Filomat 39:6 (2025), 1989–2002
https://doi.org/10.2298/FIL2506989B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Mild solutions for conformable fractional order functional evolution equations via Meir-Keeler type fixed point theorem

Fatma Berrighi[a], **Imene Medjadj**[a,b], **Erdal Karapınar**[c,d,*]

[a]*Department of Mathematics, University of Science and Technology-Mohamed Boudiaf (USTO MB) El Mnaouar, BP 1505, Bir El Djir, 31000, Oran, Algeria*
[b]*Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Ben Bella, B.P. 1524, El Mnaouar, 31000, Oran, Algeria*
[c]*Department of Mathematics, Atılım University, 06830, Incek, Ankara, Turkey*
[d]*Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan*

Abstract. In this study, we delve into the realm of mild solutions for conformable fractional order functional evolution equations, focusing on cases where the fractional order is strictly greater than 1 and less than 2 within a separable Banach space. We demonstrate the existence, uniqueness, attractivity, and controllability of these solutions under local conditions. Our approach involves leveraging a contribution of Meir-Keeler's fixed point theorem alongside the principle of measures of noncompactness. To demonstrate the practical ramifications of our theoretical finds, we provide a specific example that underscores the relevance and applications of the established results.

1. Introduction

Fractional calculus extends classical differentiation and integration to non-integer orders, thereby unifying discrete and continuous domains. It commonly employs definitions like the Riemann-Liouville and Caputo formulations (see eg. [40, 44]). Khalil *et al.* [39] proposed the conformable derivative, which shares important properties with the integer-order derivative. The conformable derivative has been applied to many sciences such as physics (see eg. [16, 41, 46]), biology (see eg. [15, 49]), chemistry (see eg. [45]), and medicine (see eg. [11, 13]).

Jaiswal *et al.* [34] studied a conformable fractional abstract initial value problem in Banach spaces, finding moderate solutions using the contraction principle. Kataria *et al.* [37] embarked on an exploration of mild solutions for impulsive integro-differential equations, harnessing the power of conformable differential operators and fixed-point theorems. Among all we may count some of the contributions to conformable fractional evolution equations have also been made by Bouaouid *et al.* [26, 28] and Boukenkoul *et al.* [29].

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. Functional differential equation, mild solution, finite delay, fixed point, condensing operator, measure of noncompactness, conformable fractional.
Received: 12 October 2024; Revised: 28 November 2024; Accepted: 02 December 2024
Communicated by Calogero Vetro
* Corresponding author: Erdal Karapınar
Email addresses: fatmaberrighi@gmail.com (Fatma Berrighi), imene.medjadj@gmail.com (Imene Medjadj), erdalkarapinar@yahoo.com (Erdal Karapınar)
ORCID iDs: https://orcid.org/0009-0002-8714-7787 (Fatma Berrighi), https://orcid.org/0009-0008-8812-0010 (Imene Medjadj), https://orcid.org/0000-0002-6798-3254 (Erdal Karapınar)

This work leverages measures of noncompactness (see eg. [12, 21, 35]) to address solution existence issues and illuminate nonlinear differential equations (see eg. [1, 9, 10, 22, 23, 24, 25, 47]).

Additionally, Baghli *et al.* [20] utilized the nonlinear Leray-Schauder alternative [32] for controllability within bounded intervals, while Agarwal *et al.* [6] addressed controllability over semi-infinite intervals using the nonlinear alternative of Avramescu [19] for equations with infinite delay.

In their study [50], Zou *et al.* established four uniqueness criteria for the initial value problem of differential equations incorporating the conformable fractional derivative. Their work extends Nagumo-type uniqueness theory and Lipschitz conditional theory. To bolster these theoretical advancements, they provide four practical examples, showcasing the efficacy of their approach in proving fractional differential equations.

Bouaouid *et al.* [27] applied Krasnoselskii's fixed point theorem to obtain mild solutions for an abstract class of conformable fractional differential equations. Building on this foundation, Atraoui *et al.* [17] ventured further by utilizing the Darbo-Sadovskii fixed point theorem.

Moreover, Atraoui *et al.* [18] ventured into the intricate realm of controllability, exploring mild solutions for a nonlocal fractional conformable Cauchy problem with keen insight by using second-order differential equations. They used the Banach contraction principle and the cosine linear operator family to arrive at their conclusions.

This work examines conformable fractional order evolutionary equations with local conditions:

$$D^c[D^c\psi(s)] = \mathfrak{P}\psi(s) + \Psi(s,\psi_s), \quad \text{a.e. } s \in I := [0,+\infty); \tag{1}$$

$$\psi(s) = \eta(s), \quad s \in \mathcal{H} := [-a,0], \text{ where } 0 < a < +\infty, \quad D^c\psi(0) = \vartheta \in \mathcal{F}; \tag{2}$$

such that, $\Psi : I \times C([-a,0],\mathcal{F}) \to \mathcal{F}$ denotes a specified function, $\eta : [-a,0] \to \mathcal{F}$ is continuous, and $\mathfrak{P} : D(\mathfrak{P}) \subset \mathcal{F} \to \mathcal{F}$ serves as the infinitesimal generator of a trongly continuous cosine function composed of bounded linear operators $\{\mathfrak{C}(s)\}_{s\in\mathbb{R}}$ such that $\mathfrak{S}(s) = \int_0^s \mathfrak{C}(x)dx$, and $\mathcal{F}$ denotes real separable Banach space generates by the norm $|\cdot|$. Furthermore, D^c is a fractional conformable derivative, where $0 < c \le 1$.

We define ψ_s for $s \ge 0$ as a continuous function from $\mathcal{H}$ to $\mathcal{F}$ given by: $\psi_s(x) = \psi(s+x)$, where $\psi_s(\cdot)$ represents the state's history from $s-a$ to the present moment s.

Subsection 3.3 investigates the attractivity of mild solutions under these settings. Subsection 3.4 delves into how mild solutions can be controlled across the interval $I = [0,+\infty)$, providing a thorough exploration of their behavior, so we consider the following problem:

$$D^c[D^c\psi(s)] = \mathfrak{P}\psi(s) + \Psi(s,\psi_s) + \mathcal{B}\mathcal{U}(s), \quad \text{a.e. } s \in I := [0,+\infty); \tag{3}$$

$$\psi(s) = \eta(s), \quad s \in \mathcal{H} := [-a,0], \text{ where } 0 < a < +\infty, \quad D^c\psi(0) = \vartheta \in \mathcal{F}; \tag{4}$$

where $\mathfrak{P}$, Ψ, η and ϑ are as in problem $(1)-(2)$, the control function $\mathcal{U}(\cdot)$ finds its place within $L^2(I,\mathcal{F})$, the Banach space housing admissible control functions, while $\mathcal{B}$ represents a bounded linear operator mapping from $\mathcal{F}$ to $\mathcal{F}$.

Ultimately, we provide an illustrative example demonstrating the abstract theory expounded in the preceding sections.

2. Preliminary Concepts

In this section, we present symbols, explanations, and fundamental principles drawn from multivalued analysis. These elements will be incorporated consistently in the subsequent sections of this paper.

