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On affine symmetries in 4−dimensional spaces of neutral signature

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Dedicated to Professor Graham S. Hall on the occasion of his 78th birthday

Abstract. The goal of this paper is to examine affine vector fields on 4−dimensional spaces endowed with a metric of signature $(+,+,-,-)$. The existence of affine symmetries is discussed on these spaces and its connection between the holonomy structure is presented. The holonomy types of such spaces admitting proper affine vector fields are obtained and some examples are given. Additionally, several remarks are made regarding the close relationship of symmetries to Ricci solitons.

1. Introduction

Examining geometric properties of symmetries has been an area of interest both in mathematics and physics. In general terms, a symmetry is described by local diffeomorphisms which preserves certain geometrical features of the manifold and that can be associated with vector fields. Some very well-known examples of these vector fields are affine, conformal, homothetic, Killing and projective vector fields. Considering these vector fields on 4−dimensional spaces and investigation of the geometric structures related to symmetries have a significant place in the literature (for example, see [1, 3, 5–9, 11, 13, 16, 20]). The object of this paper is to examine the case when a 4−dimensional manifold equipped with a neutral metric, i.e. it is of signature $(+,+,-,-)$, admits local groups of affine symmetries. It will be seen that if such a space has a vector field which is proper affine, then a parallel, symmetric tensor of second order that is not a constant multiple of the metric is permitted. Therefore, it is essential to consider such tensor fields for this signature. Fortunately, analysis of the complete set of local metrics has been carried out in [10] and all possible Segre types of parallel, symmetric tensor fields of second order are found for another purpose in this reference which will also be useful for the present study. The solutions to the problem for affine symmetries in 4−dimensional spaces of Lorentz signature $(+,+,+,-)$, referred to as *space-times*, are known and considered in [8] (see also, [3, 9]). Since the neutral signature is the most complicated metric signature as it leads to different geometric features (for instance, it admits independent pairs of orthogonal null vectors), it will be interesting to study affine vector fields for this signature as well.

This work is organized as follows: In Section 2, some necessary information about 4−dimensional spaces of neutral signature is presented together with the notations. Section 3 is devoted to an overview of the

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holonomy structure of these spaces where the related subalgebras are expressed. A brief knowledge on second order symmetric and parallel (also known as covariantly constant) tensor fields is provided and the algebraic types of these tensor fields are described. Section 4 introduces affine symmetries by examining the existence of affine vector fields over the manifold. Elements of the Lie algebra of such vector fields are characterized and their geometric interpretation is explained. The main results of the study are presented in Section 5 where the holonomy types permitting affine vector fields which are proper and the ones that cannot are also stated. In Section 6, some remarks are made regarding the link between the Ricci solitons and symmetries of 4−dimensional spaces. Finally, several examples of metrics and their affine vector fields are given in Section 7.

2. Preliminaries

Consider a structure $(\mathscr{M}, g)$ where $\mathscr{M}$ is a 4−dimensional, connected, smooth manifold and g is a metric of signature $(+,+,-,-)$ (neutral signature) on $\mathscr{M}$. Let ∇ be the covariant derivative from the Levi-Civita connection of g. Throughout the following, *Riem*, *Ricc*, C and r will denote the Riemann curvature tensor (with components $R^i{}_{jkl}$ leading to the curvature tensor of type $(0,4)$ with components $R_{ijkl} = g_{im}R^m{}_{jkl}$), Ricci tensor (with components $R_{ik} = R^j{}_{ijk}$), Weyl conformal tensor (with components $C^i{}_{jkl}$) and scalar curvature ($r = g^{ij}R_{ij}$) of $\mathscr{M}$, respectively. At $p \in \mathscr{M}$, the tangent space will be shown as $\mathscr{T}_p\mathscr{M}$ and a tangent vector $0 \neq u \in \mathscr{T}_p\mathscr{M}$ is named as *spacelike* if $g(u,u) > 0$, *timelike* if $g(u,u) < 0$ and *null* if $g(u,u) = 0$ at p. Let $\{x, y, s, t\}$ denote a pseudo-orthonormal basis for $\mathscr{T}_p\mathscr{M}$ so that $g(x,x) = g(y,y) = -g(s,s) = -g(t,t) = 1$. On the other hand, an alternative tetrad $\{l, n, L, N\}$ (referred to as *null basis*) satisfies $l = \frac{1}{\sqrt{2}}(x+t)$, $n = \frac{1}{\sqrt{2}}(x-t)$, $L = \frac{1}{\sqrt{2}}(y+s)$, $N = \frac{1}{\sqrt{2}}(y-s)$ where the only non-zero products are $g(L,N) = g(l,n) = 1$. For neutral signature, a 2−dimensional subspace $\mathscr{V}$ of $\mathscr{T}_p\mathscr{M}$ (called a 2−space) is either spacelike (for which every non-zero element of $\mathscr{V}$ is timelike or every non-zero element of $\mathscr{V}$ is spacelike) or timelike (for which $\mathscr{V}$ admits precisely two, distinct, null directions) or null (for which $\mathscr{V}$ admits precisely one null direction) or totally null (for which every non-zero element of $\mathscr{V}$ is null).

Let $\Lambda_p\mathscr{M}$ be the space of all bivectors (that is, 2−forms). Then, $0 \neq F \in \Lambda_p\mathscr{M}$ is called *simple* (respectively, *non-simple*) provided that the rank of F equals 2 (respectively, 4). The *blade* of a simple bivector F is a 2−space generated by the elements of $\mathscr{T}_p\mathscr{M}$, which is uniquely identified by F and will be indicated by $v \wedge w$ for $v, w \in \mathscr{T}_p\mathscr{M}$. According to the nature (i.e., timelike, spacelike, null or totally null) of the blade of a simple $F \in \Lambda_p\mathscr{M}$, it is said to be timelike, spacelike, null or totally null, respectively. The dual bivector of F will be denoted by $\overset{*}{F}$ which is defined by $\overset{*}{F}_{ij} = \frac{1}{2}\epsilon_{ijkl}F^{kl}$ with the operator "$*$" denoting the Hodge duality, $\epsilon_{ijkl} = \sqrt{\det g}\, \delta_{ijkl}$ being the classical pseudotensor, δ being the alternating symbol. In fact, $\Lambda_p\mathscr{M}$ is a Lie algebra with Lie bracket [ ] as product that can be decomposed into two special subalgebras denoted by $\overset{+}{S}_p$, $\overset{-}{S}_p$ which are 3−dimensional and defined, respectively, by $\overset{+}{S}_p = \{F \in \Lambda_p\mathscr{M} : \overset{*}{F} = F\}$, $\overset{-}{S}_p = \{F \in \Lambda_p\mathscr{M} : \overset{*}{F} = -F\}$. It follows that any $F \in \Lambda_p\mathscr{M}$ is written in a unique way as $F = \overset{+}{F} + \overset{-}{F}$ where $\overset{+}{F} \in \overset{+}{S}_p$, $\overset{-}{F} \in \overset{-}{S}_p$. In that case, we have $[\overset{+}{F}, \overset{-}{F}] = 0$ and $\overset{+}{F}_{ij}\overset{-}{F}{}^{ij} = 0$. Thus, one has $\Lambda_p\mathscr{M} = \overset{+}{S}_p \oplus \overset{-}{S}_p$.

