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A remark on universal marked spaces

S. D. Iliadis[a,*], Yu. V. Sadovnichy[a]

[a]*Moscow State University (M.V. Lomonosov), Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia*

Abstract. By *marked space* we mean a pair (X, a^X), where X is a topological space and a^X is a fixed point of X. By *embedding of a marked space* (Y, a^Y) *into a marked space* (X, a^X) we mean an embedding i of Y into X for which $i(a^Y) = a^X$. In this paper it is proved that: (1) in the class of all marked spaces there are universal elements and (b) in the Alexandroff cube F^τ there are points a, such that the marked space (F^τ, a) is universal in the class of all marked spaces (note that not all points of F^τ satisfy this condition), which are answers to the corresponding problems, putting in [3].

1. Introduction

All spaces are considered to be T_0-spaces of weight less than or equal to a **fixed** infinite cardinal, denoted by τ.

By *marked space* we mean a pair (X, a^X), where X is a topological space and a^X is a (fixed) point of X, called *marked* point. By *embedding* of a marked space (Y, a^Y) into a marked space (X, a^X) we mean an embedding i_X^Y of the space Y into X such that $i_X^Y(a^Y) = a^X$. Classes of marked spaces will be denoted by $(\mathbb{S}, *)$, where $\mathbb{S}$ is the class of all spaces X for which $(X, a^X) \in (\mathbb{S}, *)$ for some $a^X \in X$.

Let $(\mathbb{S}, *)$ be a class of marked spaces. A marked space (T, a^T) is called *universal in the class* $(\mathbb{S}, *)$ if:
(a) $(T, a^T) \in (\mathbb{S}, *)$ and
(b) for each element (X, a^X) of $(\mathbb{S}, *)$ there exists an embedding of (X, a^X) into (T, a^T).

Obviously, if in a class $(\mathbb{S}, *)$ of marked spaces there are universal elements, then in the class $\mathbb{S}$ of topological spaces there are also universal elements. However, the inverse is not true. For example, in the class of all locally finite-dimensional separable metrizable spaces there are universal elements (see [5]), although, it possible to prove that in the class of all marked locally finite-dimensional separable metrizable spaces there are no universal elements.

In the paper [2], universal elements are constructed, by a canonical way, for many classes of marked spaces. (We note that the marked points a^X of all elements X of the considered classes in [2] are *closed*, that is the singleton $\{a^X\}$ is a closed subset of X. Of course, this fact is not a restriction for T_1-spaces.) In particular,

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* Corresponding author: S. D. Iliadis
Email addresses: s.d.iliadis@gmail.com (S. D. Iliadis), sadovnichiy.yu@gmail.com (Yu. V. Sadovnichy)
ORCID iDs: https://orcid.org/0000-0003-4067-5793 (S. D. Iliadis), https://orcid.org/0000-0002-1575-4505 (Yu. V. Sadovnichy)

such classes are the class of all marked separable metrizable spaces and the class of all marked completely regular spaces.

For these classes universal marked spaces can be constructed also as follows. Using the fact that the Hilbert cub I^ω is homogeneous (that is, for each points $a, b \in I^\omega$ there is a homeomorphism h of I^ω onto I^ω such that $h(a) = b$) (see [4]) and universal for the class of all separable metrizable spaces it can be proved that any pair (I^ω, a^{I^ω}), where a^{I^ω} is an arbitrary point of I^ω, is an universal element in the class of all marked separable metrizable spaces. Similarly, using the homogeneity of the Tychonoff cub I^τ and its universality for the class of all completely regular spaces, it can be proved that any pair (I^τ, a^{I^τ}), where a^{I^τ} is an arbitrary point of I^τ, is an universal element in the class of all marked completely regular spaces.

We note that the Alexandroff cube F^τ (that is, the product of two-point connected spaces: $\{0_\delta, 1_\delta\}$, $\delta \in \tau$, with the topology $\{\emptyset, 0_\delta, \{0_\delta, 1_\delta\}\}$), is an universal space in the class $\mathbb{S}$ of all spaces of weight $\leq \tau$ (see [1]). The Alexandroff cube is not homogeneous. This follows from the fact that the point $a \equiv \{1_\delta : \delta \in \tau\} \in F^\tau$ is the unique point of F^τ, which is *closed*, that is the singleton $\{a\}$ is a closed subset of F^τ. Since there are spaces without closed points, the pair (F^τ, a) can not be an universal marked space for the class of all marked spaces. We note that by the same reason, the method of construction of universal marked spaces using in the paper [2] can not be used for the class of all spaces.

