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$\mathcal{Q}$-curvature tensor in spacetimes and $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{ij}\mathcal{T}^{ij}\right)$-gravity

Uday Chand De[a], Krishnendu De[b,*], Abdallah Abdelhameed Syied[c], Nasser Bin Turki[d]

[a]*Department of Pure Mathematics,University of Calcutta, 35 Ballygaunge Circular Road, Kolkata -700019, West Bengal, India*
[b]*Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India*
[c]*Department of Mathematics, Faculty of Science, Zagazig University, Egypt*
[d]*Department of Mathematics, College of Science, King Saud University, P.O. Box 2455,Riyadh 11451, Saudi Arabia*

Abstract. In this article, we characterize a spacetime endowed with $\mathcal{Q}$-curvature tensor. We acquire that a $\mathcal{Q}$-flat perfect fluid spacetime is either, vacuum or, a de-Sitter spacetime. Moreover, it is established that a $\mathcal{Q}$-flat perfect fluid spacetime becomes a Robertson-Walker spacetime. Besides, it is proved that a $\mathcal{Q}$-recurrent spacetime is a generalized Ricci recurrent spacetime. Finally, we examine the impact of $\mathcal{Q}$-flat perfect fluid spacetime solutions in $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{ij}\mathcal{T}^{ij}\right)$-gravity.

1. Introduction

A spacetime can be defined as a 4-dimensional Lorentzian manifold M^4 bearing the signature $(-,+,+,+)$ for the Lorentzian metric g, allowing for a globally time-oriented vector.

A generalized Robertson-Walker (GRW) spacetime ([2], [9], [10]) is a Lorentzian manifold M^n $(n \geq 4)$, the metric of which is

$$ds^2 = -(d\zeta)^2 + \psi^2(\zeta)\, g^*_{v_1 v_2} dx^{v_1} dx^{v_2}, \tag{1}$$

where the function ψ is dependent on ζ and $g^*_{v_1 v_2} = g^*_{v_1 v_2}(x^{v_3})$ are only functions of x^{v_3} $(v_1, v_2, v_3 = 2, 3, \ldots, n)$. Another way to structure equation (1) is by the warped product $-\mathcal{I} \times \psi^2 \tilde{M}$, where $\mathcal{I}$ is included in $\mathbb{R}$ and $\tilde{M}^{n-1}$ is a Riemannian manifold. The GRW spacetime becomes a Robertson-Walker (RW) spacetime if dimension of the Riemannian manifold $\tilde{M}$ is 3 with constant sectional curvature.

A Lorentzian manifold M^4 is defined as a perfect fluid spacetime (PFS) if the non-zero Ricci tensor $\mathcal{R}_{kl}$ obeys

$$\mathcal{R}_{kl} = c_1 g_{kl} + d_1 u_k u_l, \tag{2}$$

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* Corresponding author: Krishnendu De
Email addresses: uc_de@yahoo.com (Uday Chand De), krishnendu.de@outlook.in (Krishnendu De), a.a_syied@yahoo.com (Abdallah Abdelhameed Syied), nassert@ksu.edu.sa (Nasser Bin Turki)
ORCID iDs: https://orcid.org/0000-0002-8990-4609 (Uday Chand De), https://orcid.org/0000-0001-6520-4520 (Krishnendu De), https://orcid.org/0000-0001-5568-7661 (Abdallah Abdelhameed Syied), https://orcid.org/0000-0002-9738-026X (Nasser Bin Turki)

in which c_1, d_1 are scalars and the velocity vector u_l is a unit time-like vector. The energy-momentum tensor (EMT) in general relativity (GR) for a PFS is given by [32]

$$\mathcal{T}_{kl} = pg_{kl} + (p + \mu) u_k u_l \tag{3}$$

where μ and p are the energy density and isotropic pressure, respectively. Additionally, p and μ are linked by a state equation of the type $p = p(\mu)$, and the PFS is called isentropic. Also, if $p = \mu$, it is named by stiff matter fluid [8]. In PFS, if $p = 0$, $p + \mu = 0$, and $p = \frac{\mu}{3}$, then it is called the dust matter fluid, dark matter era (DME), and radiation era, respectively [8].