Let $C(\mathcal{H},\mathcal{F})$ be the Banach space of continuous functions with the norm

$$\|x\| = \sup\{\ |x(s)|\ :\ s \in \mathcal{H}\}.$$

$BC(I,\mathcal{F})$ denotes the Banach space comprising all functions from I to $\mathcal{F}$ that are both bounded and continuous, endowed with the norm

$$\|x\|_{BC} = \sup\{ \, |x(s)| \; : \; s \in I\}.$$

Consider the space BC_∞ defined as $\{x : [-a, +\infty) \to \mathcal{F}, \; x|_{[0,b]}$ is bounded and continuous for $b > 0\}$, with the norm

$$\|x\|_{BC_\infty} = \sup\{ \, |x(s)| \; : \; s \in [0, T]\},$$

where $T = \sup\{b > 0 : \; x|_{[0,b]}$ is bounded and continuous$\}$.

Definition 2.1. *(Khalil et al. [39])*
The conformable fractional derivative of order $0 < c \leq 1$ for a function $x(\cdot)$ is expressed as

$$D^c x(s) = \lim_{\tau \to 0} \frac{x(s + \tau s^{1-c}) - x(s)}{\tau}, \;\; s > 0;$$

$$D^c x(0) = \lim_{\tau \to 0} D^c x(\tau).$$

Additionally, the conformable fractional integral of order c of a function x is given by

$$I^c x(s) = \int_0^s t^{c-1} x(t) dt$$

provided that the limits and integrals are well defined.

Definition 2.2. *A map $f : J \times E \longrightarrow E$ is said to be Carathéodory if it satisfies :*

(i) $y \mapsto f(t, y)$ is continuous for almost all $t \in J$;

(ii) $t \mapsto f(t, y)$ is measurable for each $y \in E$.

Let us now review some key aspects of the Kuratowski noncompactness measure.

Definition 2.3. *(see eg. [12, 21, 35]) Let $\mathcal{D}_\mathcal{F}$ be the bounded subsets of $\mathcal{F}$. The map $\mathfrak{H} : \mathcal{D}_\mathcal{F} \to [0, +\infty)$ is called Kuratowski's non-compactness measure and is de ned as follows:*

$$\mathfrak{H}(\mathcal{E}) = \inf\{\alpha > 0 : \mathcal{E} \subseteq \bigcup_{j+1}^{k} \mathcal{E}_j \text{ and } diam(\mathcal{E}_j) \leq \alpha\}, \text{ here } \mathcal{E} \in \mathcal{D}_\mathcal{F}.$$

For properties and more details about the Kuratowski measure of noncompactness (see eg. [12, 21, 35]).

In this paper, we shall solve our problem by using the method of fixed point theory (see e.g. [7, 36, 8]) It belongs to us, inspired by [8], we define the following improved form of the Meir-Keeler condensing operator.

Definition 2.4. *Let $\mathcal{E}$ be a nonempty subset of the Banach space $\mathcal{F}$. Define $\mathfrak{H}$ as an arbitrary measure of noncompactness on $\mathcal{F}$. We introduce $\mathfrak{N} : \mathcal{E} \to \mathcal{F}$ as Meir-Keeler condensing operator if it meets the following criteria: $\mathfrak{N}$ is both continuous and bounded, and for every $\beta > 0$, there exists $\mu > 0$ such that whenever $\beta < \mathfrak{H}(\mathcal{R}) < \beta + \mu$, it follows that $\mathfrak{H}(\mathfrak{N}(\mathcal{R})) \leq \beta$ is true for any subset $\mathcal{R}$ which is bounded of $\mathcal{E}$.*

Lemma 2.5. *(see [33]) Consider $\mathcal{F}$ as a Banach space, and let $\mathcal{E} \subset C(I, \mathcal{F})$ that is both bounded and equicontinuous. Consequently, the map $\mathfrak{H}(\mathcal{E}(s))$ remains continuous over the interval I, and $\mathfrak{H}_I(\mathcal{E})$ equals the supremum value of $\mathfrak{H}(\mathcal{E}(s))$ for s in I.*

Theorem 2.6. *(Meir-Keeler's theorem see [8]) Let $\mathcal{E}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{F}$. If $\mathfrak{N} : \mathcal{E} \to \mathcal{E}$ is a continuous Meir-Keeler condensing operator, then $\mathfrak{N}$ guarantees at least one fixed point, and the collection of all such fixed points within $\mathcal{E}$ forms a compact set.*

Definition 2.7. *(see [43]) A function $T : \mathbb{R}_+ \to \mathbb{R}_+$ is called an L-function if $T(0) = 0$, $T(s) > 0$ for $s \in \mathbb{R}_+^*$ and for every $s \in \mathbb{R}_+^*$ there exists $\delta > 0$ such that $T(t) \leq s$ for all $t \in [s, s+\delta]$.*

Corollary 2.8. *Let $\mathcal{E}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{F}$. If $\mathfrak{N} : \mathcal{E} \to \mathcal{E}$ is a continuous operator such that*

$$\mathfrak{H}(\mathfrak{N}(\mathcal{R})) \leq T(\mathfrak{H}(\mathcal{R})).$$

For every subset $\mathcal{R}$ of $\mathcal{E}$, where $\mathfrak{H}$ is an arbitrary measure of noncompactness and T is an $L-$function, it follows that $\mathfrak{N}$ has at least one fixed point. Furthermore, the collection of all fixed points of $\mathfrak{N}$ in $\mathcal{E}$ is compact.

Proof. To prove Corollary 2.7, it is sufficient to demonstrate that the operator $\mathfrak{N}$ is a Meir-Keeler condensing operator, following steps similar to those used in Theorem 2.10 [8]. We will then conclude the proof by applying Theorem 2.5.

Definition 2.9. *(see [30]) We characterize solutions of equations $(1)-(2)$ as locally attractive if there exists a closed ball $\overline{B}(\psi^*, \sigma)$ in the space BC, centered at ψ^* and with radius σ, such that for any solutions ψ and $\widetilde{\psi}$ of Equations $(1)-(2)$ within this ball, the following convergence condition holds:*

$$\lim_{s \to +\infty} (\psi(s) - \widetilde{\psi}(s)) = 0.$$

3. Main Results

3.1. Existence results

In this section, we outline our main result concerning the existence of solutions for problem $(1)-(2)$. Before delving into the details and proof of this result, we first introduce the concept of a mild solution.

Definition 3.1. *We define the mild solution $\psi \in C([-a, +\infty), \mathcal{F})$ of the problem $(1)-(2)$ as follows*

$$\psi(s) = \begin{cases} \eta(s), & \text{if } s \in \mathcal{H}; \\ \mathfrak{C}\left(\frac{s^c}{c}\right) \eta(0) + \mathfrak{S}\left(\frac{s^c}{c}\right) \vartheta + \int_0^s x^{c-1} \mathfrak{S}\left(\frac{s^c - x^c}{c}\right) \Psi(x, \psi_x)\, dx, & \text{if } s \in I; \end{cases}$$

We must introduce the following hypotheses, which will be utilized later:

(i) The function $\Psi : I \times C(\mathcal{H}, \mathcal{F}) \to \mathcal{F}$ is carathéodory and there exists a continuous function $\mathcal{O} : I \to I$ that satisfies:

$$|\Psi(s, u)| \leq \mathcal{O}(s)\|u\|,$$
$$\mathfrak{H}(\Psi(s, \mathcal{D})) \leq \mathcal{O}(s)\mathfrak{H}(\mathcal{D}),$$

and $\mathcal{O}^* := \sup_{s\in I} \int_0^s x^{c-1}\mathcal{O}(x)dx < \infty$, for all $s \in I$, $u \in C(\mathcal{H}, \mathcal{F})$, *bounded set* $\mathcal{D} \subset C(\mathcal{H}, \mathcal{F})$ *and* $0 < c \leq 1$;