It is useful to add that the following equivalent conditions: $F \in \Lambda_p\mathscr{M}$ is simple $\Leftrightarrow$ there exists $0 \neq u \in \mathscr{T}_p\mathscr{M}$ such that $F_{ij}u^j = 0 \Leftrightarrow F_{i[j}F_{kl]} = 0 \Leftrightarrow \overset{*}{F}_{ij}F^{jk} = \overset{*}{F}_{ij}F^{ij} = 0$ where the square brackets surrounding the indices symbolize the skew-symmetrisation (for details see page 175 of [3] where these results were proven for space-times but they hold for all signatures).

3. Holonomy structure and second order symmetric tensors

Let Φ_p denote the holonomy group of $(\mathscr{M}, g)$ at the point $p \in \mathscr{M}$. It is well-known that Φ_p involves parallel transport of the elements of the tangent space $\mathscr{T}_p\mathscr{M}$ around closed curves in $\mathscr{M}$ at p, which is a linear isomorphism on $\mathscr{T}_p\mathscr{M}$. As $\mathscr{M}$ is connected, one can consider the holonomy group of the manifold, indicated by Φ, as for any $p, q \in \mathscr{M}$, Φ_p and Φ_q are isomorphic. We refer to the reference [18] for full details of the holonomy theory. Being a Lie group, Φ has a Lie algebra indicated by ϕ and each holonomy algebra has a bivector representation. In fact, ϕ is subalgebra of the orthogonal algebra of g, i.e., $o(2,2)$ as g is of signature $(+,+,-,-)$. We shall make use of the subalgebras tabulated in [20] and up to isomorphism, these subalgebras are given in Table 1.

Table 1: Holonomy types for neutral signature, parallel vector fields and solutions to $\nabla h = 0$ are indicated.

Type	Basis	Parallel vector fields	Algebraic types for h
$1(a)$	$l \wedge n$	$\langle L, N\rangle$	$h_{ij} = ag_{ij} + bL_iL_j + cN_iN_j + d(L_iN_j + N_iL_j)$
$1(b)$	$x \wedge y$	$\langle s, t\rangle$	$h_{ij} = ag_{ij} + bs_is_j + ct_it_j + d(s_it_j + t_is_j)$
$1(c)$	$l \wedge y$	$\langle l, s\rangle$	$h_{ij} = ag_{ij} + bl_il_j + cs_is_j + d(l_is_j + s_il_j)$
$1(d)$	$l \wedge L$	$\langle l, L\rangle$	$h_{ij} = ag_{ij} + bl_il_j + cL_iL_j + d(l_iL_j + L_il_j)$
$2(a)$	$l \wedge n - L \wedge N, l \wedge N$	—	$h_{ij} = ag_{ij}$
$2(b)$	$l \wedge n, L \wedge N$	—	$h_{ij} = ag_{ij} + b(l_in_j + n_il_j)$
$2(c)$	$l \wedge n - L \wedge N, l \wedge L + n \wedge N$	—	$h_{ij} = ag_{ij} + b(n_iN_j + N_in_j - l_iL_j - L_il_j)$
$2(d)$	$l \wedge n - L \wedge N, l \wedge L$	—	$h_{ij} = ag_{ij} + b(l_iL_j + L_il_j)$
$2(e)$	$x \wedge y, s \wedge t$	—	$h_{ij} = ag_{ij} + b(x_ix_j + y_iy_j)$
$2(f)$	$l \wedge N + n \wedge L, l \wedge L$	—	$h_{ij} = ag_{ij} + b(l_il_j + L_iL_j)$
$2(g)$	$l \wedge N, l \wedge L$	$\langle l\rangle$	$h_{ij} = ag_{ij} + bl_il_j$
$2(h)$	$l \wedge N$, $\alpha(l \wedge n) + \beta(L \wedge N)$	— $\langle l\rangle$ $\langle N\rangle$	$h_{ij} = ag_{ij}$ $(\alpha\beta \neq 0)$, $h_{ij} = ag_{ij} + bl_il_j$ $(\alpha = 0)$ (or $h_{ij} = ag_{ij} + bN_iN_j$) $(\beta = 0)$
$2(j)$	$l \wedge N$, $\alpha(l \wedge n - L \wedge N) + \beta(l \wedge L)$	—	$h_{ij} = ag_{ij}$
$2(k)$	$l \wedge y, l \wedge n$ or $l \wedge s, l \wedge n$	$\langle s\rangle$ or $\langle y\rangle$	$h_{ij} = ag_{ij} + bs_is_j$ or $h_{ij} = ag_{ij} + by_iy_j$
$3(a)$	$l \wedge n, l \wedge N, L \wedge N$	—	$h_{ij} = ag_{ij}$
$3(b)$	$l \wedge n - L \wedge N, l \wedge N, l \wedge L$	—	$h_{ij} = ag_{ij}$
$3(c)$	$x \wedge y, x \wedge t, y \wedge t$ or $x \wedge s, x \wedge t, s \wedge t$	$\langle s\rangle$ or $\langle y\rangle$	$h_{ij} = ag_{ij} + bs_is_j$ or $h_{ij} = ag_{ij} + by_iy_j$
$3(d)$	$l \wedge N, l \wedge L, \alpha(l \wedge n) + \beta(L \wedge N)$	— $\langle l\rangle$	$h_{ij} = ag_{ij}$ $(\alpha \neq 0)$, $h_{ij} = ag_{ij} + bl_il_j$ $(\alpha = 0, \beta \neq 0)$
$4(a)$	$\overset{+}{S}, l \wedge n + L \wedge N$	—	$h_{ij} = ag_{ij}$
$4(b)$	$\overset{+}{S}, l \wedge L + n \wedge N$	—	$h_{ij} = ag_{ij}$
$4(c)$	$l \wedge L, l \wedge N, l \wedge n, L \wedge N$	—	$h_{ij} = ag_{ij}$
5	$\overset{+}{S}, \overset{-}{B}$	—	$h_{ij} = ag_{ij}$
6	$o(2,2)$	—	$h_{ij} = ag_{ij}$