Thus, in view of the above considerations, the following questions naturally arise (see [3]):

(1) Are there universal elements in the class of all marked spaces?

(2) Is there a point $a^{F^\tau} \in F^\tau$ such that the pair (F^τ, a^{F^τ}) is an universal element in the class of all marked spaces?

In the present note, we shall give the positive answers to these questions.

2. The results

Concerning the Alexandroff cube F^τ, we shall use the following notions and notations. For each point $a \in F^\tau$ we denote by a_δ the δ-coordinate of a, that is

$$a = \{a_\delta : \delta \in \tau\}.$$

For each $\delta \in \tau$ we denote by U_δ the set of all points $a \in F^\tau$, whose δ-coordinate is equal to 0_δ. The set U_δ is an open subset of F^τ and the set

$$B \equiv \{\emptyset, F^\tau\} \cup \{U_\delta : \delta \in \tau\}$$

is a *subbase* of F^τ (that is, its finite intersections is a base for F^τ).

Let κ be a cardinal, $\kappa \leq \tau$. We denote by F^τ_κ the set of all points a of F^τ such that the set of all $\delta \in \tau$ for which $a_\delta = 0_\delta$ is equal to κ. Obviously, for each κ, $0 \leq \kappa \leq \tau$, the set F^τ_κ is not empty and $F^\tau_{\kappa_1} \cap F^\tau_{\kappa_2} = \emptyset$ if $\kappa_1 \neq \kappa_2$. Also, if $0 \leq \kappa < \tau$, then for each point $a \in F^\tau_\kappa$ the cardinality of the set of all $\delta \in \tau$ for which $a_\delta = 1_\delta$ is equal to τ. However, if $\kappa = \tau$, then the cardinality of this set may be any cardinal ν, $0 \leq \nu \leq \tau$. We shall denote by $F^\tau_{\tau,\nu}$ the set of all points a for which the cardinality of the set $\{\delta \in \tau : a_\delta = 0_\delta\}$ is equal to τ and the cardinality of the set $\{\delta \in \tau : a_\delta = 1_\delta\}$ is equal to ν.

Let X be an arbitrary space and

$$B^X = \{U^X_\delta : \delta \in \tau\}$$

an indexed base for the open subsets of X. Let $i^X_{F^\tau} : X \to F^\tau$ be a mapping defined as follows: for each point x of X the point $i^X_{F^\tau}(x)$ is the point a of F^τ for which $a_\delta = 0_\delta$ if $x \in U^X_\delta$ and $a_\delta = 1_\delta$ if $x \notin U^X_\delta$. This mapping is an embedding of X into F^τ (see [1]). (Indeed, since X is a T_0-space, the mapping $i^X_{F^\tau}$ is one-to-one. On the other hand, it is easy to prove that $i^X_{F^\tau}(U^X_\delta) = U_\delta \cap i^X_{F^\tau}(X)$, which shows that $i^X_{F^\tau}$ is an embedding of X into F^τ.)

Let (X, a^X) be a marked space. We say that *marked character of X at the point a^X*, denoted by $mc(X, a)$, is equal to a cardinal κ (it may be finite) if κ is the minimal cardinal for which there exists a base B^X for the open subsets of X such that the number of elements of B^X, containing the point a^X, is equal to κ.

Similarly, we say that *marked co-character of X at the point a^X*, denoted by $mcc(X, a)$, is equal to a cardinal ν if ν is the minimal cardinal for which there exists a base B^X for the open subsets of X such that the number of elements of B^X, which do not contain the point a^X, is equal to ν. It is easy to prove that the marked character of X at the point a^X coincides with the character of X at the point a^X if the space X is a T_1-space. However, this is not true if the space X is not T_1-space. For example, for the marked space $(C, 0)$, where $C \equiv \{0, 1\}$ is the above mentioned two-point connected space with the topology $\{\emptyset, 0.\{0, 1\}\}$, the character of C at the point 0 is 1 and the marked character is 2.