Einstein's field equations (EFEs) without a cosmological constant is written by

$$\mathcal{R}_{kl} - \frac{1}{2} g_{kl}\mathcal{R} = \kappa\mathcal{T}_{kl}, \tag{4}$$

where κ indicates gravitational constant, $R = g^{kl}R_{kl}$ stands for the Ricci scalar.
In a 4-dimensional Riemannian or semi-Riemannian manifold, Mantica and Suh [31] presented a novel curvature tensor, indicated by Q and defined as

$$Q^h_{ijk} = \mathcal{R}^h_{ijk} - \frac{\phi}{3} \left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\} \tag{5}$$

where $\mathcal{R}^h_{ijk}$ indicates the curvature tensor and ϕ is an arbitrary scalar function. Several researchers, including ([27], [40]) and numerous others, examined Q^h_{ijk} in Riemannian and semi Riemannian manifolds.

Multiplying (5) with g_{lh}, we get

$$Q_{lijk} = \mathcal{R}_{lijk} - \frac{\phi}{3} \left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}. \tag{6}$$

In GR theory, when studying wormholes and black holes in different modified gravity systems, like $f(\mathcal{R})$, $f(\mathcal{T})$, $f(G)$, $f(\mathcal{R}, G)$, $f(\mathcal{R}, L_m)$ and $f(\mathcal{R}, \mathcal{T})$ gravity ([3], [4], [6], [11], [12], [13], [15], [16], [17], [22], [38]), the energy conditions (ECs) are main resources. The ECs are systematically structured using the Raychaudhuri equations [33], which express the intriguing nature of gravity through the requirement $\mathcal{R}_{kl}v^k v^l \geq 0$ (positivity condition), where v^k is a null vector. The geometric condition $\mathcal{T}_{kl}v^k v^l \geq 0$ is identical to the null EC (NEC) on matter in GR theory. In particular, the weak EC (WEC) requires a positive local energy density and states that $\mathcal{T}_{kl}u^k u^l \geq 0$, for any timelike vector u^k. Several modifications and in-depth research have been done on EFE in [7]. The generalization of $f(\mathcal{R}, \mathcal{T})$-gravity [23] that consists of the Ricci scalar $\mathcal{R}$ and the trace of the EMT $\mathcal{T} = g^{kl}\mathcal{T}_{kl}$, an explicit first order coupling between the Ricci tensor $\mathcal{R}_{kl}$ and the matter EMT $\mathcal{T}_{kl}$, is called $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ gravity. The creators of this modified gravity theory are Harko et al. [21]. In this paper, we investigate "$f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ gravity theory"and choose a model $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right) = \mathcal{R} + \alpha\left(\mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ (α is constant) [21], which is constructed to explain different ECs.

We set up our paper as:

Following some preliminary work in Section 3, we examine the characteristics of PFS admitting the Q-curvature tensor. In Section 4, Q-recurrent spacetimes are analysed. In conclusion, we give Q-flat PFS solutions for $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ gravity.

2. Preliminaries

The spacetime is called Q-flat if the Q-curvature tensor vanishes at every point in the spacetime. Hence equation (6) yields

$$\mathcal{R}_{lijk} = \frac{\phi}{3} \left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}, \quad R_{lijk} = g_{hl}R^h_{ijk} \tag{7}$$

which becomes a space of constant sectional curvature.

Multiplying (7) with g^{ij}, we obtain

$$\mathcal{R}_{lk} = \phi g_{lk}. \tag{8}$$

Again, multiplying (8) with g^{lk}, we have

$$\mathcal{R} = 4\phi. \tag{9}$$

Using (9) in (7) infers

$$\mathcal{R}_{lijk} = \frac{\mathcal{R}}{12}\left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}, \tag{10}$$

which implies that the spacetime is of constant curvature.