In Table 1, the dimensions and nomenclature of each subalgebra can be read from the first column and the generators of these subalgebras in bivector notation are given in the second column. Columns 3 and 4 will be explained later. All possible *metric* holonomy types are included in this table. Moreover, a spanning set is denoted by the symbol $\langle\,\rangle$ and one has the subalgebras $\langle l \wedge n + L \wedge N, l \wedge L\rangle = \overset{-}{B}$, $\langle l \wedge n - L \wedge N, l \wedge N\rangle = \overset{+}{B}$ (subalgebras of $\overset{-}{S}_p$ and $\overset{+}{S}_p$, respectively). Besides, for type $2(j)$, $\beta \neq 0 \neq \alpha$ and for types $3(d)$ and $2(h)$, $\alpha \neq \pm\beta$ $(\alpha, \beta \in \mathbb{R})$. Additionally, the bivector $l \wedge s$ could also be a generator for the holonomy type $1(c)$ and the holonomy type $4(c)$ is spanned by $\langle \overset{+}{B}, \overset{-}{B}\rangle$.

A smooth, global tensor h over $\mathscr{M}$ is said to be *recurrent* if $\nabla h = h \otimes \lambda$ holds for some smooth λ (*recurrence* $1-$*form*) over $\mathscr{M}$. The case when λ vanishes over connected, open subset $\emptyset \neq \mathscr{U} \subset \mathscr{M}$, h is named as *parallel* over $\mathscr{U}$. It will be seen in Section 4 that second order symmetric and parallel tensor fields play an important role in detecting affine vector fields for each holonomy type. All such tensors with this property were found in [10] for another purpose but it will be useful here. A symmetric tensor field of second order may be classified at $p \in \mathscr{M}$ according to its Jordan canonical forms when one studies the linear map on $\mathscr{T}_p\mathscr{M}$ by considering its complex or real eigenvectors and eigenvalues. Such a classification has been given in [7] where all possible Segre types are presented. Note that the Segre type of a nowhere-zero recurrent, second order symmetric tensor field h (including any degeneracies) is the same at every point of $\mathscr{U}$ and that the eigenvalues of h may be considered as constant functions over $\mathscr{U}$, [14]. The algebraic types of symmetric and parallel tensor fields of second order are indicated in column 4 of Table 1 with constants a, b, c, d. It can be deduced from Table 1 that if $\mathscr{M}$ contains a nowhere-zero, second order symmetric, parallel tensor h that is not proportional to g, the holonomy type of $\mathscr{M}$ is either $1(a)$–$1(d)$, $2(b)$–$2(g)$, $2(h)$ $(\alpha\beta = 0)$, $2(k)$, $3(c)$, $3(d)$ $(\alpha = 0, \beta \neq 0)$. It is also noted that the solution of $\nabla h = 0$ gives rise to find a full set of alternative metrics for a given Levi-Civita connection and holonomy type. Moreover, such a metric need not be of signature $(+, +, -, -)$ (see, [10, 20]).

It should be pointed out that if a non-zero element u of $\mathscr{T}_p\mathscr{M}$ is an eigenvector of every bivector of ϕ, a (local) smooth recurrent vector field exists on a neighbourhood of p agreeing with u at p and such a vector field can be selected as parallel when the corresponding eigenvalues of common eigenvectors are all zero (see [3]). Based on this information, one gets possible (local) parallel vector fields for all holonomy types (if any) which are indicated in the third column of Table 1 and which will have an important place in determining proper affine vector fields.

Remark 3.1. *Note that there is also a notion called local holonomy group indicated by Φ_p^* at $p \in \mathscr{M}$ [18], being the holonomy group of $(\mathscr{U}, g)$ where $\mathscr{U} \neq \emptyset$ is a certain open, connected neighbourhood of p with metric restricted from g. The remainder of this work, we assume that each Φ_p^* is of the same dimension, so every Φ_p^* is isomorphic to the (restricted) holonomy group of $\mathscr{M}$. As a result, it possesses the identical Lie algebra ϕ as Φ (see [18]). The aim of this adoption is to prevent the holonomy from reducing locally on a certain open subset of $\mathscr{M}$, thus ensuring that the local holonomy features over $\mathscr{M}$ are uniform (see also, [3, 9, 10]).*

4. Affine symmetries

Let $\psi : \mathscr{M} \to \mathscr{M}$ be a bijection such that both itself and its inverse are smooth. Then ψ is said to be an *affine transformation* if the condition $\tilde{\psi}^\star\omega = \omega$ is satisfied where ω is the connection 1−form over the frame bundle of $\mathscr{M}$ associated with g, $\tilde{\psi}$ is the natural map on the frame bundle associated with ψ and "$\star$" denotes the pullback of $\tilde{\psi}$. Here, it is discussed the local Lie groups of local affine transformations described by a finite-dimensional Lie algebra of global affine vector fields on $\mathscr{M}$ which are defined as follows:

Definition 4.1. *Let ξ be a global smooth vector field on $\mathscr{M}$. If each of the smooth local diffeomorphisms associated with ξ preserves the geodesics of $\mathscr{M}$* and *their affine parameters, then ξ is called an affine vector field. Equivalently, the relation $\mathscr{L}_\xi\nabla = 0$ holds where $\mathscr{L}$ denotes the Lie derivative.*

In a local adaptation, one can describe affine vector fields by the following equations:

$$\nabla_j\xi_i = \frac{1}{2}h_{ij} + F_{ij}, \qquad \nabla_k h_{ij} = 0, \qquad \nabla_k F_{ij} = R_{ijkl}\xi^l \tag{1}$$

where ξ_i and R_{ijkl} indicate the components of ξ and *Riem*, respectively. Furthermore, $F \in \Lambda_p\mathscr{M}$ having components $F_{ij} = -F_{ji}$ in (1) is referred to as the *affine bivector* of ξ and h is a symmetric, parallel tensor of second order with components $h_{ij} = \mathscr{L}_\xi g_{ij}$ (for details, we refer to [3]).

Consider the vector space of affine vector fields on $\mathscr{M}$, which is symbolized by $\mathscr{A}(\mathscr{M})$ called the *affine algebra* of $\mathscr{M}$. It can be shown that $\mathscr{A}(\mathscr{M})$ is a Lie algebra under the Lie bracket operation. Any member $\xi \in \mathscr{A}(\mathscr{M})$ is uniquely identified by ξ, F and h at $p \in \mathscr{M}$ as defined in (1) yielding a first order system of differential equations and thus $\dim \mathscr{A}(\mathscr{M}) \leq 4 + 6 + 10 = 20$. It is remarked that the causal character (spacelike, timelike, null, totally null or non-simple) of the affine bivector is constant along the integral curves of ξ (see pages 288-289 in [3]).