Theorem 2.1. *Let $a \in F^\tau_\kappa$, $0 < \kappa < \tau$. Then, the marked space (F^τ, a) is the universal element in the class of all marked spaces (X, a^X) for which $mc(X, a^X) \leq \kappa$.*

Proof. Denote by $0(a)$ the set of all elements δ of τ for which $a_\delta = 0_\delta$. Since $a \in F^\tau_\kappa$ and $0 < \kappa$ this set is not empty and since $\kappa < \tau$, $|\tau \setminus 0(a)| = \tau$. Let (X, a^X) be a marked space, where $mc(X, A^X) < \kappa$. Then, there exists an indexed base $B^X \equiv \{U^X_\delta : \delta \in \tau\}$ for the open subsets of X such that the cardinality of the set $\Delta(a^X)$ of all elements δ of τ for which $a^X \in U^X_\delta$ is equal to a cardinal $\nu \leq \kappa$. Therefore, $|\tau \setminus \Delta(a^X)| = \tau$. Let α be an one-to-one mapping of τ into τ such that $\alpha(\Delta(a^X)) \subset 0(a)$ and $\alpha(\tau \setminus \Delta(a^X)) = \tau \setminus 0(a)$. We note that $\alpha(\tau) = \tau \setminus (0(a) \setminus \alpha(\Delta(a^X)))$.

Now, we construct a mapping $h^X_{F^\tau} \equiv h$ of X into F^τ as follows. Let $x \in X$. In order to define the point $h(x) \equiv b$ it suffices to define the δ-coordinate of b for each $\delta \in \tau$. If $\delta \in 0(a) \setminus \alpha(\Delta(a^X))$ we put $b_\delta = 0_\delta = a_\delta$. If $\delta \notin 0(a) \setminus \alpha(\Delta(a^X))$, then there exists $\delta' \in \tau$ such that $\alpha(\delta') = \delta$. In this case, we put $b_\delta = 0_\delta$ if $x \in U^X_{\delta'}$ and $b_\delta = 1_\delta$ if $x \notin U^X_{\delta'}$. Obviously, the point b is uniquely determined. We shall prove that $h^X_{F^\tau}$ is an embedding of the marked space (X, a^X) into the marked space (F^τ, a), which will complete the proof of the theorem.

Indeed, first we prove that $h^X_{F^\tau}$ is one-to-one. Let x, y be two distinct points of X. Since X is a T_0-space there exists $\delta' \in \tau$ such that the open set $U^X_{\delta'}$ contains one of the point x, y and does not contain the other, which means that δ-coordinates of the points $h^X_{F^\tau}(x)$ and $h^X_{F^\tau}(y)$ are distinct, where $\delta = \alpha(\delta') \notin (0(a) \setminus \alpha(\Delta(a^X)))$, proving that $h^X_{F^\tau}$ is one-to-one.

We note that if $\delta \in (0(a) \setminus \alpha(\Delta(a^X)))$, then for each $b \in h^X_{F^\tau}(X)$ we have $b_\delta = 0_\delta$, which means that $h^X_{F^\tau}(X) \subset U_\delta$. On the other hand, we have

$$\{U_\delta : \delta \in \tau\} = \{U_\delta : \delta \in (0(a) \setminus \alpha(\Delta(a^X)))\} \cup$$

$$\cup \{U_\delta : \delta \in \tau \setminus (0(a) \setminus \alpha(\Delta(a^X)))\}.$$