Proposition 2.1. *A Q-flat spacetime represents a spacetime of constant curvature.*

Remark 2.2. *Though this result occurs in [31], we include this proof for the sake of completeness.*

Equations (8) and (9) together imply

$$\mathcal{R}_{lk} = \frac{\mathcal{R}}{4}\, g_{lk}. \tag{11}$$

Thus we write:

Proposition 2.3. *A Q-flat spacetime represents an Einstein spacetime.*

It is well-known that

$$\nabla_k C^k_{lij} = \frac{1}{2}\left[\left\{\nabla_j \mathcal{R}_{li} - \nabla_i \mathcal{R}_{lj}\right\} - \frac{1}{2(n-1)}\left\{g_{li}\nabla_j \mathcal{R} - g_{lj}\nabla_i \mathcal{R}\right\}\right] \tag{12}$$

in which C^k_{lij} is the Weyl tensor. Considering that the Q-flat spacetime transforms into an Einstein spacetime, the Ricci scalar $\mathcal{R}$ remains constant and therefore, from equation (12),

$$\nabla_k C^k_{lij} = 0 \quad \text{and} \quad \nabla_k \mathcal{R} = 0. \tag{13}$$

If the metric of a Lorentzian manifold M^n satisfies the following Yang's equation,

$$\nabla_l \mathcal{R}_{ij} = \nabla_j \mathcal{R}_{il}, \tag{14}$$

it is called a Yang pure space [20].

Obviously, equation (14) is obeyed by the Einstein spacetime, hence we conclude:

Theorem 2.4. *A Q-flat spacetime represents a Yang pure space.*

Definition 2.5. *[19] For a scalar β, a vector field ξ is named a conformal collineation* (in short, CC)*, conformal Ricci collineation* (in short, CRC) *and Ricci inheritance vector* (in short, RIV) *if it fulfills*

$$\pounds_\xi g_{lk} = 2\beta g_{lk}, \tag{15}$$
$$\pounds_\xi \mathcal{R}_{lk} = 2\beta \mathcal{R}_{lk}, \tag{16}$$
$$\pounds_\xi \mathcal{R}_{lk} = 2\beta g_{lk}, \tag{17}$$

respectively. Particularly, for $\beta = 0$, the equation (17) *reduces to Ricci collineation and equation* (15) *turns into Killing equation.*

After taking into account the Lie derivative on each side of equation (11), we arrive at

$$£_\xi \mathcal{R}_{lk} = \frac{\mathcal{R}}{4} £_\xi g_{lk}. \tag{18}$$

If ξ is CRC, hence the equations (17) and (18) together imply

$$£_\xi g_{lk} = 2\psi_1 g_{lk}, \quad \text{where} \quad \psi_1 = \frac{4\beta}{\mathcal{R}}. \tag{19}$$

Also, taking ξ as a CC, the equations (15) and (18) give us

$$£_\xi \mathcal{R}_{lk} = 2\psi_2 g_{lk}, \quad \text{where} \quad \psi_2 = \frac{\beta\mathcal{R}}{4}. \tag{20}$$

From the above we provide:

Theorem 2.6. *For the vector field ξ, a $\mathcal{Q}$-flat spacetime admits* CC *if and only if it admits* CRC*.*

Taking ξ as RIV, from equations (11) and (16), we have

$$£_\xi \mathcal{R}_{lk} = 2\psi_3 g_{lk}, \quad \text{in which} \quad \psi_3 = \frac{\mathcal{R}\beta}{4}. \tag{21}$$

Therefore, we write:

Theorem 2.7. *For a $\mathcal{Q}$-flat spacetime,* RIV *turns into* CRC*.*

If a symmetric vector ξ of a spacetime leaves the matter tensor invariant, that is,

$$£_\xi \mathcal{T}_{lk} = 0, \tag{22}$$

then we declare that there is a symmetry in the spacetime known as matter collineation (in short, MC). Using (11) in (4), we find

$$-\frac{\mathcal{R}}{4} g_{lk} = \kappa \mathcal{T}_{lk}. \tag{23}$$

Taking Lie derivative on equation (23) and using $\mathcal{R}$ = constant, we obtain

$$£_\xi \mathcal{T}_{lk} = -\frac{\mathcal{R}}{4\kappa} £_\xi g_{lk}, \quad \text{since} \quad \mathcal{R} = \text{constant}. \tag{24}$$

If $£_\xi \mathcal{T}_{lk} = 0$, then from (24) we get

$$£_\xi g_{lk} = 0, \tag{25}$$

that is, ξ is Killing. If a M^4 admits a Killing time-like vector, it is named a stationary spacetime ([34], [37], p. 283).