The affine algebra $\mathscr{A}(\mathscr{M})$ possesses two important subalgebras called *Killing* and *homothetic* algebras which are defined as follows: If $h \equiv 0$ in (1), then ξ is said to be a *Killing vector field* and F is called the *Killing bivector* of ξ. Equivalently, a member $\xi \in \mathcal{K}(\mathscr{M})$ satisfies $\mathscr{L}_\xi g = 0$ where the *Killing algebra*, i.e., the Lie algebra of Killing vector fields is indicated by $\mathcal{K}(\mathscr{M})$. For 4−dimensional spaces equipped with a neutral metric, a comprehensive study of Killing symmetries has been done in [11] where several results on the theory of Killing orbits are obtained and also the isotropy subalgebras and the possible algebraic forms of the Ricci and Weyl tensors are determined for these spaces. A similar work for space-times was presented in [5] and an extension of [11] to each metric signature was carried out in [16]. On the other hand, if $h = 2cg$ in (1) with a constant real number c, one has $\mathscr{L}_\xi g = 2cg$ and ξ is called *homothetic* and F is said to be the *homothetic bivector* of ξ. The *homothetic algebra* consists of all homothetic vector fields over $\mathscr{M}$, which is also a finite-dimensional Lie algebra shown as $\mathcal{H}(\mathscr{M})$. Obviously, if $c = 0$, $\xi \in \mathcal{K}(\mathscr{M}) \subseteq \mathcal{H}(\mathscr{M})$. Besides, ξ is referred to as *proper homothetic* if $c \neq 0$. For 4−dimensional spaces endowed with a neutral metric, the zeros of proper homothetic vector fields have been explored recently in [12]. Finally, it is noted that the case when $\xi \in \mathscr{A}(\mathscr{M})$ is neither homothetic nor Killing will be referred to as *proper affine*. Holonomy types that may contain such vector fields will be discussed in Section 5. Additionally, examples will be given in Section 7.

5. Results on affine symmetries in 4−dimensional spaces of neutral signature

In this section, various results related to affine vector fields will be presented by considering the existence of symmetric tensor fields of second order. Several remarks will also be given for some special holonomy types. Let the set $\mathscr{S}(\mathscr{M})$ denote the collection of all second order symmetric, parallel tensor fields over $\mathscr{M}$. Take into account the linear map $f : \mathscr{A}(\mathscr{M}) \to \mathscr{S}(\mathscr{M})$ associating $\xi \in \mathscr{A}(\mathscr{M})$ with $h \in \mathscr{S}(\mathscr{M})$. Let $f(\mathscr{A}(\mathscr{M}))$ be denoted by $\mathscr{F}$ which is a subspace of $\mathscr{S}(\mathscr{M})$. Suppose that $g \notin \mathscr{F}$ and let $\dim \mathscr{F} = m$. In that case, there exists a basis $\{h_1, h_2, \ldots, h_m\}$ of $\mathscr{F}$ satisfying $f(\varphi_i) = h_i$ where $\varphi_i \in \mathscr{A}(\mathscr{M})$ for $i = 1, 2, \ldots, m$. It then follows from the work given in [2, 3, 8] that $\varphi_1, \varphi_2, \ldots, \varphi_m$ are *proper affine* vector fields and m is the maximum number of independent proper members of $\mathscr{A}(\mathscr{M})$ that $(\mathscr{M}, g)$ may admit. Moreover, the fact that the Killing algebra is the kernel of f yields that if $\dim \mathscr{A}(\mathscr{M}) = m'$ $(\geq m)$, then $\dim \mathcal{K}(\mathscr{M}) = m' - m$. In addition, if $g \in \mathscr{F}$, one can choose $h_1 = g$ so that φ_1 is a proper homothetic vector field and $\varphi_2, \ldots, \varphi_m$ are proper affine. With the help of these, the following results can now be given:

Theorem 5.1. *Let $\mathscr{M}$ be a connected* 4−*dimensional space equipped with a neutral metric. If the holonomy type of $\mathscr{M}$ is* $2(a)$, $2(h)$ $(\beta \neq 0 \neq \alpha)$, $2(j)$, $3(a)$, $3(b)$, $3(d)$ $(\alpha \neq 0)$, $4(a)$, $4(b)$, $4(c)$, 5 *or* 6, *no proper members of $\mathscr{A}(\mathscr{M})$ exist over $\mathscr{M}$.*

Proof. As mentioned above, if a 4−dimensional space of neutral signature contains a *proper* affine vector field, then $(\mathscr{M}, g)$ has a second order symmetric, parallel tensor field h that is not a constant multiple of g. On the other hand, for another purpose, it was proven in [10] that the local solutions to $\nabla h = 0$ can be found algebraically by solving the equation (2) expressed as

$$h_{ik}F^k{}_j + h_{jk}F^k{}_i = 0 \tag{2}$$

for each $F \in \phi$. These solutions in question are expressed in column 4 of Table 1. Thus, in the case when the only solution of $\nabla h = 0$ is the trivial solution $h = ag$ for some constant a, no (global) proper members of $\mathscr{A}(\mathscr{M})$ exist over $\mathscr{M}$ and Table 1 provides the result. Accordingly, from the assumption made in Remark 3.1 and Table 1, one gets holonomy types $2(a)$, $2(h)$ $(\beta \neq 0 \neq \alpha)$, $2(j)$, $3(a)$, $3(b)$, $3(d)$ $(\alpha \neq 0)$, $4(a)$, $4(b)$, $4(c)$, 5 or 6 and thus the proof is completed. □

Theorem 5.2. *Let $\mathscr{M}$ be a connected* 4−*dimensional space equipped with a neutral metric. In this case, the following hold:*

(i) *For all* 1−*dimensional holonomy types, if four or more affine vector fields exist over $\mathscr{M}$, some linear combination of them is homothetic.*

(ii) *If the holonomy type of* $\mathscr{M}$ *is* $2(b)$, $2(c)$, $2(d)$, $2(e)$, $2(f)$, $2(g)$, $2(h)$ $(\alpha\beta = 0)$, $2(k)$, $3(c)$, $3(d)$ $(\alpha = 0, \beta \neq 0)$ *and if two or more affine vector fields exist over* $\mathscr{M}$, *some linear combination of them is homothetic.*

Proof. First of all, consider holonomy type $1(a)$. For this type, it can be read from Table 1 that L and N are parallel. In this case, by keeping the assumption made in Remark 3.1, the solutions to $\nabla h = 0$ gives [10]

$$h_{ij} = ag_{ij} + bL_iL_j + cN_iN_j + d(L_iN_j + N_iL_j) \tag{3}$$

with a, b, c, d denoting constants. It then follows from (3) that there are four independent parallel, symmetric tensor fields of second order given by g_{ij}, L_iL_j, N_iN_j and $L_iN_j + N_iL_j$ yielding a basis for $\mathscr{S}(\mathscr{M})$ (see also the beginning of this section). Hence, $\mathscr{S}(\mathscr{M})$ is 4−dimensional. Thus, if four affine vector fields exist over $\mathscr{M}$, there exist $h_1, h_2, h_3, h_4 \in \mathscr{S}(\mathscr{M})$. If these are dependent, some linear combination of them is zero and the corresponding vector field must be Killing (and so, it is homothetic as desired). On the other hand, if they are independent, they form a basis for $\mathscr{S}(\mathscr{M})$. For this case, some linear combination of them equals $g \in \mathscr{S}(\mathscr{M})$ resulting a proper homothetic vector field. Similar analysis can be made for other 1−dimensional types (i.e., $1(b)$, $1(c)$, $1(d)$) which complete part (i).