(ii) The cosine operator $\mathfrak{C}(s)_{s\in\mathbb{R}}$ is uniformly continuous and there exist constants $\mathcal{M}_c^{\mathfrak{C}}$, $\mathcal{M}_c^{\mathfrak{S}}$ both greater than zero, such that

$$\sup_{s\in I} \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\| \leq \mathcal{M}_c^{\mathfrak{C}} \text{ and } \sup_{s\in I} \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\| \leq \mathcal{M}_c^{\mathfrak{S}}.$$

Theorem 3.2. *Under the assumptions $(i)-(ii)$ and if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* < 1$, then the system $(1)-(2)$ is guaranteed to have a mild solution within the space $BC([-a, +\infty), \mathcal{F})$.*

Proof. We reformulate the problem (1) − (2) as fixed-point issue. Let us define the operator $\mathfrak{N} : BC([-a,+\infty),\mathcal{F}) \to BC([-a,+\infty),\mathcal{F})$ as follows:

$$\mathfrak{N}(\psi)(s) = \begin{cases} \eta(s), & \text{if } s \in \mathcal{H}; \\ \mathfrak{C}\left(\frac{s^c}{c}\right)\eta(0) + \mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta + \int_0^s x^{c-1}\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\Psi(x,\psi_x)\,dx, & \text{if } s \in [0,+\infty). \end{cases}$$

The operator $\mathfrak{N}$ maps $BC([-a,+\infty),\mathcal{F})$ into $BC([-a,+\infty),\mathcal{F})$. Specifically, for $\psi \in BC([-a,+\infty),\mathcal{F})$ and for any $s \in I$ we have:

$$\begin{aligned} |\mathfrak{N}(\psi)(s)| &\le \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\||\eta(0)| + \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\||\vartheta| + \int_0^s x^{c-1}\|\mathfrak{S}\Big(\frac{s^c-x^c}{c}\Big)\|\,|\Psi(x,\psi_x)|dx \\ &\le \mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)\|\psi_x\|dx \\ &\le \mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\|\psi\|_{BC}. \end{aligned}$$

So, $\mathfrak{N} \in BC([-a,+\infty),\mathcal{F})$. Furthermore, suppose $l \ge \frac{\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|}{1-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}$, and let B_l denote the closed ball in $BC([-a,+\infty),\mathcal{F})$ centered at the origin with radius l. consider $\psi \in B_l$ and $s \in I$, we get

$$|\mathfrak{N}(\psi)(s)| \le \mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* l.$$

Thus, we get that

$$\|\mathfrak{N}(\psi)\|_{BC} \le l.$$

We embark on the validation process to ensure that $\mathfrak{N} : B_l \to B_l$ satisfies the criteria outlined in Meir Keeler's fixed-point Theorem 2.6.

Firstly, we ascertain the continuity of $\mathfrak{N}$ within B_l. Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$ in B_l. We observe that:

$$|\mathfrak{N}(\psi_n)(s) - \mathfrak{N}(\psi)(s)| \le \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}|\Psi(x,(\psi_x)_n) - \Psi(x,\psi_x)|dx.$$

According to (i), we have $\Psi(x,(\psi_x)_n) \to \Psi(x,\psi_x)$ as $n \to +\infty$ for almost every $s \in I$. Application of the Theorem of convergence dominated by Lebesgue yields:

$$\|\mathfrak{N}(\psi_n) - \mathfrak{N}(\psi)\|_{BC} \to 0, \; as \; n \to \infty.$$

Consequently, $\mathfrak{N}$ exhibits continuity.

So, by the previous steps we can thus infer that $\mathfrak{N}(B_l) \subset B_l$.

Next, we confirm the equicontinuity of $\mathfrak{N}(B_l)$ over every compact interval J of $[0,+\infty)$. Let $x_1,\ x_2 \in J$ with $x_2 > x_1$, we have

$$\begin{aligned} |\mathfrak{N}(\psi)(x_1) - \mathfrak{N}(\psi)(x_2)| &\le \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})}\,\|\eta\| \\ &+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})}\,\|\vartheta\| \\ &+ \int_0^{x_1} x^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c-x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c-x^c}{c}\Big)\|_{B(\mathcal{F})}\,|\Psi(x,\psi_x)|dx \\ &+ \int_{x_1}^{x_2} x^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c-x^c}{c}\Big)\|_{B(\mathcal{F})}\,|\Psi(x,\psi_x)|dx \\ &\le \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})}\,\|\eta\| \end{aligned}$$

$$+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})} \|\vartheta\|$$
$$+ \int_0^{x_1} x^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - x^c}{c}\Big)\|_{B(\mathcal{F})} |\Psi(x,\psi_x)|dx$$
$$+ \mathcal{M}_c^{\mathfrak{S}} \int_{x_1}^{x_2} x^{c-1}|\Psi(x,\psi_x)|dx.$$

As $x_1 \to x_2$, the uniformly continuity property of the operators $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous enequality converges to zero. This confirms the equicontinuity of $\mathfrak{N}$.

Additionally, we establish the equiconvergence of $\mathfrak{N}(B_l)$. For $s \in I$ and $\psi \in B_l$, we find

$$|\mathfrak{N}(\psi)(s)| \le \mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + \mathcal{M}_c^{\mathfrak{S}} l \int_0^s x^{c-1}\mathcal{O}(x)dx.$$

Consequently,

$$|\mathfrak{N}(\psi)(s)| \to l', \; as\; s \to +\infty,$$

where $l' \le \mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + \mathcal{M}_c^{\mathfrak{S}} l \; \mathcal{O}^*$. Here $\mathcal{O}^* := \sup_{s\in I} \int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{N}(\psi)(s) - \mathfrak{N}(\psi)(+\infty)| \to 0, \; s \to +\infty.$$

Lastly, we prove that the condition of Meir-Keeler's theorem is satisfied. For any given $\beta > 0$, we demonstrate the existence of a $\mu > 0$ such that:

$$\beta < \mathfrak{H}_I(\Delta) < \beta + \mu \Rightarrow \mathfrak{H}_I(\mathfrak{N}(\Delta)) \le \beta, \text{ for any } \Delta \subset B_l,$$

where $\mathfrak{H}_I(\Delta) = \sup_{s\in I} \mathfrak{H}(\Delta(s))$.
By using the characteristics of the measure of noncompactness (see eg. [12, 21, 35]) and the associated Lemma 2.5 [33], we get

$$\begin{aligned}
\mathfrak{H}(\mathfrak{N}(\Delta)(s)) &\le \mathcal{M}_c^{\mathfrak{S}} \int_0^s x^{c-1}\mathcal{O}(x)\mathfrak{H}(\Delta(x))dx \\
&\le \mathcal{M}_c^{\mathfrak{S}} \int_0^s x^{c-1}\mathcal{O}(x)dx\mathfrak{H}_I(\Delta) \\
&\le \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\mathfrak{H}_I(\Delta).
\end{aligned}$$

Since $\mathfrak{N}(\Delta)$ is bounded and equicontinuous for all $\Delta \subset B_l$ then:

$$\mathfrak{H}_I(\mathfrak{N}(\Delta)) = \sup_{s\in I} \mathfrak{H}(\mathfrak{N}(\Delta)(s)).$$

Therefore, $\mathfrak{H}_I(\mathfrak{N}(\Delta)) \le \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\mathfrak{H}_I(\Delta) \le \beta \Rightarrow \mathfrak{H}_I(\Delta) \le \frac{\beta}{\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}$. Then, for given $\beta > 0$ and taking $\mu = \Big(\frac{1-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}{\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}\Big)\beta - \epsilon$ such that $\epsilon > 0$, we obtain:

$$\beta < \mathfrak{H}_I(\Delta) < \beta + \mu \Rightarrow \mathfrak{H}_I(\mathfrak{N}(\Delta)) \le \beta, \text{ for any } \Delta \subset B_l.$$

We conclude that $\mathfrak{N}$ is a Meir-Keeler operator with condensing properties.
Through these steps, we ensure that the conditions required for Meir-Keeler's fixed-point Theorem 2.6 are satisfied by $\mathfrak{N} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{N}$ has a fixed point ψ that provides a mild solution to the problem $(1) - (2)$.