Thus, we provide:

Theorem 2.8. *If a $\mathcal{Q}$-flat spacetime fulfilling* EFE *with non-zero Ricci scalar, admits* MC *with respect to a time-like vector ξ, then the spacetime becomes a stationary spacetime.*

3. $\mathcal{Q}$-flat perfect fluid spacetimes

Here, we consider a $\mathcal{Q}$-flat PFS obeying EFE.

Equations (3), (4) and (11) together provide

$$\kappa (p + \mu) u_l u_k + \left(\kappa p + \frac{\mathcal{R}}{4}\right) g_{lk} = 0. \tag{26}$$

Multiplying equation (26) with g^{lk} gives

$$\mathcal{R} + 3\kappa p - \kappa\mu = 0. \tag{27}$$

Also, multiplying the equation (26) with u^l, we acquire

$$\mathcal{R} = 4\kappa\mu. \tag{28}$$

The equations (27) and (28) are jointly reveal

$$p + \mu = 0, \tag{29}$$

which implies a DME [8] and hence we state:

Theorem 3.1. *A $\mathcal{Q}$-flat* PFS *satisfying* EFE *becomes a DME.*

From equations (3) and (4), we find

$$\mathcal{R}_{kl} = \left(\kappa p + \frac{\mathcal{R}}{2}\right) g_{kl} + \kappa (p + \mu) u_k u_l. \tag{30}$$

Multiplying (30) with $u^k u^l$, we acquire

$$\mathcal{R}_{kl} u^k u^l = \kappa\mu - \frac{\mathcal{R}}{2}. \tag{31}$$

Therefore, equations (28) and (31) together imply

$$\mathcal{R}_{kl} u^k u^l = -\kappa\mu. \tag{32}$$

The strong energy condition (SEC) of a spacetime is satisfied if $\mathcal{R}_{hj} v^h v^j \geq 0$ holds for any timelike vector v [19]. In this case, we assume that the spacetime under study satisfies SEC. Hence,

$$\kappa\mu \leq 0. \tag{33}$$

Given that $\mu \geq 0$ and $\kappa > 0$, then the equations (28) and (33) give

$$\mathcal{R} = 0. \tag{34}$$

Thus equation (10) provides $\mathcal{R}_{lijk} = 0$, that is, $R_{lk} = 0$ which implies vacuum spacetime.
We provide:

Theorem 3.2. *A $\mathcal{Q}$-flat* PFS *obeying the* SEC, *is vacuum spacetime.*

Since $\mu \geq 0$, equation (28) infers that

$$\mathcal{R} \geq 0, \tag{35}$$

which entails $\mathcal{R} > 0$ or, $\mathcal{R} = 0$.

Case 1. For $\mathcal{R} > 0$, equation (10) implies that the spacetime is of positive constant curvature. Thus, it represents a de-Sitter spacetime [19].

Case 2. For $\mathcal{R} = 0$, (10) infers $\mathcal{R}_{lijk} = 0$ and hence, the spacetime is vacuum. Therefore, we conclude:

Theorem 3.3. *A Q-flat* PFS *is either, vacuum or, a de-Sitter spacetime.*

Mantica et al. prove the subsequent in [28]:
Theorem A. A PFS with $\nabla_k C^k_{lij} = 0$ and $\nabla_k \mathcal{R} = 0$ is a GRW spacetime.

In light of the foregoing Theorem and equation (13), we can draw the conclusion as:

Corollary 3.4. *The Q-flat* PFS *represents a* GRW *spacetime.*

Theorem B. [29] For a GRW spacetime, the Weyl tensor C^h_{lij} is purely electric [24], that is, $u^k C_{lijk} = 0$.
Thus, from equation (13), Corollary 1 and Theorem B, we say:

Corollary 3.5. *For a Q-flat* PFS*, C^h_{lij} is purely electric.*

We know that in a spacetime if C^h_{lij} is purely electric admitting a unit time-like vector, then it belongs to Petrov classification I, D or O ([37], p. 73). Therefore, we get:

Corollary 3.6. *A Q-flat* PFS *belongs to Petrov classification I, D or O.*

In 4 dimensions, $u^i C_{jkli} = 0$ is similar to $u_h C_{lijk} + u_l C_{ihjk} + u_i C_{hljk} = 0$ ([26], p. 128), where $C_{lijk} = g_{hl} C^h_{ijk}$. Multiplying with u^h we have $C_{lijk} = 0$. The authors of [5] proved that a GRW spacetime is conformally flat if and only if it is a RW spacetime.
Hence, by Corollary 1, we state:

Corollary 3.7. *A Q-flat* PFS *becomes a* RW *spacetime.*

Theorem C. [36] A PFS with harmonic Weyl tensor and the EoS $p = p(\mu)$ is conformally flat, the metric is RW, the flow is geodesic, irrotational and shear-free.
In view of Theorem C and equation (13), we conclude the subsequent:

Corollary 3.8. *A Q-flat* PFS *with the* EoS *$p = p(\mu)$ is conformally flat, the metric is* RW*, the flow is geodesic, irrotational and shear-free.*

Theorem D. [35] A PFS admitting a proper conformal Killing vector with harmonic Weyl tensor is conformally flat.
In light of equation (13) and Theorem D, we can state:

Corollary 3.9. *A Q-flat* PFS *admitting a proper conformal Killing vector is conformally flat.*

4. Q-recurrent spacetimes

Definition 4.1. *[14] A generalized Ricci recurrent (inshort, $(GR)_n$) spacetime is defined by*

$$\nabla_h \mathcal{R}_{lk} = \omega_h \mathcal{R}_{lk} + \tau_h g_{lk}, \tag{36}$$

in which ω_h and τ_h are non-zero vectors.

Definition 4.2. *[39] For a non-zero vector θ_l,* M^n *is named a recurrent spacetime if $\mathcal{R}^h_{ijk}$ obeys*

$$\nabla_l \mathcal{R}^h_{ijk} = \theta_l \mathcal{R}^h_{ijk}. \tag{37}$$

Definition 4.3. *For a unit time-like vector λ_l, a* M^4 *is named a Q-recurrent spacetime if*

$$\nabla_l Q^h_{ijk} = \lambda_l Q^h_{ijk}. \tag{38}$$

By putting equation (5) to (38), we may infer

$$\nabla_l \mathcal{R}^h_{ijk} - \frac{\nabla_l \phi}{3}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\} = \lambda_l \left[\mathcal{R}^h_{ijk} - \frac{\phi}{3}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}\right]. \tag{39}$$

Contracting h and k in equation (39), we deduce

$$\nabla_l \mathcal{R}_{ij} - g_{ij}\nabla_l \phi = \lambda_l \left\{\mathcal{R}_{ij} - \phi g_{ij}\right\}. \tag{40}$$

Therefore,

$$\nabla_l \mathcal{R}_{ij} = \lambda_l \mathcal{R}_{ij} + \mu_l g_{ij}, \quad \text{where} \quad \mu_l = \nabla_l \phi - \lambda_l \phi. \tag{41}$$

In other words, the spacetime is a generalized Ricci recurrent spacetime.

Therefore, we state:

Theorem 4.4. *A $\mathcal{Q}$-recurrent spacetime represents a $(GR)_n$ spacetime.*

A generalized Ricci recurrent GRW spacetime is known to be an Einstein spacetime [18]. Thus, we can determine that $\nabla_k C^k_{lij} = 0$ using equation (12). The authors of [30] also showed that a GRW spacetime with $\nabla_k C^k_{lij} = 0$ turns into a PFS.

Hence, we arrive:

Corollary 4.5. *A $\mathcal{Q}$-recurrent* GRW *spacetime represents a* PFS.

5. $\mathcal{Q}$-flat PFS solutions obeying $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$-gravity

Here, we concentrate on a certain class of $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ modified gravity model. The associated field equations have been studied for a variety of special cases (different metric form) of $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$-gravity. Throughout this case, we choose

$$f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right) = \mathcal{R} + \alpha\left(\mathcal{R}_{lk}\mathcal{T}^{lk}\right). \tag{42}$$