For holonomy type $2(b)$, there exist *recurrent vector fields* l, n, L, N satisfying $\nabla_j l_i = l_i\gamma_j$, $\nabla_j n_i = -n_i\gamma_j$, $\nabla_j L_i = L_i\sigma_j$, $\nabla_j N_i = -N_i\sigma_j$ for smooth 1−forms γ and σ (for details, see [10]). In this case, we get $\nabla_k(l_in_j + n_il_j) = 0$ and as indicated in Table 1, the solutions to $\nabla h = 0$ gives

$$h_{ij} = ag_{ij} + b(l_in_j + n_il_j)$$

where a, b are constants. In that case, there are two independent parallel, symmetric tensor fields of second order given by g_{ij} and $l_in_j + n_il_j$ yielding a basis for $\mathscr{S}(\mathscr{M})$ and the result follows from similar steps carried out for $1(a)$. Other cases are obtained from column 4 of Table 1 as well and thus the proof is completed. □

Now, let us focus on special holonomy types and consider the dimension of $\mathscr{A}(\mathscr{M})$. For the case when the metric is of Lorentz signature, a detailed study was made in [2] (see also, [3, 8]). It will be useful to identify similar situations in neutral signature as well. In general, although $\dim \mathscr{A}(\mathscr{M}) \leq 20$, certain lower and upper limits can be determined on the dimension of $\mathscr{A}(\mathscr{M})$ for some holonomy types. Firstly, consider holonomy type $1(b)$ (it is similar to type R_4 in the Lorentz signature). From Table 1, one has timelike vector fields s and t that are parallel. It follows that the covectors corresponding to s and t satisfy $s_i = \partial_i q$ and $t_i = \partial_i\tilde{q}$ for smooth functions q and $\tilde{q}$ on $\mathscr{U} \subset \mathscr{M}$. In that case, the independent vector fields s, t and $\tilde{q}s - qt$ are members of $\mathcal{K}(\mathscr{M})$. Moreover, it can be read from Table 1 that $\mathscr{S}(\mathscr{M})$ is generated by tensors g_{ij}, s_is_j, t_it_j and $s_it_j + t_is_j$. The work carried out in Theorem 5.2 yields that three independent proper members of $\mathscr{A}(\mathscr{M})$, which are qs, $\tilde{q}t$ and $\tilde{q}s + qt$, are always contained. Combining all these, we get $6 \leq \dim \mathscr{A}(\mathscr{M})$. On the other hand, to find the upper limit for $\dim \mathscr{A}(\mathscr{M})$, choose an open neighbourhood $\widetilde{U} = U' \times U''$ of arbitrary $p \in \mathscr{M}$ so that U' is flat and s and t are tangent to U'. Therefore, $\mathscr{A}(\widetilde{U}) = \mathscr{A}(U') \oplus \mathcal{H}(U'')$, so it is obtained that $6 \leq \dim \mathscr{A}(\widetilde{U}) \leq 9$ and $3 \leq \dim \mathcal{K}(\widetilde{U}) \leq 6$. Hence, one gets $6 \leq \dim \mathscr{A}(\mathscr{M}) \leq 9$. In fact, one can decompose a member $\xi \in \mathscr{A}(\mathscr{M})$ as $\xi = v + v'$ such that $v' = \kappa s + \kappa' t$ where $\kappa = -g(\xi, s)$, $\kappa' = -g(\xi, t)$, and v is orthogonal to s and t. This gives rise to affine vector fields v and v' over $\mathscr{U} \subset \mathscr{M}$. The examination for $1(a)$ is carried out in a similar manner.

For holonomy type $2(b)$, $(\mathscr{M}, g)$ does not contain parallel vector fields. Nevertheless, the holonomy establishes a pair of mutually orthogonal 2−dimensional timelike distributions. Moreover, $\dim \mathscr{S}(\mathscr{M}) = 2$ by Table 1 and $\xi \in \mathscr{A}(\mathscr{U})$ $(\mathscr{U} \subset \mathscr{M})$ satisfies

$$\nabla_j\xi_i = a'g_{ij} + b'(l_in_j + n_il_j) + F_{ij} \tag{4}$$

where $a', b' \in \mathbb{R}$ and F denotes the affine bivector. Using the completeness relation $g_{ij} = Q_{ij} + Q'_{ij}$ in (4) where $Q_{ij} = l_in_j + n_il_j$ and $Q'_{ij} = L_iN_j + N_iL_j$, we get

$$\nabla_j\xi_i = (a' + b')Q_{ij} + a'Q'_{ij} + F_{ij}.$$

For this case, one can define affine vector fields by decomposing $\xi \in \mathscr{A}(\mathscr{U})$ (uniquely) as $\xi = v + v'$ where $v^i = Q^i{}_j\xi^j$ and $v'^i = Q'^i{}_j\xi^j$. It can be shown that v and v' are both affine being tangent to timelike submanifolds of decomposition. Consider now $\mathscr{U} = \mathscr{U}_1 \times \mathscr{U}_2$ where $\mathscr{U}_1$ and $\mathscr{U}_2$ are both timelike. In this case, the vector fields v and v' can be regarded as homothetic over $\mathscr{U}_1$ and $\mathscr{U}_2$. Because of this feature, $\mathscr{A}(\mathscr{U})$ is isomorphic to $\mathcal{H}(\mathscr{U}_1) \oplus \mathcal{H}(\mathscr{U}_2)$. Hence, $\dim \mathscr{A}(\mathscr{U}) = \dim \mathcal{H}(\mathscr{U}_1) + \dim \mathcal{H}(\mathscr{U}_2)$. As these 2−dimensional submanifolds are not flat, $\dim \mathcal{H}(\mathscr{U}_1) \leq 3$ and $\dim \mathcal{H}(\mathscr{U}_2) \leq 3$. Therefore, $\dim \mathscr{A}(\mathscr{U}) \leq 6$. On the other hand, these submanifolds may not contain non-trivial homothetic vector field in general and thus the lower bound for $\dim \mathscr{A}(\mathscr{U})$ is zero. Similar steps can be done for type $2(e)$, where the holonomy establishes a pair of mutually orthogonal 2−dimensional spacelike distributions.

