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The Prabhakar fractional q-integral and q-differential operators, and their properties

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Abstract. In this paper, we have introduced the Prabhakar fractional q-integral and q-differential operators. We first study the semigroup property of the Prabhakar fractional q-integral operator, which allowed us to introduce the corresponding q-differential operator. Formulas for compositions of q-integral and q-differential operators are also presented. We show the boundedness of the Prabhakar fractional q-integral operator in the class of q-integrable functions.

1. Introduction

Fractional calculus is the area of mathematical analysis that deals with the study and application of integrals and derivatives of arbitrary order. In recent decades, fractional calculus has become of increasing significance because of its applications in many fields of science and engineering. For example, it has many applications in viscoelasticity, signal processing, electromagnetics, fluid mechanics, and optics. For more information on this research we refer the readers to [25], [23], [22], [20], [16], [21], [2], [17], [37] and the references therein. On application of the Prabhakar derivative in diffusion and diffusion-wave processes, we refer the readers to [19], [34].

An interesting and distinctive feature of the Fractional Calculus is that it is possible to present different definitions of fractional-order integrals and derivatives; furthermore, many instances of those definitions are being applied and discussed to analyze specific processes [33]. As an example, we can take the Riemann-Liouville and Caputo fractional-order integral-differential operators which have been used widely

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to describe mathematical models of many natural phenomena (see the references [36], [28]). Recently, researchers have focused on generalizing fractional-order operators. On the one hand, it can be explained by the fact that in mathematical modeling of some real-life processes we get such a type of generalization of fractional operators. On the other hand, it is an inner need of the theory of the Fractional Calculus. We wish to focus on the Prabhakar fractional q-differential and differential operators among these operators.

First, we would like to explain the classical Prabhakar fractional calculus in brief. The theory of Prabhakar fractional calculus [14] has been studied more intensively in recent years, and as a result, certain differential equations involving Prabhakar operators became an intensive target, which is interesting both for their pure mathematical properties [31],[10], [12], and for their real-world applications in topics such as viscoelasticity, anomalous dielectrics, and option pricing [8], [13], [35], [27].

We should note that Prabhakar fractional operators can be considered a broad category that includes various named operators within fractional calculus, covering both singular and non-singular types[11]. Moreover, Prabhakar's function is crucial in empirical laws for anomalous dielectrics such as Davidson–Cole [9] and Havriliak–Negami [15] models.

The origin of the q-difference calculus can be traced back to the works [18] and [6]. Recently, W. Alsalam [3] and R.P.Agarwal [1] proposed the fractional q difference calculus. In [29] the Caputo q-differential operators and Riemann-Liuvill q-differential operators are considered and their properties are studied. Nowadays new developments in the theory of fractional q-difference calculus have been addressed extensively by several researchers (see [30], [32] and the references therein).

These advancements in the field raise the question: Is it possible to introduce Prabhakar fractional q-integral and differential operators, and do they possess the same properties as in the classical case? Our approach shows that in some cases the answer is yes; however, there are nuances specific to fractional q cases.

In the present work, our aim is to introduce and study some properties of Prabhakar fractional q-integral and differential operators.

We note that the short note of this investigation was announced in [24].

2. Preliminaries

First, we recall some elements of the q-calculus for the sequel. For more information, we note the works [7], [4], and the references therein. From now on, we assume that $0 < q < 1$ and $0 \le a < b < \infty$.

Let $\alpha \in \mathbb{R}$. A q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}.$$

And also, the q-shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0; \\ (1-a)(1-aq)\dots\left(1-aq^{n-1}\right), & n \in \mathbb{N}. \end{cases}$$

The q-analogue of the factorial is

$$[n]_q! = [1]_q[2]_q[3]_q \dots [n]_q = \frac{(q;q)_n}{(1-q)^n}, n \in \mathbb{N}, \quad [0]_q! = 1. \tag{1}$$

For q-binomial coefficients we have the following formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\dots(1-q)^{n-k+1}}{(q;q)_k} = \frac{[n]_q!}{[n-k]_q![k]_q!}. \tag{2}$$

Also, the q-analogue of the power $(a-b)_q^k$ is defined by

$$(a-b)_q^0 = 1, \quad (a-b)_q^k = \prod_{i=0}^{k-1}\left(a - bq^i\right), \quad k \in \mathbb{N}.$$