3.2. Uniqueness results

We now introduce our primary result on the existence and uniqueness of solutions for problem $(1)-(2)$. Before demonstrating this result, we first present the following condition.

(iii) The given function $\Psi : I \times C(\mathcal{H},\mathcal{F}) \to \mathcal{F}$ is carathéodory function and there exist a continuous function $\mathcal{O} : I \to I$ that satisfies

$$|\Psi(s,u)-\Psi(s,v)| \leq \mathcal{O}(s)\|u-v\|,$$

$$\Psi^* = \sup_{s\in I}\int_0^s x^{c-1}\Psi(x,0)dx < \infty,$$

$$\mathfrak{H}(\Psi(s,\mathcal{D})) \leq \mathcal{O}(s)\mathfrak{H}(\mathcal{D}),$$

and $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx < \infty$, for all $s \in I,\ u,v \in C(\mathcal{H},\mathcal{F})$, *bounded set* $\mathcal{D} \subset C([-a,+\infty),\mathcal{F})$ *and* $0 < c \leq 1$.

Theorem 3.3. *Under the conditions (ii) and (iii) being satisfied, if the product $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* < 1$, then the system defined by equations $(1)-(2)$ possesses a unique mild solution over the space $BC([-a,+\infty),\mathcal{F})$.*

Proof: We observe that following analogous steps to those in the proof of Theorem 3.2, we establish the existence of a mild solution with the modified radius estimate $l \geq \frac{\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*}{1-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}$.

Next, we establish uniqueness: Assume ψ_1 and ψ_2 are two mild solutions to problem $(1)-(2)$. Then,

$$\begin{aligned}
|\psi_1(s)-\psi_2(s)| &= |\mathfrak{N}\psi_1(s)-\mathfrak{N}\psi_2(s)| \\
&\leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}|\Psi(x,\psi_{1_x})-\Psi(x,\psi_{2_x})|dx \\
&\leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)|\psi_{1_x}-\psi_{2_x}|dx \\
&\leq \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\|\psi_1-\psi_2\|_{BC}.
\end{aligned}$$

This implies $(1-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\|\psi_1-\psi_2\|_{BC} \leq 0$, and consequently $\psi_1 = \psi_2$. Thus, we have established the uniqueness of the mild solution.

3.3. Attractiveness of Mild Solutions

In this section, we explore the local attractiveness of solutions to problem (1)-(2).

Theorem 3.4. *Assuming conditions (ii) and (iii) are satisfied, and if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* < 1$, let ψ^* be a solution of $(1)-(2)$, and $\overline{B}(\psi^*,\tau)$ be the closed ball in BC such that*

$$\tau \geq \frac{\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*}{1-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}.$$

Then, the problem (1)-(2) *exhibits attractiveness.*

Proof. Considering $\psi \in \overline{B}(\psi^*,\tau)$, utilizing (ii) and (iii), we obtain:

$$\begin{aligned}
|\mathfrak{N}(\psi)(s)-\psi^*(s)| &= |\mathfrak{N}(\psi)(s)-\mathfrak{N}(\psi^*)(s)| \\
&\leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}|\Psi(x,\psi_x^*)-\Psi(x,\psi_x)|dx \\
&\leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)\|\psi_x^*-\psi_x\|dx
\end{aligned}$$

$$\leq \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\|\psi^* - \psi\|_{BC}$$
$$\leq \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\tau$$
$$\leq \tau.$$

Consequently, $\mathfrak{N}(\overline{B}(\psi^*,\tau)) \subset \overline{B}(\psi^*,\tau)$ Thus, for any solutions $\psi,\ \widetilde{\psi} \in \overline{B}(\psi^*,\tau)$ of $(1)-(2)$ and $s \in I$, we have

$$|\psi(s) - \widetilde{\psi}(s)| \leq \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\ \|\widetilde{\psi} - \psi\|_{BC},$$

then

$$(1 - \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\|\widetilde{\psi} - \psi\|_{BC} \leq 0.$$

Hence

$$\|\widetilde{\psi} - \psi\|_{BC} = 0.$$

As a result, the problem solutions $(1)-(2)$ are locally attractive.

3.4. Controllability results

In this section, we present a controllability result for the system (3)-(4). Prior to that, we introduce a specific type of solutions for problem (3)-(4).

Definition 3.5. *We define the mild solution $\psi \in C([-a,+\infty),\mathcal{F})$ of the problem $(3)-(4)$ as follows*

$$\psi(s) = \begin{cases} \eta(s), & \text{if } s \in \mathcal{H}; \\ \mathfrak{C}\left(\frac{s^c}{c}\right)\eta(0) + \mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta + \int_0^s x^{c-1}\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\mathcal{B}\mathcal{U}(x)dx & \\ +\int_0^s x^{c-1}\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\Psi(x,\psi_x)\,dx, & \text{if } s \in I. \end{cases}$$

Definition 3.6. *The system $(3)-(4)$ is considered controllable if, for every initial function $\eta \in C(\mathcal{H},\mathcal{F})$ and $\widehat{\psi} \in \mathcal{F}$, there exists some $n \in \mathbb{N}$ and some control $\mathcal{U} \in L^2([0,n],\mathcal{F})$ such that the mild solution $\psi(\cdot)$ of this problem satisfies the terminal condition $\psi(n) = \widehat{\psi}$.*

We will consider the assumptions $(i)-(ii)$ from Section 3, and we introduce the following additional assumptions:

- (iv) For all n integer, the linear operator $\mathfrak{V} : L^2([0,n],\mathcal{F}) \to \mathcal{F}$ defined by

$$\mathfrak{V}\mathcal{U} = \int_0^n x^{c-1}\mathfrak{S}\Big(\frac{n^c - x^c}{c}\Big)\,\mathcal{B}\mathcal{U}(x)dx,$$

possesses a pseudo-invertible operator $\widetilde{\mathfrak{V}}^{-1}$, which maps functions from $L^2([0,$ $n],\mathcal{F})$ to the space $L^2([0,n],\mathcal{F})$ excluding the kernel of $\mathfrak{V}$, and is bounded. Additionally, $\mathcal{B}$ is bounded, satisfying:

$$\|\mathcal{B}\| \leq \widetilde{\mathcal{N}} \text{ and } \|\widetilde{\mathfrak{V}}^{-1}\| \leq \widetilde{\mathcal{N}_1}.$$

- (v) There exists a continuous function function $\mathcal{K}_{\mathfrak{V}} : [0,n] \to \mathbb{R}_+$ such that: for any bounded subset $\mathcal{D} \subset \mathcal{F}$, we have : $\mathfrak{H}(\widetilde{\mathfrak{V}}^{-1}(\mathcal{D})(s)) \leq \mathcal{K}_{\mathfrak{V}}(s)\mathfrak{H}(\mathcal{D})$, $s \in I$ and $\mathcal{K}' := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{K}_{\mathfrak{V}}(x)dx < \infty$ for all $0 < c \leq 1$.