Regarding holonomy type $3(c)$ (it is similar to the R_{13} holonomy type in the Lorentz signature), $\nabla s = 0$ and so s is a member of $\mathcal{K}(\mathscr{M})$. Furthermore, the covector corresponding to s satisfies $s_i = \partial_i q$ for smooth function q on $\mathscr{U} \subset \mathscr{M}$ and it yields that the vector field qs is proper affine. In light of these data, one gets $\dim \mathscr{A}(\mathscr{M}) \geq 2$. On the other hand, for an affine vector field ξ, the first relation of (1) gives

$$\nabla_j \xi_i = a' g_{ij} + b' s_i s_j + F_{ij} \tag{5}$$

where F denotes the affine bivector and $a', b' \in \mathbb{R}$. Consider now a smooth function ρ over $\mathscr{M}$ which is defined by $\rho = g(\xi, s)$. Deriving the relation $\rho = \xi^i s_i$ twice and using equations (1) and (5), it can be obtained that $\nabla_m \nabla_j \rho = 0$ which means that $d\rho$ is parallel. Therefore, $d\rho$ is proportional to s, i.e., $\rho_i = \varsigma s_i$ for some $\varsigma \in \mathbb{R}$. Moreover, one gets $\rho_i s^i = \nabla_i \xi^j s_j s^i = b' - a' = -\varsigma$ and contracting (5) over s^i yields that $F_{ij}s^i = 0$. Hence, the affine bivector is simple and it can be spacelike, timelike or null (cf. holonomy type R_{13} in Lorentz case). In this case, we can determine affine vector fields v and v' over $\mathscr{M}$ given by $v' = \rho s$ and $v = \xi + v'$ satisfying $g(s, v) = 0$. Let $\mathcal{J}$ indicate the 3−dimensional submanifold (having Lorentz signature and) emerging from the holonomy invariant distribution described by the subspace of $\mathscr{T}_p\mathscr{M}$ orthogonal to s at $p \in \mathscr{M}$. Besides, v is everywhere tangent to $\mathcal{J}$ for which s is orthogonal. As $\mathcal{J}$ is not flat, $\dim \mathcal{J} \leq 6$ which gives rise to $\dim \mathscr{A}(\mathscr{M}) \leq 8$. Similar comments can be made for type $2(k)$. In conclusion, the following results are proven:

Theorem 5.3. *Let $\mathscr{M}$ be a connected 4−dimensional space of neutral signature. In that case, one has the following:*

(i) *For holonomy types $1(a)$ and $1(b)$, $6 \leq \dim \mathscr{A}(\mathscr{M}) \leq 9$.*
(ii) *For holonomy types $2(b)$ and $2(e)$, $0 \leq \dim \mathscr{A}(\mathscr{M}) \leq 6$.*
(iii) *For holonomy types $2(k)$ and $3(c)$, $2 \leq \dim \mathscr{A}(\mathscr{M}) \leq 8$.*

In addition to the cases examined above, it is also useful to add some remarks for the holonomy types containing parallel vector fields in Table 1. The proof of the following corollary can be seen in column 3 of Table 1 and the methods used in the proof of Theorem 5.3.

Corollary 5.4. *Let $\mathscr{M}$ be a connected 4−dimensional space of neutral signature. In that case, one has the following:*

(i) *If the holonomy type of $\mathscr{M}$ is $1(c)$, there exist proper affine vector fields $\tilde{a}l$, $\tilde{b}s$ and $\tilde{a}s + \tilde{b}l$ where the covectors $\tilde{l}$ and $\tilde{s}$ corresponding to parallel vector fields l and s respectively satisfy $\tilde{l} = d\tilde{a}$ and $\tilde{s} = d\tilde{b}$ for some functions $\tilde{a}$ and $\tilde{b}$ that are smooth over $\mathscr{U} \subset \mathscr{M}$.*
(ii) *If the holonomy type of $\mathscr{M}$ is $1(d)$, there exist proper affine vector fields $\tilde{a}l$, $\tilde{b}L$ and $\tilde{a}L + \tilde{b}l$ where the covectors $\tilde{l}$ and $\tilde{L}$ corresponding to parallel vector fields l and L respectively satisfy $\tilde{l} = d\tilde{a}$ and $\tilde{L} = d\tilde{b}$ for some functions $\tilde{a}$ and $\tilde{b}$ that are smooth over $\mathscr{U} \subset \mathscr{M}$.*
(iii) *If the holonomy type of $\mathscr{M}$ is $2(g)$, $2(h)$ $(\alpha = 0)$ or $3(d)$ $(\alpha = 0)$, there exists a proper affine vector field $X \equiv ql$ (or $X \equiv qN$ for $2(h)$ $(\beta = 0)$) so that the covector $\tilde{l}$ (or $\tilde{N}$) corresponding to the (null) parallel vector field l (or N) satisfies $\tilde{l} = dq$ (or $\tilde{N} = dq$) for some smooth function q over $\mathscr{U} \subset \mathscr{M}$.*

6. Remarks on symmetries and Ricci solitons

In this section, further remarks are given regarding the close relationship of symmetries with Ricci solitons and some situations for neutral signature are discussed. Ricci solitons, which have been intensively studied recently, are important for their connection with Ricci flows introduced by R. Hamilton [15]. A *Ricci soliton* is described by the following equation

$$\frac{1}{2}\mathscr{L}_\eta g + Ricc = \lambda g \tag{6}$$

where $\mathscr{L}_\eta$ symbolizes the Lie derivative along the vector field η which is named as *potential field* and λ indicates a constant. Such a structure will be shown as $(\mathscr{M}, g, \eta, \lambda)$. Ricci solitons have a close relationship with special vector fields that appear in the concept of symmetry. In this regard, we can give the following theorem for 4−dimensional manifolds equipped with a neutral metric:

Theorem 6.1. *Consider a Ricci soliton $(\mathscr{M}, g, \eta, \lambda)$ on a 4−dimensional space of neutral signature. In this case, the following conditions are valid:*

(i) *If $\xi \in \mathcal{K}(\mathscr{M})$, then $[\xi, \eta] \in \mathcal{K}(\mathscr{M})$ where $[\ ,\ ]$ indicates the Lie bracket of ξ and η.*