There is the following relationship between them:

$$(a-b)_q^0 = 1; \quad (a-b)_q^k = a^k (b/a;q)_k, \quad a \neq 0, \quad k \in \mathbb{N},$$

as well as

$$(a-b)_q^\alpha = a^\alpha \frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}, \quad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \quad (a;q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right).$$

For $x > 0$ the q-analogue of the Gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{3}$$

It has the following property

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x). \tag{4}$$

The (Jackson) q-derivative of a function $f(x)$ is defined by

$$\left(D_q f\right)(x) = \frac{f(x) - f(qx)}{x(1-q)}, \quad (x \neq 0)$$

and q-derivatives $D_q^n f$ of higher-order is defined inductively as follows:

$$D_q^0 f = f, \quad D_q^n f = D_q\left(D_q^{n-1} f\right) \quad (n = 1, 2, 3, ...).$$

Moreover,

$$D_q[(x-b)_q^\alpha] = [\alpha]_q (x-b)_q^{\alpha-1}, \tag{5}$$

$$D_q[(a-x)_q^\alpha] = -[\alpha]_q (a-qx)_q^{\alpha-1}. \tag{6}$$

The q-integral (Jackson integral) is defined by

$$\left(I_{q,0+} f\right)(x) = \int_0^x f(t)\, d_q t = x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right) q^k$$

and

$$\left(I_{q,a+} f\right)(x) = \int_a^x f(t)\, d_q t = \int_0^x f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

For the n-th order integral operator $I_{q,a}^n$ we have

$$(I_{q,a+}^0 f)(x) = f(x), \ (I_{q,a+}^n f)(x) = I_{q,a+}\left(I_{q,a+}^{n-1} f\right)(x) \qquad (n = 0, 1, 2, \cdots).$$

And also between q-integral and q-derivative operators, we have the following relations:

$$\left(D_q I_{q,a+} f\right)(x) = f(x), \ \left(I_{q,a+} D_q f\right)(x) = f(x) - f(a).$$

For $\alpha, \beta > 0$ and $z \in \mathbb{R}$, a q-analogue of the Mittag–Leffler function is defined as follows ([4]):

$$e_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}, \qquad \left(\left|z(1-q)^\alpha\right| < 1\right). \tag{7}$$

Definition 2.1. *[26] Let $\alpha, \beta, \gamma, z \in \mathbb{R}$ such that $\alpha, \beta > 0$. Then the q-Prabhakar function $e^{\gamma}_{\alpha,\beta}(z;q)$ is defined by*

$$e^{\gamma}_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q} z^n}{\Gamma_q(\alpha n + \beta)}, \qquad |z(1-q)^{\alpha}| < 1, \tag{8}$$

where

$$(\gamma)_{n,q} := \frac{(q^{\gamma};q)_n}{(q;q)_n}. \tag{9}$$

Lemma 2.2. *[4] Let α and β be two complex numbers. Then*

$$(q^{\alpha+\beta};q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k\beta}(q^{\alpha};q)_k (q^{\beta};q)_{n-k}, \qquad (n = 0, 1, 2, \cdots). \tag{10}$$

Proposition 2.3. *Let $\gamma, \sigma \in \mathbb{C}$. Then the following equality is valid*

$$\sum_{k=0}^{n} (\gamma)_{n-k,q}\, q^{\gamma k}\, (\sigma)_{k,q} = (\gamma + \sigma)_{n,q}, \qquad (n = 0, 1, 2, \cdots). \tag{11}$$

Proof. Taking (9) into account, we rewrite (11) in the form

$$\sum_{k=0}^{n} q^{k\gamma} \frac{(q^{\gamma};q)_{n-k}}{(q;q)_{n-k}} \frac{(q^{\sigma};q)_k}{(q;q)_k} = \frac{(q^{\gamma+\sigma};q)_n}{(q;q)_n}.$$

To prove Proposition 2.3 it is sufficient to show the validity of the last equality. For this aim, we multiply both sides of the last equality by $(q;q)_n$ and considering (1) and (2), we obtain

$$\begin{aligned}(q^{\gamma+\sigma};q)_n &= \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} q^{k\gamma}(q^{\sigma};q)_k (q^{\gamma};q)_{n-k} = \sum_{k=0}^{n} \frac{\frac{(q;q)_n}{(1-q)^n}}{\frac{(q;q)_{n-k}}{(1-q)^{n-k}} \frac{(q;q)_k}{(1-q)^k}} q^{k\gamma}(q^{\sigma};q)_k (q^{\gamma};q)_{n-k} \\ &= \sum_{k=0}^{n} \frac{[n]_q!}{[n-k]_q![k]_q!} q^{k\gamma}(q^{\sigma};q)_k (q^{\gamma};q)_{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k\gamma}(q^{\sigma};q)_k (q^{\gamma};q)_{n-k}.\end{aligned}$$

Using the result of Lemma 2.2 when $\alpha = \sigma$ and $\beta = \gamma$, we get the proof of Proposition 2.3. □

Now, we introduce a generalized q-Prabhakar function, which will be used further.