Theorem 3.7. *Assuming that $(i)-(ii)$ and $(iv)-(v)$ hold. If*

$$\max\{\mathcal{M}_c^{\mathfrak{S}}[\mathcal{O}^* + \widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c}(1+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)],\ \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')\} < 1,$$

then the problem $(3)-(4)$ is controllable on $[-a,+\infty)$.

Proof. We reformulate the problem $(3)-(4)$ into a fixed-point issue. Define the operator $\mathfrak{N}: BC_\infty \rightarrow BC_\infty$ as follows:

$$\mathfrak{N}(\psi)(s)=\begin{cases}\eta(s), & \text{if } s\in\mathcal{H};\\ \mathfrak{C}\left(\frac{s^c}{c}\right)\eta(0)+\mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta+\int_0^s x^{c-1}\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\mathcal{B}\mathcal{U}(x)dx & \\ +\int_0^s x^{c-1}\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\Psi(x,\psi_x)\,dx, & \text{if } s\in[0,+\infty).\end{cases}$$

Utilizing assumption (iv), we can define the control for any arbitrary function $\psi(\cdot)$:

$$\mathcal{U}_\psi(s)=\widetilde{\mathfrak{B}}^{-1}\left[\widehat{\psi}-\mathfrak{C}\left(\frac{n^c}{c}\right)\eta(0)-\mathfrak{S}\left(\frac{n^c}{c}\right)\vartheta-\int_0^n x^{c-1}\mathfrak{S}\left(\frac{n^c-x^c}{c}\right)\Psi(x,\psi_x)\,dx\right](s).$$

Such that, we have

$$\begin{aligned}|\mathcal{U}_\psi(s)|&\leq\|\widetilde{\mathfrak{B}}^{-1}\|\left[|\widehat{\psi}|+\mathcal{M}_c^{\mathfrak{C}}|\eta(0)|+\mathcal{M}_c^{\mathfrak{S}}|\vartheta|+\mathcal{M}_c^{\mathfrak{S}}\int_0^n x^{c-1}\mathcal{O}(x)\,|\psi_x|dx\right]\\&\leq\widetilde{\mathcal{N}_1}\left[|\widehat{\psi}|+\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\,\|\psi\|_{BC_\infty}\right]\end{aligned}$$

The operator $\mathfrak{N}$ maps BC_∞ into BC_∞. Specifically, the mapping $\mathfrak{N}(\psi)$ is continuous on $[-a,n]$ for any $\psi\in BC_\infty$ we have:

$$\begin{aligned}|\mathfrak{N}(\psi)(s)|&\leq\|\mathfrak{C}\left(\frac{s^c}{c}\right)\||\eta(0)|+\|\mathfrak{S}\left(\frac{s^c}{c}\right)\||\vartheta(0)|+\int_0^s x^{c-1}\|\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\||\Psi(x,\psi_x)|dx\\&+\int_0^s x^{c-1}\|\mathfrak{S}\left(\frac{s^c-x^c}{c}\right)\|\|\mathcal{B}\|\,|\mathcal{U}_\psi(x)|dx\\&\leq\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)\|\psi_x\|dx\\&+\mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\left[|\widehat{\psi}|+\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\,\|\psi\|_{BC_\infty}\right]dx\\&\leq\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\|\psi\|_{BC_\infty}+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c}\left[|\widehat{\psi}|+\mathcal{M}_c^{\mathfrak{C}}\|\eta\|\right.\\&\left.+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\,\|\psi\|_{BC_\infty}\right]\\&\leq(\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c})+\mathcal{M}_c^{\mathfrak{S}}\|\psi\|_{BC_\infty}\left[\mathcal{O}^*+\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c}(1+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\right].\end{aligned}$$

So, $\mathfrak{N}\in BC_\infty$.

Furthermore, suppose $l\geq\frac{(\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c})}{1-\mathcal{M}_c^{\mathfrak{S}}\left[\mathcal{O}^*+\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c}(1+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\right]}$, and let B_l denote the closed ball in BC_∞ centered at the origin with radius l. Let $\psi\in B_l$ and $s\in I$, we get

$$|\mathfrak{N}(\psi)(s)|\leq(\mathcal{M}_c^{\mathfrak{C}}\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c})+\mathcal{M}_c^{\mathfrak{S}}l\left[\mathcal{O}^*+\widetilde{\mathcal{N}}\widetilde{\mathcal{N}_1}\frac{n^c}{c}(1+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\right].$$

Thus, we get that

$$\|\mathfrak{N}(\psi)\|_{BC_\infty}\leq l.$$

We embark on the validation process to ensure that $\mathfrak{N}: B_l\rightarrow B_l$ satisfies the criteria outlined in Meir Keeler's fixed-point Theorem 2.6.

Firstly, we ascertain the continuity of $\mathfrak{N}$ within B_l. Let $\{\psi_k\}$ be a sequence such that $\psi_k \to \psi$ in B_l. We observe that:

$$\begin{aligned}
|\mathfrak{N}(\psi_k)(s) - \mathfrak{N}(\psi)(s)| &\leq \mathcal{M}_c^{\mathfrak{S}} \int_0^s x^{c-1} |\Psi(x, (\psi_x)_k) - \Psi(x, \psi_x)| dx \\
&+ \mathcal{M}_c^{\mathfrak{S}} \widetilde{\mathcal{N}} \int_0^s x^{c-1} |\mathcal{U}_{\psi_k}(x) - \mathcal{U}_{\psi}(x)| dx \\
&\leq \mathcal{M}_c^{\mathfrak{S}} \int_0^s x^{c-1} |\Psi(x, (\psi_x)_k) - \Psi(x, \psi_x)| dx \\
&+ \mathcal{M}_c^{\mathfrak{S}} \widetilde{\mathcal{N}} \widetilde{\mathcal{N}_1} \int_0^s x^{c-1} \Big[|\widehat{\psi}_k - \widehat{\psi}| \\
&+ \mathcal{M}_c^{\mathfrak{S}} \int_0^n \tau^{c-1} |\Psi(\tau, (\psi_\tau)_k) - \Psi(\tau, \psi_\tau)| d\tau \Big] dx \\
&\leq \mathcal{M}_c^{\mathfrak{S}} \Big(1 + \mathcal{M}_c^{\mathfrak{S}} \widetilde{\mathcal{N}} \widetilde{\mathcal{N}_1} \frac{n^c}{c} \Big) \int_0^n x^{c-1} |\Psi(x, (\psi_x)_k) - \Psi(x, \psi_x)| dx \\
&+ \mathcal{M}_c^{\mathfrak{S}} \widetilde{\mathcal{N}} \widetilde{\mathcal{N}_1} \frac{n^c}{c} |\widehat{\psi}_k - \widehat{\psi}|.
\end{aligned}$$

According to (i), we have $\Psi(x, (\psi_x)_k) \to \Psi(x, \psi_x)$ as $k \to +\infty$ for almost every $x \in [0, n]$. Application of the Lebesgue dominated convergence Theorem yields:

$$\|\mathfrak{N}(\psi_k) - \mathfrak{N}(\psi)\|_{BC_\infty} \to 0, \ as \ k \to \infty.$$

Consequently, $\mathfrak{N}$ exhibits continuity.