The gravitational action term is

$$\mathcal{S} = \int \left\{\frac{f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)}{16\pi G} + L_m\right\} \sqrt{-g}d^4x, \tag{43}$$

in which L_m indicates the matter Lagrangian density depends on the metric g_{lk}. Equation (43), yields the commonly used gravitational field equations of $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$-gravity as

$$\begin{aligned}&\{f_{\mathcal{R}} - L_m f_{\mathcal{R}\mathcal{T}}\} G_{lk} + \left\{L_m f_{\mathcal{T}} + \Box f_{\mathcal{R}} + \frac{1}{2}\nabla_i\nabla_j\left(f_{\mathcal{R}\mathcal{T}}\mathcal{T}^{ij}\right) + \frac{\mathcal{R}}{2}f_{\mathcal{R}} - \frac{1}{2}f\right\} g_{lk} + 2f_{\mathcal{R}\mathcal{T}}\mathcal{R}_{i(l}\mathcal{T}^i_{k)} - \nabla_i\nabla_{(l}\left[\mathcal{T}^i_{k)}f_{\mathcal{R}\mathcal{T}}\right]\\ &- \nabla_l\nabla_k f_{\mathcal{R}} + \frac{1}{2}\Box\left(f_{\mathcal{R}\mathcal{T}}\mathcal{T}_{lk}\right) - \left(\frac{\mathcal{R}}{2}f_{\mathcal{R}\mathcal{T}} + f_{\mathcal{T}} + 8\pi G\right)\mathcal{T}_{lk} - 2\left(f_{\mathcal{R}\mathcal{T}}\mathcal{R}^{ij} + f_{\mathcal{T}}g^{ij}\right)\frac{\partial^2 L_m}{\partial g^{lk}\partial g^{ij}} = 0\end{aligned} \tag{44}$$

in which $G_{lk} = \mathcal{R}_{lk} - \frac{\mathcal{R}}{2}g_{lk}$ and $\Box$ represents the d'Alembert operator.

In the set up of $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ modified gravity, the ECs are derived using the modified gravitational field equations and these are the outcomes

$$\text{NEC} \Leftrightarrow p + \mu \geq 0, \tag{45}$$

$$\text{WEC} \Leftrightarrow p + \mu \geq 0 \quad \text{and} \quad \mu \geq 0, \tag{46}$$

$$\text{DEC} \Leftrightarrow \mu \pm p \geq 0 \quad \text{and} \quad \mu \geq 0, \tag{47}$$

$$\text{SEC} \Leftrightarrow p + \mu \geq 0 \quad \text{and} \quad 3p + \mu \geq 0, \tag{48}$$

in which dominant energy condition is indicated by DEC.

Here, we investigate PFS solutions to the $f\left(\mathcal{R},\mathcal{T},\mathcal{R}_{lk}\mathcal{T}^{lk}\right)$-gravity in which the EMT has the shape (3).

Using (42) in (44), we acquire

$$\mathcal{R}_{lk}-\frac{f}{2}g_{lk}-8\pi G\mathcal{T}_{lk}=0. \tag{49}$$

Equations (3), (11) and (49) yield that

$$\left\{\frac{\mathcal{R}}{4}-\frac{f}{2}-8\pi Gp\right\}g_{lk}-8\pi G\left(\mu+p\right)u_l u_k=0. \tag{50}$$

Multiplying (50) with u^l, we have

$$8\pi G\mu=\frac{f}{2}-\frac{\mathcal{R}}{4}. \tag{51}$$

Multiplying (50) with g^{lk} and using (51), we arrive

$$8\pi Gp=\frac{\mathcal{R}}{4}-\frac{f}{2}. \tag{52}$$

From (3), it follows that

$$\mathcal{T}^{lk}=pg^{lk}+\left(p+\mu\right)u^l u^k \tag{53}$$

and

$$\mathcal{T}=3p-\mu. \tag{54}$$

Equations (11), (53) and (54) together imply

$$\mathcal{R}_{lk}\mathcal{T}^{lk}=\frac{\mathcal{R}\mathcal{T}}{4}. \tag{55}$$

Adopting (42) and (55), we find

$$f\left(\mathcal{R},\mathcal{T},\mathcal{R}_{lk}\mathcal{T}^{lk}\right)=\left(1+\frac{\alpha\mathcal{T}}{4}\right)\mathcal{R}. \tag{56}$$

Equations (51), (52), (54) and (56) give us

$$\mu=\frac{\mathcal{R}}{32\pi G+2\alpha\mathcal{R}} \tag{57}$$

and

$$p=-\frac{\mathcal{R}}{32\pi G+2\alpha\mathcal{R}}. \tag{58}$$

Equations (57) and (58) together produce $\mu+p=0$.