Definition 2.4. *Let $\alpha, \beta, \gamma, \omega, \delta, z, s \in \mathbb{R}$ be such that $\alpha, \beta > 0$ and $s < z$. Then the generalized q-Prabhakar function $e^{\gamma}_{\alpha,\beta}$ is defined by*

$$e^{\gamma}_{\alpha,\beta}\left[\omega(z-s)^{\delta}_q; q\right] := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q}\, \omega^n (z-s)^{\delta n}_q}{\Gamma_q(\alpha n + \beta)}, \tag{12}$$

where $|\omega(z-s)^{\delta}_q| < (1-q)^{-\alpha}$.

We note that (12) can be considered a generalization of some known functions. For example, if $\delta = \omega = 1, s = 0$ then from (12) we get Definition 2.1 of the q-Prabhakar function. And also when $\gamma = 0$ and $\delta = \omega = 1$ then from (12) we get formula (7) for the q-Mittag-Leffler function.

Now, we give basic concepts of the q-fractional calculus.

Definition 2.5. *[30] The Riemann-Liouville q-fractional integral $I^{\alpha}_{q,a+}$ of order $\alpha > 0$ is defined by*

$$\left(I^{\alpha}_{q,a+}f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)_q^{\alpha-1} f(t)\, d_qt. \tag{13}$$

Definition 2.6. *[30] The Riemann-Liouville q-fractional differential operator $D^{\alpha}_{q,a+}f$ of order $\alpha > 0$ is defined by*

$$\left(D^{\alpha}_{q,a+}f\right)(x) = \left(D^{\lceil\alpha\rceil}_{q,a+} I^{\lceil\alpha\rceil-\alpha}_{q,a+} f\right)(x), \tag{14}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α.

Notice that for $\lambda \in (-1, +\infty)$, we have

$$I^{\alpha}_{q,a+}\left((x-a)_q^{\lambda}\right) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x-a)_q^{\alpha+\lambda}. \tag{15}$$

For $1 \leq p < \infty$ the space $L^p_q[a,b]$ is defined by [32]

$$L^p_q[a,b] = \left\{ f : [a,b] \to \mathbb{C} : \left(\int_a^b \left|f(x)\right|^p d_qx \right)^{1/p} < \infty \right\}.$$

Definition 2.7. *[4] A function $f : [a,b] \to \mathbb{R}$ is called q-absolutely continuous if $\exists \varphi \in L^1_q[a,b]$ such that*

$$f(x) = f(a) + \int_a^x \varphi(t) d_qt$$

for all $x \in [a,b]$.

The set of all q-absolutely continuous functions defined in $[a,b]$ is denoted by $AC_q[a,b]$. Moreover, $AC^n_q[a,b]$ ($n \in \mathbb{N}$) is the space of real-valued functions $f(x)$ which have q-derivatives up to order $n-1$ on $[a,b]$ such that $D^{n-1}_q f \in AC_q[a,b]$, i.e.

$$AC^n_q[a,b] = \left\{ f : [a,b] \to \mathbb{R}; (D^{n-1}_q f)(x) \in AC_q[a,b] \right\}.$$

Lemma 2.8. *[32] a) Let $\alpha > 0$, $\beta > 0$ and $1 \leq p < \infty$. Then the q-fractional integral has the following semi-group property*

$$\left(I^{\alpha}_{q,a+} I^{\beta}_{q,a+} f\right)(x) = \left(I^{\alpha+\beta}_{q,a+} f\right)(x)$$

for all $x \in [a,b]$ and $f \in L^p_q[a,b]$.

b) Let $\alpha > \beta > 0$, $1 \leq p < \infty$ and $f \in L^p_q[a,b]$. Then the following equalities

$$\left(D^{\alpha}_{q,a+} I^{\alpha}_{q,a+}\right)(x) = f(x), \quad \left(D^{\beta}_{q,a+} I^{\alpha}_{q,a+} f\right)(x) = \left(I^{\alpha-\beta}_{q,a+} f\right)(x)$$

hold for all $x \in [a,b]$.