So by the previous steps we can conclude that $\mathfrak{N}(B_l) \subset B_l$.

Next, we confirm the equicontinuity of $\mathfrak{N}(B_l)$ over every compact interval $J = [0, n]$. let $x_1, \ x_2 \in J$ with $x_2 > x_1$ we have

$$\begin{aligned}
|\mathfrak{N}(\psi)(x_1) - \mathfrak{N}(\psi)(x_2)| &\leq \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\eta\| + \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\vartheta\| \\
&+ \int_0^{x_1} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ |\Psi(x, \psi_x)| dx \\
&+ \int_{x_1}^{x_2} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ |\Psi(x, \psi_x)| dx \\
&+ \int_0^{x_1} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\mathcal{B}\| \|\mathcal{U}_\psi(x)\| dx \\
&+ \int_{x_1}^{x_2} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\mathcal{B}\| \|\mathcal{U}_\psi(x)\| dx \\
&\leq \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\eta\| + \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\vartheta\| \\
&+ \int_0^{x_1} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ |\Psi(x, \psi_x)| dx \\
&+ \mathcal{M}_c^{\mathfrak{S}} \int_{x_1}^{x_2} x^{c-1} \ |\Psi(x, \psi_x)| dx \\
&+ \int_0^{x_1} x^{c-1} \|\mathfrak{S}\Big(\frac{x_2^c - x^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - x^c}{c}\Big)\|_{B(\mathcal{F})} \ \|\mathcal{B}\| \|\mathcal{U}_\psi(x)\| dx \\
&+ \mathcal{M}_c^{\mathfrak{S}} \widetilde{\mathcal{N}} \int_{x_1}^{x_2} x^{c-1} \ \|\mathcal{U}_\psi(x)\| dx.
\end{aligned}$$

As $x_1 \to x_2$, the uniformly continuity property of $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous inequality converges to zero. This confirms the equicontinuity of $\mathfrak{N}$.

Additionally, we establish the equiconvergence of $\mathfrak{N}(B_l)$. For $s \in J$ and $\psi \in B_l$, we find

$$|\mathfrak{N}(\psi)(s)| \leq (\mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + \mathcal{M}_c^{\mathfrak{S}} l\Big[\int_0^s x^{c-1}\mathcal{O}(x)dx + \widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}(1 + \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)dx)\Big].$$

Consequently,

$$|\mathfrak{N}(\psi)(s)| \to l', \text{ as } s \to +\infty.$$

where $l' \leq (\mathcal{M}_c^{\mathfrak{C}}\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + \mathcal{M}_c^{\mathfrak{S}} l\Big[\mathcal{O}^* + \widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}(1 + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\Big]$. Here $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{N}(\psi)(s) - \mathfrak{N}(\psi)(+\infty)| \to 0, \ s \to +\infty.$$

Lastly, we validate the satisfaction of the Meir-Keeler's type condition. For all $\beta > 0$ given. we prove that there exists $\mu > 0$ such that:

$$\beta < \mathfrak{H}_I(\Delta) < \beta + \mu \Rightarrow \mathfrak{H}_I(\mathfrak{N}(\Delta)) \leq \beta, \text{ for any } \Delta \subset B_l,$$

where

$$\mathfrak{H}_I(\Delta) = \sup_{s\in I}\mathfrak{H}(\Delta(s)).$$

By using the properties of the measure of noncompactness (see eg. [12, 21, 35]) and Lemma 2.5 [33], we get

$$\mathfrak{H}(\mathfrak{N}(\Delta)(s)) \leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)\mathfrak{H}(\Delta(x))dx + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s x^{c-1}\mathfrak{H}(\mathcal{U}_\Delta(x))dx.$$

We have,

$$\begin{aligned}\mathfrak{H}(\mathcal{U}_\Delta(x)) &\leq \mathcal{K}_{\mathfrak{B}}(x)\mathcal{M}_c^{\mathfrak{S}}\int_0^n \tau^{c-1}\mathfrak{H}(\Psi(\tau, \Delta(\tau)))d\tau\\ &\leq \mathcal{K}_{\mathfrak{B}}(x)\mathcal{M}_c^{\mathfrak{S}}\int_0^n \tau^{c-1}\mathcal{O}(\tau)\mathfrak{H}(\Delta(\tau))d\tau\\ &\leq \mathcal{K}_{\mathfrak{B}}(x)\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\mathfrak{H}_I(\Delta),\end{aligned}$$

which implies

$$\begin{aligned}\mathfrak{H}(\mathfrak{N}(\Delta)(s)) &\leq \mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)dx\mathfrak{H}_I(\Delta) + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s x^{c-1}\mathcal{K}_{\mathfrak{B}}(x)\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\mathfrak{H}_I(\Delta)ds\\ &\leq \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*\mathfrak{H}_I(\Delta) + \mathcal{M}_c^{\mathfrak{S}2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}'\ \mathfrak{H}_I(\Delta)\\ &\leq (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{M}_c^{\mathfrak{S}2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}'\ )\mathfrak{H}_I(\Delta).\end{aligned}$$

Since $\mathfrak{N}(\Delta)$ is bounded and equicontinuous for all $\Delta \subset B_l$ then:

$$\mathfrak{H}_I(\mathfrak{N}(\Delta)) = \sup_{s\in I}\mathfrak{H}(\mathfrak{N}(\Delta)(s)).$$

Therefore, $\mathfrak{H}_I(\mathfrak{N}(\Delta)) \leq (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{M}_c^{\mathfrak{S}2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}'\ )\mathfrak{H}_I(\Delta) \leq \beta \Rightarrow \mathfrak{H}_I(\Delta) \leq \frac{\beta}{\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{M}_c^{\mathfrak{S}2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}'}$. Then, for given $\beta > 0$ and taking $\mu = \Big(\frac{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{M}_c^{\mathfrak{S}^2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}')}{\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{M}_c^{\mathfrak{S}^2}\widetilde{\mathcal{N}}\mathcal{O}^*\ \mathcal{K}'}\Big)\beta - \epsilon$ such that $\epsilon > 0$, we obtain:

$$\beta < \mathfrak{H}_I(\Delta) < \beta + \mu \Rightarrow \mathfrak{H}_I(\mathfrak{N}(\Delta)) \leq \beta, \text{ for any } \Delta \subset B_l.$$

Hence $\mathfrak{N}$ is a Meir-Keeler condensing operator.
Through these steps, we ensure that the conditions required for Meir-Keeler's fixed-point Theorem 2.6 are satisfied by $\mathfrak{N} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{N}$ has a fixed point ψ that provides a mild solution to the problem $(3) - (4)$.