Hence we provide the result:

Theorem 5.1. *A Q-flat* PFS *solutions obeying $f\left(\mathcal{R},\mathcal{T},\mathcal{R}_{lk}\mathcal{T}^{lk}\right)=\mathcal{R}+\alpha\left(\mathcal{R}_{lk}\mathcal{T}^{lk}\right)$ represents a DME.*

We now look at the ECs for the model (42). It is now possible to discuss the ECs for this configuration using equations (57) and (58).

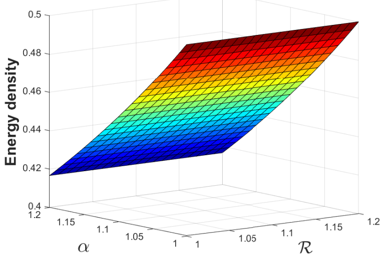

Fig. 1: Evolution of μ

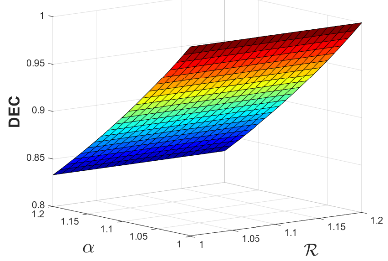

Fig. 2: Development of DEC

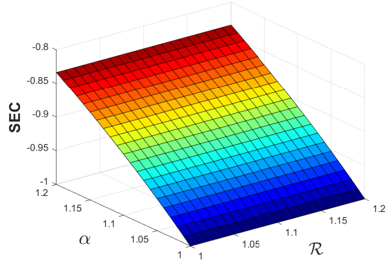

Fig. 3: Evolution of SEC

μ, DEC, and SEC's profiles are displayed in Figs. 1, 2, and 3, respectively. Under these circumstances, $\mu + p$ equals zero. When $\mathcal{R} > 0$ and $\alpha > 0$ are used as parameter ranges, the energy density always positive. Fig. 1 illustrates that the energy density is greater for larger values of α and $\mathcal{R}$. NEC and WEC are satisfied because NEC is a part of WEC. The DEC profile with a positive range for its value, is depicted in Fig. 2. SEC violates and this result demonstrates the late-time acceleration of Universe [25]. Moreover, the EoS for this formulation is $\frac{p}{\mu} = -1$. Moreover, each result is consistent with the ΛCDM model [1].

Remark 5.2. *The* EoS *is* $p = -\mu$, *that is,* $|\mu| + |-p| = 0$, *which entails that* $\mu = |p|$, *since* $\mu \geq 0$. *Therefore, in a* $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$*-gravity obeying the model* (42)*, a* $\mathcal{Q}$*-flat* PFS *fulfills the* DEC*. Hence in a* $\mathcal{Q}$*-flat* PFS *obeying* $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right) = \mathcal{R} + \alpha\left(\mathcal{R}_{lk}\mathcal{T}^{lk}\right)$, *the speed of light is faster than the speed of matter [19].*

6. Discussion

The investigation of $\mathcal{Q}$-flat PFS solutions in connection with $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right)$-gravity has been the primary emphasis of this paper. Our results have been evaluated using both analytical and graphical approaches in this instance. To build our formulation and evaluate the stability of the cosmological model, we employed the analytical technique, as in the following formula: $f\left(\mathcal{R}, \mathcal{T}, \mathcal{R}_{lk}\mathcal{T}^{lk}\right) = \mathcal{R} + \alpha\left(\mathcal{R}_{lk}\mathcal{T}^{lk}\right)$. Regarding our model, Figs. 1, 2, and 3 display the ECs' profiles. For the parameters $\mathcal{R} > 0$ and $\alpha > 0$, it has been found that the development of μ is positive. NEC, WEC, and DEC were satisfied, but SEC disobeyed the agreement. The DME is indicated by the EoS, which is $\frac{p}{\mu} = -1$. These results, however, agree with the ΛCDM model.

Declarations

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