Lemma 2.9. *[32] Let $\alpha > 0$ and $1 \leq p < \infty$. Then the q-fractional integral operator $I^{\alpha}_{q,a+}$ is bounded in $L^p_q[a,b]$:*

$$\left\|I^{\alpha}_{q,a+} f\right\|_{L^p_q[a,b]} \leq K \left\|f\right\|_{L^p_q[a,b]}, \tag{16}$$

where $K = \frac{(b-qa)_q^{\alpha}}{\Gamma_q(\alpha+1)}$.

3. The definitions and the main properties of the Prabhakar fractional q- integral and q-differential operators

In this section, using Definition 2.4 of the generalized q-Prabhakar function, we introduce Prabhakar fractional q-integral and q-differential operators.

Definition 3.1. *Let $f \in L_q^1[a,b]$ and $\alpha,\beta,\gamma,\omega \in \mathbb{R}$ be such that $\alpha,\beta > 0$. Then the Prabhakar fractional q-integral operator is defined by*

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x) := \int_a^x (x-qt)_q^{\beta-1} e_{\alpha,\beta}^{\gamma}\left[\omega\left(x-q^{\beta}t\right)_q^{\alpha};q\right] f(t)d_qt. \tag{17}$$

From this and for the rest of the paper we denote $\omega' = q^{\gamma}\omega$.

Proposition 3.2. *Let $\alpha,\beta,\gamma,\mu,\sigma,\omega \in \mathbb{R}$ be such that $\alpha,\beta,\mu > 0$ and $x,s \in \mathbb{R}^+$, $x > s$. Then*

$${}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}\left\{g_{\sigma,\omega'}^{\alpha,\mu}(x,s)\right\} = g_{\sigma+\gamma,\omega}^{\alpha,\mu+\beta}(x,s), \tag{18}$$

where

$$g_{\sigma,\omega'}^{\alpha,\mu}(x,s) := (x-qs)_q^{\mu-1} e_{\alpha,\mu}^{\sigma}\left[\omega'(x-q^{\mu}s)_q^{\alpha};q\right]. \tag{19}$$

Proof. By using (17), we have

$${}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}\left\{g_{\sigma,\omega'}^{\alpha,\mu}(x,s)\right\} = \int_{qs}^{x} g_{\gamma,\omega}^{\alpha,\beta}(x,t) g_{\sigma,\omega'}^{\alpha,\mu}(x,s)d_qt.$$

Considering (12) and using Definition 2.5, we have

$${}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}\left\{g_{\sigma,\omega'}^{\alpha,\mu}(x,s)\right\} = \sum_{n=0}^{\infty}(\gamma)_{n,q}\omega^n \sum_{k=0}^{\infty}\frac{q^{\gamma k}\omega^k(\sigma)_{k,q}}{\Gamma_q(\alpha k+\mu)} I_{q,qs+}^{\alpha n+\beta}(x-qs)_q^{\alpha k+\mu-1}.$$

Hence, applying (15) we obtain

$${}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}\left\{g_{\sigma,\omega'}^{\alpha,\mu}(x,s)\right\} = \sum_{n=0}^{\infty}(\gamma)_{n,q}\omega^n \sum_{k=0}^{\infty}(\sigma)_{k,q}q^{\gamma k}\omega^k \frac{(x-qs)_q^{\alpha k+\alpha n+\beta+\mu-1}}{\Gamma_q(\alpha n+\alpha k+\beta+\mu)}.$$

Using the Cauchy product formula [5] and then considering (11), and also the expansion (12) of the generalized q-Prabhakar function, we derive

$$\begin{aligned}{}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}\left\{g_{\sigma,\omega'}^{\alpha,\mu}(x,s)\right\} &= \sum_{n=0}^{\infty}\frac{(x-qs)_q^{\alpha n+\beta+\mu-1}\omega^n}{\Gamma_q(\alpha n+\beta+\mu)}\sum_{k=0}^{n}(\gamma)_{n-k,q}q^{\gamma k}(\sigma)_{k,q} = \sum_{n=0}^{\infty}\frac{(\gamma+\sigma)_{n,q}\omega^n}{\Gamma_q(\alpha n+\beta+\mu)}(x-qs)_q^{\alpha n+\beta+\mu-1}\\ &= (x-qs)_q^{\beta+\mu-1} e_{\alpha,\beta+\mu}^{\gamma+\sigma}\left[\omega\left(x-q^{\beta+\mu}s\right)_q^{\alpha};q\right] = g_{\sigma+\gamma,\omega}^{\alpha,\mu+\beta}(x,s),\end{aligned}$$