Since $\{U_\delta : \delta \in \tau\} \cup \{\emptyset, F^\tau\}$ is a subbase of F^τ, the set

$$\{U_\delta \cap h^X_{F^\tau}(X) : \delta \in \tau\} \cup \{\emptyset, F^\tau \cap h^X_{F^\tau}(X)\} = \{U_\delta \cap h^X_{F^\tau}(X) : \delta \in (0(a) \setminus \alpha(\Delta(a^X)))\} \cup$$

$$\cup \{U_\delta \cap h^X_{F^\tau}(X) : \delta \in \tau \setminus (0(a) \setminus \alpha(\Delta(a^X)))\} \cup \{\emptyset, F^X_\tau \cap h^X_{F^\tau}(X)\}$$

is a subbase for $h^X_{F^\tau}(X)$. The condition $U_\delta \cap h^X_{F^\tau}(X) = h^X_{F^\tau}(X)$ for each $\delta \in (0(a) \setminus \alpha(\Delta(a^X)))$ implies that the set

$$\{U_\delta \cap h^X_{F^\tau}(X) : \delta \in \tau \setminus (0(a) \setminus \alpha(\Delta(a^X)))\} \cup \{\emptyset, F^X_{F^\tau}(X)\} =$$

$$\{U_{\alpha(\delta')} \cap h^X_{F^\tau}(X) : \delta' \in \tau\} \cup \{\emptyset, F^X_{F^\tau}(X)\} \tag{2.1}$$

is a subbase for $h^X_{F^\tau}(X)$.

Now we prove that for every $\delta' \in \tau$ we have

$$h^X_{F^\tau}(U^X_{\delta'}) = U_{\alpha(\delta')} \cap h^X_{F^\tau}(X). \tag{2.2}$$

Let $\delta' \in \tau$. If $x \in U^X_{\delta'}$, then by the definition of the mapping $h^X_{F^\tau}$, $h^X_{F^\tau}(x) \in U_{\alpha(\delta')}$, proving that the left side of the relation (2.2) belongs to the right. Let $b \in U_{\alpha(\delta')} \cap h^X_{F^\tau}(X)$. Then, there exists $x \in X$ such that $h^X_{F^\tau}(x) = b$ and $b_{\alpha(\delta')} = 0_{\alpha(\delta')}$. By the definition of the mapping $h^X_{F^\tau}$, $x \in U^X_{\delta'}$. Since (2.1) is a subbase for $h^X_{F^\tau}(X)$ and $\{U^X_{\delta'} : \delta' \in \tau\}$ is a base for X, relation (2.2) shows that $h^X_{F^\tau}$ is an embedding.

Finally, we prove that $h^X_{F^\tau}(a^X) = a$. It suffices to prove that for each $\delta \in \tau$ we have $b_\delta = a_\delta$, where $b = h^X_{F^\tau}(a^X)$. Let $\delta \in \tau$. If $\delta \in (0(a) \setminus \alpha(\Delta(a^X)))$, then by the definition of the mapping $h^X_{F^\tau}$, $b_\delta = a_\delta = 1_\delta$.

Let $\delta \in \tau \setminus (0(a) \setminus \alpha(\Delta(a^X)))$. Then, there exists $\delta' \in \tau$ such that $\alpha(\delta') = \delta$. If $\delta' \in \Delta(a^X)$ and, therefore, $a^X \in U^X_{\delta'}$, then by the definition of the mapping $h^X_{F^\tau}$, $b_\delta = 0_\delta$. Since $\delta \in \alpha(\Delta(a^X)) \subset 0(a)$, $a_\delta = 0_\delta$ and, therefore, $b_\delta = a_\delta$. If $\delta' \notin \Delta(a^X)$ and, therefore, $a^X \notin U^X_{\delta'}$, then by the definition of the mapping $h^X_{F^\tau}$, $b_\delta = 1_\delta$. Since $\delta \in \alpha(\tau \setminus \Delta(a^X)) = \tau \setminus 0(a)$, $a_\delta = 1_\delta = b_\delta$. Hence, for each $\delta \in \tau$, $b_\delta = a_\delta$, proving that $h^X_{F^\tau}(a^X) = a$ and completing the proof of the theorem. □

Theorem 2.2. *Let $a \in F^\tau_{\tau,\nu}$, $0 \leq \nu \leq \tau$. Then, the marked space (F^τ, a) is an universal element in the class of all marked spaces (X, a^X) for which $mc(X, a^X) = \tau$ and $mcc(X, a^X) \leq \nu$.*