(ii) *Ricc is a parallel tensor field, i.e., $\nabla Ricc \equiv 0$ if the potential field $\eta \in \mathscr{A}(\mathscr{M})$. Accordingly, possible algebraic types for Ricc in Segre notation (respectively, corresponding holonomy types) are $\{(1111)\}$ (in which case, the condition of being a (proper) Einstein space is fulfilled) $2(b)$, $2(c)$, $2(e)$, $2(f)$, $3(a)$, $3(b)$ or one of the holonomy types whose dimension is greater than or equal to 4) $\{1(111)\}$ ($3(c)$), $\{(11)(11)\}$ (spacelike eigenspaces) ($1(b)$ or $2(e)$), $\{(11)(11)\}$ (timelike eigenspaces) ($1(a)$ or $2(b)$), $\{(zz)(\bar{z}\bar{z})\}$ ($2(c)$), $\{(211)\}$ ($1(c)$ or $2(g)$), $\{(22)\}$ ($2(d)$ or $2(f)$).*

(iii) *If $\xi \in \mathcal{H}(\mathscr{M})$, then $[\xi, \eta] \in \mathscr{A}(\mathscr{M})$ provided that $\nabla Ricc \equiv 0$. Particularly, in case of $(\mathscr{M}, g)$ being an Einstein space, no proper homothetic vector field exists, in other words any homotheties are Killing.*

(iv) *If η belongs to any of the sets $\mathcal{K}(\mathscr{M})$, $\mathcal{H}(\mathscr{M})$ or $C(\mathscr{M})$, then the condition of being an Einstein space is fulfilled, that is, Ricc is of Segre type $\{(1111)\}$ and so the possible holonomy types for this algebraic form which is stated as the first case in (ii) are achieved.*

Proof. (i) Assume that ξ is a Killing vector field on $(\mathscr{M}, g, \eta, \lambda)$. Then, one has $\mathscr{L}_\xi g = 0$ and $\mathscr{L}_\xi Ricc = 0$. Taking the Lie derivative of (6) with respect to ξ and considering these facts, it is found that $\mathscr{L}_\xi(\mathscr{L}_\eta g) = 0$. Considering the latter equation and the identity given by

$$\mathscr{L}_{[\xi,\eta]} g = \mathscr{L}_\xi(\mathscr{L}_\eta g) - \mathscr{L}_\eta(\mathscr{L}_\xi g), \tag{7}$$

it can be obtained that $\mathscr{L}_{[\xi,\eta]} g = 0$ and this implies $[\xi, \eta] \in \mathcal{K}(\mathscr{M})$ as desired.

(ii) Suppose that η is a member of $\mathscr{A}(\mathscr{M})$. In this case, the conditions (1) are satisfied for η together with $\mathscr{L}_\eta g_{ij} = h_{ij}$ where h is a parallel tensor field of second order. As shown in [17] for Lorentz signature, taking the covariant derivative of (6) and using (1), it is achieved that $\nabla Ricc = 0$, i.e., Ricci tensor must be parallel (cf. [19]). Then, the result follows from Theorem 3 in [10].

(iii) Now, let us assume that the vector field ξ is homothetic over $(\mathscr{M}, g, \eta, \lambda)$. Then, one has $\mathscr{L}_\xi g = 2cg$ for some constant c and $\mathscr{L}_\xi Ricc = 0$. Taking the Lie derivative of (6) with respect to ξ and considering these facts, it yields that $\mathscr{L}_\xi(\mathscr{L}_\eta g) = 4\lambda cg$. Plugging these into (7) and using (6) give

$$\begin{aligned} \mathscr{L}_{[\xi,\eta]} g &= 4\lambda cg - \mathscr{L}_\eta(2cg) \\ &= 4\lambda cg - 2c(2\lambda g - 2Ricc) \\ &= 4cRicc. \end{aligned} \tag{8}$$

It is clear from (8) that if $\nabla Ricc = 0$, $[\xi, \eta]$ is an affine vector field. Besides, the case when $(\mathscr{M}, g)$ is an Einstein space, in other words, if $Ricc = \kappa g$ for some constant κ, it follows from (6) that $\mathscr{L}_\eta g = 2(\lambda - \kappa)g$, i.e., $\eta \in \mathcal{H}(\mathscr{M})$. In this case, $[\xi, \eta] \in \mathcal{K}(\mathscr{M})$ for any $\xi \in \mathcal{H}(\mathscr{M})$ satisfying $\mathscr{L}_\xi g = 2cg$ and hence $c = 0$ from (8). Therefore, $\xi \in \mathcal{K}(\mathscr{M})$ meaning that any homotheties on $(\mathscr{M}, g, \eta, \lambda)$ are Killing.

(iv) Let η be an element of the conformal algebra $C(\mathscr{M})$ which is the set of all conformal vector fields. In this case, the condition $\mathscr{L}_\eta g = 2\tau g$ is satisfied for some function $\tau : \mathscr{M} \to \mathbb{R}$ which is smooth. After substituting this into (6), Einstein space condition holds, which is known as the trivial situation in literature, and so τ must be a constant. Thus, η must be a member of $\mathcal{H}(\mathscr{M})$ and *Ricc* is of Segre type $\{(1111)\}$ satisfying $\nabla Ricc = 0$. Therefore, from (ii) one gets the potential holonomy types. The other cases for η are trivial.

Hence, the proof is completed. □

Remark 6.2. *Note that items (i) and (iii) of Theorem 6.1 are fairly general and are valid for any dimension and metric signature.*

Remark 6.3. *It is also useful to note that item (i) of Theorem 6.1 is a special property of the Ricci solitons where the potential field η need not be a member of $\mathcal{K}(\mathscr{M})$. More generally, it is clear that if ξ and ν are members of $\mathcal{K}(\mathscr{M})$, then $[\xi, \nu] \in \mathcal{K}(\mathscr{M})$. Nevertheless, the case when ξ and $[\xi, \nu]$ are Killing vector fields, it does not require ν to be an element of $\mathcal{K}(\mathscr{M})$. To see this, for example, consider $\mathscr{M} = \mathbb{R}^4$ with coordinates (x, y, s, t) admitting the metric*

$$-dt^2 + \exp(x)(dx^2 + dy^2 - ds^2)$$

for which $\dim \mathcal{K}(\mathscr{M}) = 4$ and in particular, the timelike vector field $\xi \equiv -\frac{\partial}{\partial t}$ is Killing. Now, let $\nu = \frac{\partial}{\partial y} + t\frac{\partial}{\partial t}$. It can be checked that $[\xi, \nu] = \xi \in \mathcal{K}(\mathscr{M})$ but $\mathscr{L}_\nu g = -2dtdt \neq 0$ and hence $\nu \notin \mathcal{K}(\mathscr{M})$.

