées, **4**, (1813—1814), 61–71.

[33] W. Jones, *Synopsis Palmariorum Matheseos: Or, a new Introduction to the Mathematics. . . .*, Printed by Jeff. Wale at the Angel in St. Paul's Church-Yard, London, 1706.

[34] J. Ivory, *A new series for the rectification of the Ellipsis: together with some observations of the evolution of the formula* $(a^2 + b^2 - 2abcos\phi)^n$, Transactions of the Royal Society of Edinburgh, **4** (1798), 177–190.

[35] G.W. Leibniz, *Specimen novum analyseos pro scientia infini, circa Summas & Quadraturas*, Acta eruditorum, **May** (1702), 210–219.

[36] Ab. de Moivre, *Aequationum Quarundam Potestatis Tertiae, Quintae, Septimae, Nonae,& Superiorum, ad Infinitum Usque Pergendo, in Terminis Finitis, ad Instar Regularum pro Cubicis Quae Vocantur Cardani, Resolutio Analytica*, Philisophical Transactions, **305-312** (1706/1707), 2368–2371.

[37] Ab. de Moivre, *De Sectione Anguli*, Philisophical Transactions, **374** (1722), 228–230.

[38] P.J. Nahin, *Dr. Euler's Fabulous Formula*, Princeton University Press, Princeton and Oxford, 2006.

[39] J. Napier, *Mirifici Logarithmorum Canonis Descriptio: Ejusque usus, in utraque Trigonometria; ut etiam in omni Logistica Mathematica, Amplissimi, Facillimi, & expeditißimi explicatio*, Hart, Edinburgi, 1614.

[40] I. Newton, *De analysi per aequationes numero terminorum infinitas. Sent by Dr. Barrow to Mr. Collins in a Letter dated July 31. 1669*, in: I. Newton. Analysis per Quantitatum Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis, Pearson, London (1711).

[41] I. Newton, *Philosophiae Naturalis Principia Mathematica*, J. Streater, London, 1687.

[42] G. Peacock, *A treatise on algebra*, J. & J.J. Deighton, London, Cambridge, 1830.

[43] L. Pepe, *La formazione filosofica e scientifica di Giulio Carlo de' Toschi di Fagnano*, Conferenze e seminari Associazione Subalpina Mathesis a cura di E. Gallo, L. Giacardi, C.S. Roero. Torino (2000) 82–94.

[44] G. Schubring, *Argand and the Early Work on Graphical Representation: New Sources and Interpretations*, Around Caspar Wessel and the Geometric Representation of Complex Numbers. Proceedings of the Wassel Symposium at The Royal Danish Academy of Sciences and Letters, Copenhagen, August 11–15 1998 / J. Lützen (Ed.). Kongelige Danske Videnskabernes Selskab, Copenhagen. (2001), 125–146.

[45] G. I. Sinkevich, *On the Development of a Complex Number Interpretation from the 16th to the End of the 19th Century*, Archives Internationales d'Histoire des Sciences **73/1** (2023), 172–207.

[46] E. Torricellii, *Opera geometrica Evangelistae Torricellii*, Masse & de Landis, Florentiae, 1644.

[47] A. P. Yushkevich, *Istoriya matematiki (History of Mathematics)*, **Vol. 3**, Matematika XVIII stoletiya, Nauka, Moscow, 1972 (in Russian).

[48] P. Varignon, *Nouvelle formation de spirales*, Histoire de l'académie royale des science, Paris (1704), 69–131.

[49] *Virorum Celeberr. Got. Gul. Leibnitii Et Johan. Bernoullii Commercium Philosophicum Et Mathematicum*, **Vol. 2**, Sumpt. M.M. Bousquet et socior, Lausannae et Genevae, 1745.

[50] J. Wallis, *A treatise of algebra, both historical and practical*, printed by John Playford, London, 1685. Separate pagination.

[51] C. Wessel, *On the analytical Representation on Direction; an Attempt, applied Chiefly to the Solution of Plane and Spherical Polygons*, in: Smith D.E. A source book in Mathematics. **Vol. 3**, Dover publications, New York (1950), 55–66.

[52] D. Wells, *Which is the most beautiful?*, Mathematical Intelligencer, **Vol.3, No. 4** (1988), 30–31.

[53] D. Wells, *Are These the Most Beautiful?*, Mathematical Intelligencer, **Vol.12, No. 3** (1990), 37–41.