4. Examples

Example 4.1. *To showcase the practical application of our results, let $\mathcal{E}$ denote a nonempty bounded open set in $\mathbb{R}^2$. We explore the following conformable fractional differential equation:*

$$D_s^{\frac{1}{3}}[D_s^{\frac{1}{3}}\psi(s,x)] = D_x^2\psi(s,x) + \Psi(s,\psi(s-a,x)),\ x \in \mathcal{E},\ s \in [0,+\infty); \tag{5}$$

$$\psi(s,x) = 0,\ s \in [0,+\infty),\ x \in \partial\mathcal{E}; \tag{6}$$

$$\psi(s,x) = \eta(s,x);\ \ D_s^{\frac{1}{3}}[\psi(0,x)] = \vartheta,\ s \in [-a,0],\ x \in \mathcal{E}. \tag{7}$$

Here, $a > 0$ and we have

$$\Psi(s,\psi(s-a,x)) = \frac{\exp -s}{7}\sin\psi(s-a,x).$$

Taking $\mathcal{F} = L^2(\mathcal{E})$ and defining $\mathfrak{P}$ as follows:

$$D(\mathfrak{P}) = \{\psi \in \mathcal{H}(\mathcal{F}),\ \psi(x)|_{x\in\partial\mathcal{E}} = 0\},$$

$$\mathfrak{P}\psi = D_x^2\psi,\ \psi \in D(\mathfrak{P}).$$

It is well know the operator $\mathfrak{P}$ generates a cosine family $((\mathfrak{C}(s))_{s\in\mathbb{R}}, (\mathfrak{S}(s))_{s\in\mathbb{R}})$. Additionally, it follows that

$$\|\mathfrak{C}(s)\| \leq 1 \text{ and } \|\mathfrak{S}(s)\| \leq 1,\ for\ all\ s \in [0,+\infty).$$

Thus, to apply our theorems on existence and attractivity, we require $\mathcal{O}^* < 1$.
The function $\Psi(s,\psi(s-a,x)) = \frac{\exp -s}{7}\sin\psi(s-a,x)$ is Carathéodory and

$$|\Psi(s,\psi_1(s-a,x)) - \Psi(s,\psi_2(s-a,x))| \leq \frac{\exp -s}{7}|\psi_1(s-a,x) - \psi_2(s-a,x)|,$$

thus $\mathcal{O}(s) = \frac{\exp -s}{7}$. Moreover, we have

$$\mathcal{O}^* = \sup\{\int_0^s x^{-\frac{2}{3}}\frac{\exp -x}{7}dx,\ s \in [0,+\infty)\} = \frac{\Gamma(\frac{1}{3})}{7} \simeq 0.3827 < 1,\ \Psi_0 = 0.$$

Then, by [21, 31], the problem (1)-(3) is an abstract formulation of the problem (5)-(7), and conditions $(i) - (iii)$ are satisfied. Theorem 3.3 implies that the problem (5)-(7) has a unique mild solution on BC, which is attractive by Theorem 3.4.

Example 4.2. *To showcase the practical application of our results, let $\mathcal{E}$ denote a nonempty bounded open set in $\mathbb{R}^2$. We explore the following conformable fractional differential equation:*

$$D_s^{\frac{1}{3}}[D_s^{\frac{1}{3}}\psi(s,x)] = D_x^2\psi(s,x) + \Psi(s,\psi(s-a,x)) + g(x)\mathcal{U}(s),\ x \in \mathcal{E},\ s \in [0,+\infty); \tag{8}$$

$$\psi(s,x) = 0,\ s \in [0,+\infty),\ x \in \partial\mathcal{E}; \tag{9}$$

$$\psi(s,x) = \eta(s,x);\ \ D_s^{\frac{1}{3}}[\psi(0,x)] = \vartheta,\ s \in [-a,0],\ x \in \mathcal{E}. \tag{10}$$

Here, $\mathfrak{P}$, Ψ $\mathcal{Y}$ and η are as in problem $(5)-(6)$, $g : \mathcal{E} \to \mathcal{F}$ is a continuous function and $\mathcal{U} : I \to \mathcal{F}$ is a given control.
Then, by [21, 31], the problem (3)-(4) is an abstract formulation of the problem (8)-(10), and conditions $(i)-(iii)$ are satisfied. Theorem 3.7 implies that the problem (8)-(10) is controllable on $[-a,+\infty)$.

References

[1] S. Abbas, B. Ahmad, M. Benchohra and A. Salim, Fractional Difference, Differential Equations, and Inclusions: Analysis and Stability. Elsevier., 2024. https://doi.org/10.1016/C2023-0-00030-9
[2] T. Abdeljawad, On conformable fractional calculus, Comput. Appl. Math. 279 (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016
[3] R. S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Mathematical Methods in the Applied Sciences. https://doi.org/10.1002/mma.6652
[4] R. S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions , RACSAM. (2021) 115-155; https://doi.org/10.1007/s13398-021-01095-3
[5] R. S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On The Solutions Of Fractional Differential Equations Via Geraghty Type Hybrid Contractions, Appl. Comput. Math. 20 (2) (2021),313-333
[6] R. P. Agarwal, S. Baghli, M. Benchohra, Controllability of mild solutions on semi-infinite interval for classes of semi-linear functional and neutral functional evolution equations with infinite delay, Appl. Math. Optim. 60 (2009), 253–274. http://dx.doi.org/10.1007/s00245-009-9073-1
[7] R. P. Agarwal, E. Karapınar, D. O'Regan, A. Rold´an-L´opez-de-Hierro, (2015). Fixed point theory in metric type spaces. Cham: Springer.
[8] A. Aghajani, M. Mursaleen, A. Shole Haghighi, Fixed point theorems for Meir–Keeler condensing operators via measure of noncompactness, Acta Math. Sci. Ser. B Engl. Ed. 35 (2015), 552–566. https://doi.org/10.1016/S0252-9602(15)30003-5
[9] H. Ahmed, Conformable fractional stochastic differential equations with control function, Syst. Control Lett. 158 (2021), 10506. https://doi.org/10.1016/j.sysconle.2021.105062
[10] H. Ahmed, Construction controllability for conformable fractional stochastic evolution system with non-instantaneous impulse and nonlocal condition, Stat. Probab. Lett. 190 (2022), 109618. https://doi.org/10.1016/j.spl.2022.109618
[11] S. A. Ahmed, R. Saadeh, A. Qazza, T. M. Elzaki, Modified conformable double Laplace–Sumudu approach with applications, Heliyon. 9 (2023), e15891. https://doi.org/10.1016/j.heliyon.2023.e15891
[12] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina, B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser Verlag, Basel. (1992). https://doi.org/10.1007/978-3-0348-5727-7
[13] A. A. Al-Shawba, F. A. Abdullah, A. Azmi, Compatible extension of the (G′/G)-expansion approach for equations with conformable derivative, Heliyon. 9 (2023), e15717. http://dx.doi.org/10.1016/j.heliyon.2023.e15717
[14] S. Al-Sharif, M. Al Horani, R. Khalil, The Hille-Yosida theorem for conformable fractional semigroups of operators, Missouri J. Math. Sci. 33 (1) (2021), 18–26. http://dx.doi.org/10.35834/2021/3301018
[15] A. A. M. Arafa, Z. Z. Rashed, S. E. Ahmed, Radiative flow of non-Newtonian nanofluids within inclined porous enclosures with time fractional derivative, Sci. Rep. 11 (2021), 5338. https://www.nature.com/articles/s41598-021-84848-9
[16] Y. Asghari, M. Eslami, H. Rezazadeh, Novel optical solitons for the Ablowitz–Ladik lattice equation with conformable derivatives in optical fibers, Opt. Quant. Electron. 55 (2023), 930. http://dx.doi.org/10.1007/s11082-023-04953-z
[17] M. Atraoui, M. Bouaouid, On the existence of mild solutions for nonlocal differential equations of the second order with conformable fractional derivative, Adv. Differ. Equ. 447 (2021). https://advancesindifferenceequations.springeropen.com/articles/10.1186/s13662-021-03593-5
[18] M. Atraoui, M. Bouaouid, M. Johri, Controllability of mild solutions for a nonlocal fractional conformable Cauchy problem of differential equations of the second order, Adv. Math. Models Appl. 8 (2023).
[19] C. Avramescu, Some remarks on a fixed point theorem of Krasnoselskii, Electron. J. Qual. Theory Differ. Equ. 5 (2003), 1–15. http://dx.doi.org/10.14232/ejqtde.2003.1.5
[20] S. Baghli, M. Benchohra, Kh. Ezzinbi, Controllability results for semilinear functional and neutral functional evolution equations with infinite delay, Surveys Math. Appl. 4 (2009), 15–39.
[21] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Marcel Dekker, New York, 1980. https://doi.org/10.1112/blms/13.6.583b
[22] M. Benchohra, S. Bouriah, A. Salim, and Y. Zhou, Fractional differential equations: a coincidence degree approach. Walter de Gruyter GmbH and Co KG. 12 (2023). https://doi.org/10.1515/9783111334387
[23] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, Advanced Topics in Fractional Differential Equations. https://doi.org/10.1007/978-3-031-26928-8
[24] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, Fractional Differential Equations, 2023. https://doi.org/10.1007/978-3-031-34877-8