7. Examples

Example 7.1. *First of all, consider $\mathscr{M} = \mathbb{R}^4$ with coordinates (u, v, y, s) and the line element*

$$v^2du^2 + 2dudv + v^2dy^2 - ds^2 \tag{9}$$

where $v > 0$. Regarding the metric (9), the timelike vector field $\frac{\partial}{\partial s}$ is parallel and so $(\mathscr{M}, g)$ contains a global proper affine vector field $s\frac{\partial}{\partial s}$. On the other hand, Ricc is found as

$$R_{ij} = 2(g_{ij} + s_is_j) \tag{10}$$

where s_i are the components of the covector corresponding to $\frac{\partial}{\partial s}$ and hence it is of Segre type $\{1(111)\}$. Since $\frac{\partial}{\partial s}$ is parallel, it follows from (10) that $\nabla Ricc = 0$. Moreover, the manifold is conformally flat and $r = 6$. The holonomy group is of type $3(c)$. Now, define $\eta = 2s\frac{\partial}{\partial s}$. Then, η is an affine vector field and $\mathscr{L}_\eta g_{ij} = -4s_is_j$. In this case, by using (6) and (10), we get

$$\frac{1}{2}\mathscr{L}_\eta g + Ricc = 2g$$

which means that $(\mathscr{M}, g, \eta, \lambda)$ is a Ricci soliton with an affine potential field η and $\lambda = 2$. Since $\lambda > 0$, it is referred to as a shrinking Ricci soliton. Hence, Theorem 6.1 (ii) is satisfied for $(\mathscr{M}, g)$ (where the Segre type of Ricc is $\{1(111)\}$ and holonomy type is $3(c)$ as expressed earlier). It can also be checked that $\dim \mathcal{K}(\mathscr{M}) = 7$ and for $\xi \in \mathcal{K}(\mathscr{M})$, $[\xi, \eta] \in \mathcal{K}(\mathscr{M})$. In conclusion, Theorem 6.1 (i) holds as well. Additionally, one gets $\dim \mathscr{A}(\mathscr{M}) = 8$ and so Theorem 5.3 (iii) is fulfilled for holonomy type $3(c)$.

Example 7.2. *Let us consider $\mathscr{M} = \mathbb{R}^4$ with coordinates (u, v, y, s) and the following metric:*

$$s^2du^2 + 2dudv + dy^2 - ds^2. \tag{11}$$

For the metric (11), Riem is nowhere-zero on the manifold and the vector field $l := \frac{\partial}{\partial v}$ is parallel and null. The Weyl tensor C satisfies $C_{ijkh}l^h = 0$ and its type is $(\mathbf{N}, \mathbf{N})$ everywhere on $\mathscr{M}$ with repeated principal null direction (pnd) generated by l (for details of the Weyl types in neutral signature, we refer to [4]). The components of Ricc can be expressed as $R_{ij} = l_il_j$ which means that its Segre type is $\{(211)\}$ with eigenvalue zero and also $\nabla Ricc = 0$. On the *other hand,* $\dim \mathcal{K}(\mathscr{M}) = 6$ *and the vector field* $\xi := 2v\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} + s\frac{\partial}{\partial s}$ *is proper homothetic. Note that* $(\mathscr{M}, g)$ *also contains proper affine vector field* $y\frac{\partial}{\partial y}$ *where* $\frac{\partial}{\partial y}$ *is spacelike and parallel.*

Next, define the vector field $\eta := (2v - u)\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} + s\frac{\partial}{\partial s}$. *Then, it can be calculated that* $\mathscr{L}_\eta g = 2(g - Ricc)$ *and the left hand side of (6) gives*

$$\frac{1}{2}\mathscr{L}_\eta g + Ricc = g$$

yielding that $(\mathscr{M}, g, \eta, \lambda)$ *is a shrinking Ricci soliton with the potential field* η *and* $\lambda = 1 > 0$.

Example 7.3. *Consider now* $\mathscr{M} = \mathbb{R}^4$ *with coordinates* (u, v, y, s) *and the following metric:*

$$\exp(uy)dudv + dy^2 - ds^2 \tag{12}$$

which has also been considered in [12] for detecting the zeros of the homothetic vector fields. Regarding the metric (12), Ricc is of type $\{31\}$ *on* $\mathscr{M}$ *and C is of type* $(\mathbf{III}, \mathbf{III})$ *everywhere on* $\mathscr{M}$.

On the other hand, the vector field $\xi := 2u\frac{\partial}{\partial u} - 6v\frac{\partial}{\partial v} - 2y\frac{\partial}{\partial y} - 2s\frac{\partial}{\partial s}$ *is proper homothetic. Moreover,* $\dim \mathcal{K}(\mathscr{M}) = 2$ *so that the independent Killing vector fields are* $\frac{\partial}{\partial v}$ *and* $\frac{\partial}{\partial s}$. *Besides,* $\frac{\partial}{\partial s}$ *is a parallel timelike vector field and so one can derive the global proper affine vector field* $\eta := s\frac{\partial}{\partial s}$. *In that case, it can be calculated that* $\mathscr{L}_\eta g_{ij} = -2s_i s_j = h_{ij}$ *and* $\nabla h = 0$.

Example 7.4. *Let us take into account* $\mathscr{M} = \mathbb{R}^4$ *with coordinates* (x, y, s, t) *for which the line element is given by*

$$s^2dx^2 + s^2dy^2 - ds^2 - dt^2 \tag{13}$$

where $s > 0$. *For the metric (13), Riem is nowhere-zero on* $\mathscr{M}$ *and Ricc can be expressed as*

$$R_{ij} = \frac{1}{s^2}\left(g_{ij} + s_i s_j + t_i t_j\right) \tag{14}$$

where s_i *and* t_i *are the components of the covectors corresponding to* $\frac{\partial}{\partial s}$ *and* $\frac{\partial}{\partial t}$. *It yields from (14) that Ricc is of Segre type* $\{(11)(11)\}$. *Moreover,* $\dim \mathcal{K}(\mathscr{M}) = 4$, *more precisely,* $(\mathscr{M}, g)$ *admits* $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ *as Killing vector fields. Besides,* $\eta := -s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}$ *satisfies* $\mathscr{L}_\eta g = -2g$, *i.e., the vector field* η *is proper homothetic where* $c = -2$ *as described in Section 4. Furthermore,* $\frac{\partial}{\partial t}$ *is a timelike, parallel vector field. Accordingly, for the vector field* $\xi := t\frac{\partial}{\partial t}$, *one can get* $R_{ijkl}\xi^l = 0$ *and* $\mathscr{L}_\xi g_{ij} = -2t_i t_j$ *where* $h_{ij} = -2t_i t_j$ *is a parallel tensor field. Hence,* ξ *is a proper affine vector field.*

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