Proof. Let $1(a)$ and $0(a)$ be the sets of all $\delta \in \tau$ such that $a_\delta = 1_\delta$ and $a_\delta = 0_\delta$, respectively. Let (X, a^X) be a marked space such that $mc(X, a^X) = \tau$ and $mcc(X, a^X) = \mu \leq \nu$. Then, there exists an indexed base $B^X \equiv \{U^X_\delta : \delta \in \tau\}$ of X such that the cardinality of the set $\Delta(a^X)$ of all $\delta \in \tau$ for which $a^X \in U^X_\delta$ is equal to τ and the cardinality of the set $E(a^X)$ of all $\delta \in \tau$ for which $a^X \notin U^X_\delta$ is equal to μ. We have $|\Delta(a^X)| = |0(a^X)] = \tau$. Let α be an one-to-one mapping of τ into τ such that $\alpha(\Delta(a^X)) = 0(a)$ and $\alpha(E(a^X)) \subset 1(a)$.

Similarly to the corresponding construction of Theorem 2.1, we construct a mapping $h^X_{F^\tau} \equiv h$ of X into F^τ as follows. Let $x \in X$, $h(x) \equiv b$ and $\delta \in \tau$. We define the δ-coordinate of b setting $b_\delta = 1_\delta = a_\delta$ if $\delta \in 1(a) \setminus \alpha(E(a^X))$. If $\delta \notin 1(a) \setminus \alpha(E(a^X))$, then there exists $\delta' \in \tau$ such that $\alpha(\delta') = \delta$. In this case, we put $b_\delta = 0_\delta$ if $x \in U^X_{\delta'}$ and $b_\delta = 1_\delta$ if $x \notin U^X_{\delta'}$. As in Theorem 2.1, we can prove that $h^X_{F^\tau}$ is one-to-one. Further, by the same manner as in Theorem 2.1, we can prove that the set

$$\{U_{\alpha(\delta')} \cap h^X_{F^\tau}(X) : \delta' \in \tau\} \cup \{\emptyset, F^X_{F^\tau}(X)\} \tag{2.3}$$

is a subbase for $h^X_{F^\tau}(X)$ and that for each $\delta' \in \tau$ it is true the relation

$$h^X_{F^\tau}(U^X_{\delta'}) = U_{\alpha(\delta')} \cap h^X_{F^\tau}(X). \tag{2.4}$$

Since (2.3) is a subbase for $h^X_{F^\tau}(X)$ and $\{U^X_{\delta'} : \delta' \in \tau\}$ is a base for X, relation (2.4) shows that $h^X_{F^\tau}$ is an embedding.

Finally, we prove that $h^X_{F^\tau}(a^X) = a$, that is for each $\delta \in \tau$ we have $b_\delta = a_\delta$, where $b = h^X_{F^\tau}(a^X)$. Let $\delta \in \tau$. If $\delta \in (1(a) \setminus \alpha(E(a^X)))$ (and, therefore, $a_\delta = 1_\delta$), then by the definition of the mapping $h^X_{F^\tau}$, $b_\delta = 1_\delta = a_\delta$. Let $\delta \in \tau \setminus (1(a) \setminus \alpha(E(a^X)))$. Then, there exists $\delta' \in \tau$ such that $\alpha(\delta') = \delta$. If $\delta' \in E(a^X)$ and, therefore, $a^X \notin U^X_{\delta'}$, then by the definition of $h^X_{F^\tau}$, $b_\delta = 1_\delta$. Since $\delta \in \alpha(E(a^X)) \subset 1(a)$, $a_\delta = 1_\delta$ and, therefore, $b_\delta = a_\delta$. If $\delta' \notin E(a^X)$ and, therefore, $a^X \in U^X_{\delta'}$, then by the definition of $h^X_{F^\tau}$, $b_\delta = 0_\delta$. Since $\delta \in \alpha(\tau \setminus E(a^X)) = \tau \setminus 1(a)$, $a_\delta = 0_\delta$ and, therefore, $b_\delta = a_\delta$. Hence, for each $\delta \in \tau$, $b_\delta = a_\delta$, proving that $h^X_{F^\tau}(a^X) = a$ and completing the proof of the theorem. □

Corollary 2.3. *Let $a \in F^\tau$. Then, the marked space (F^τ, a) is an universal element in the class of all marked spaces if and only if $a \in F^\tau_{\tau,\tau}$.*

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