[25] M. Benchohra, A. Salim and Y. Zhou, Integro-Differential Equations: Analysis, Stability and Controllability. Walter de Gruyter GmbH and Co KG., 2024. https://doi.org/10.1515/9783111437910

[26] M. Bouaouid, Kh. Hilal, M. Hannabou, Integral solutions of nondense impulsive conformable-fractional differential equations with nonlocal condition, J. Appl. Anal. 27 (2) (2021), 187–197. https://doi.org/10.1515/jaa-2021-2045

[27] M. Bouaouid, K. Hilal, S. Melliani, Sequential evolution conformable differential equations of second order with nonlocal condition, Adv. Differ. Equ., 2019, Paper No. 21, 13 pages.
https://advancesindifferenceequations.springeropen.com/articles/10.1186/s13662-019-1954-2

[28] M. Bouaouid, A. Kajouni, K. Hilal, S. Melliani, A class of nonlocal impulsive differential equations with conformable fractional derivative, CUBO. 24 (3) (2022), 439–455. http://dx.doi.org/10.56754/0719-0646.2403.0439

[29] A. Boukenkoul, M. Ziane, Conformable functional evolution equations with nonlocal conditions in Banach spaces, Surveys Math. Appl. 18 (2023), 83–95.

[30] B. C. Dhage, V. Lakshmikantham, On global existence and attractivity results for nonlinear functional integral equations, Nonlinear Anal. 72 (2010), 2219–2227. https://doi.org/10.1016/j.na.2009.10.021

[31] K. Ezzinbi, M. Ziat, Nonlocal integro-differential equations without the assumption of equicontinuity on the resolvent operator in Banach space, Differ. Equ. Dyn. Syst. 30 (2022), 315–333. http://dx.doi.org/10.1007/s12591-018-0423-9

[32] M. Frigon, A. Granas, Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, Ann. Sci. Math. Québec. 22 (2) (1998), 161–168.

[33] H. P. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Analysis. Theory, Method and Applications. 7 (1983), 1351-1371. https://doi.org/10.1016/0362-546X(83)90006-8

[34] A. Jaiswal and D. Bahuguna, Semilinear Conformable Fractional Differential Equations in Banach Spaces, Differ Equ Dyn Syst. 27 (2019), 313–325. http://dx.doi.org/10.1007/s12591-018-0426-6

[35] M. Kamenskii, V. Obukhovskii and P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, De Gruyter., Berlin, 2001. http://dx.doi.org/10.1515/9783110870893

[36] E.Karapınar, R.P. Agarwal, Fixed Point Theory in Generalized Metric Spaces, (2023), Synthesis Lectures on Mathematics & Statistics, Springer Cham, doi:10.1007/978-3-031-14969-6.

[37] H.R. Kataria, P.H. Patel, and V. Shah, Impulsive integro-differential systems involving conformable fractional derivative in Banach space, Int. J. Dynam. Control., 2023. http://dx.doi.org/10.1007/s40435-023-01224-3

[38] R. Khalil, M. Al Horani and M. A. Hamma, Geometric meaning of conformable derivative via fractional cords, J. Math. Comput. Sci. 19 (2019), 241–245. http://dx.doi.org/10.22436/jmcs.019.04.03

[39] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivatives, J. Comput. Appl. Math. 264 (2014), 65–70. http://dx.doi.org/10.1016/j.cam.2014.01.002

[40] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier., 2006. http://dx.doi.org/10.1016/S0304-0208(06)80001-0

[41] G. Li, Q. Wanyan, Z. Li, et al., A Fractional-Order Creep Model of Water-Immersed Coal, Appl. Sci. 13 (2023), 12839. https://doi.org/10.3390/app132312839

[42] J. Liang, Y. Mu, and T.J. Xiao, Impulsive differential equations involving general conformable fractional derivative in Banach spaces, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 116 (3) (2022), 114. http://dx.doi.org/10.1007/s13398-022-01260-2

[43] T. C. Lim, On characterizations of Meir-Keeler contractive maps, Nonlinear Anal. 46 (2001), 113–120. http://dx.doi.org/10.1016/S0362-546X(99)00448-4

[44] I. Podlubny, Fractional differential equations, 1999.

[45] O. Rosario Cayetano, A. Fleitas Imbert, J.F. Gómez-Aguilar and A.F. Sarmiento Galán, Modeling Alcohol Concentration in Blood via a Fractional Context, Symmetry. 12 (2020), 459. http://dx.doi.org/10.3390/sym12030459

[46] M. Sadaf, S. Arshed, G. Akram and Iqra, A variety of solitary wave solutions for the modified nonlinear Schrödinger equation with conformable fractional derivative, Opt Quant Electron. 55 (2023), 372. http://dx.doi.org/10.1007/s11082-023-04628-9

[47] O.P.K. Sharma, R.K. Vats and A. Kumar, A note on existence and exact controllability of fractional stochastic system with finite delay, Int. J. Dynam. Control., 2023. http://dx.doi.org/10.1007/s40435-023-01258-7

[48] O.P.K. Sharma, R.K. Vats and A. Kumar, Existence and exact controllability results of nonlocal integro-differential neutral stochastic system with finite delay, J Anal., 2023. http://dx.doi.org/10.1007/s41478-023-00675-3

[49] N. H. Tuan, V. T. Nguyen and C. Yang, On an initial boundary value problem for fractional pseudo-parabolic equation with conformable derivative, Math. Biosci. Eng. 19 (2022), 11232-11259. https://doi.org/10.3934/mbe.2022524

[50] Y. Zou and Y. Cui, Uniqueness criteria for initial value problem of conformable fractional differential equation, Electronic Research Archive. 31 (7) (2023). https://doi.org/10.3934/era.2023207