which completes the proof of the statement. □

Lemma 3.3. *Let $f \in L_q^p[a,b]$ and $\alpha,\beta,\gamma,\omega,\mu,\sigma \in \mathbb{R}$ be such that $\alpha,\beta,\mu > 0$. Then the following relation*

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}\,{}^{P}I_{q,a+}^{\alpha,\mu,\sigma,\omega'} f\right)(x) = \left({}^{P}I_{q,a+}^{\alpha,\beta+\mu,\gamma+\sigma,\omega} f\right)(x) \tag{20}$$

holds for all $x \in [a,b]$.

In particular

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}\,{}^{P}I_{q,a+}^{\alpha,\mu,-\gamma,\omega'} f\right)(x) = \left(I_{q,a+}^{\beta+\mu} f\right)(x). \tag{21}$$

Proof. By Definition 3.1 of the Prabhakar fractional q-integral operator and taking the notation (19) into account, we have

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}{}^{P}I_{q,a+}^{\alpha,\mu,\sigma,\omega'}f\right)(x)=\int_a^x g_{\gamma,\omega}^{\alpha,\beta}(x,t)\int_a^t g_{\sigma,\omega'}^{\alpha,\mu}(x,s)f(s)d_qsd_qt.$$

Hence, by changing the order of integration and using Definition 3.1, we get

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}{}^{P}I_{q,a+}^{\alpha,\mu,\sigma,\omega'}f\right)(x)=\int_a^x f(s)d_qs\int_{qs}^x g_{\gamma,\omega}^{\alpha,\beta}(x,t)g_{\sigma,\omega'}^{\alpha,\mu}(x,s)d_qt=\int_a^x {}^{P}I_{q,qs+}^{\alpha,\beta,\gamma,\omega}[g_{\sigma,\omega'}^{\alpha,\mu}(x,s)]f(s)d_qs.$$

Applying Proposition 3.2 and considering Definition 3.1, we obtain

$$\left({}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}{}^{P}I_{q,a+}^{\alpha,\mu,\sigma,\omega'}f\right)(x)=\int_a^x g_{\gamma+\sigma,\omega}^{\alpha,\beta+\mu}(x,s)f(s)d_qs=\left({}^{P}I_{q,a+}^{\alpha,\beta+\mu,\gamma+\sigma,\omega}f\right)(x).$$

By putting $\sigma=-\gamma$ and taking $e_{\alpha,\beta}^{0}(z)=1/\Gamma_q(\beta)$ into account from the last equality one can easily obtain (21).

The proof of Lemma 3.3 is complete. □

Proposition 3.4. *Let $\alpha,\beta,\gamma,\omega\in\mathbb{R}$ be such that $\alpha,\beta>0$ $|\gamma|<1$, $|\omega(b-q^{\beta+1}a)_q^{\alpha}|<(1-q)^{\alpha}$ and $1\le p<\infty$. Then the Prabhakar fractional q-integral operator ${}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}$ is bounded in $L_q^p[a,b]$:*

$$\left\|{}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}f\right\|_{L_q^p[a,b]}\le M\left\|f\right\|_{L_q^p[a,b]}, \tag{22}$$

where

$$M=(b-qa)_q^{\beta}e_{\alpha,\beta+1}[(b-q^{\beta+1}a)_q^{\alpha};q]. \tag{23}$$

Proof. Taking Definition 3.1 and notation (19) into account, we have

$$\left\|{}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega}f\right\|_{L_q^p[a,b]}^p=\int_a^b\left|\int_a^x g_{\gamma,\omega}^{\alpha,\beta}(x,t)f(t)d_qt\right|^p d_qx\le\int_a^b J_2(x)d_qx, \tag{24}$$

where

$$J_2(x):=\left\{\int_a^x g_{\gamma,|\omega|}^{\alpha,\beta}(x,t)|f(t)|d_qt\right\}^p.$$

For $p>1$, we will define p' from the equality $\frac{1}{p}+\frac{1}{p'}=1$. Applying the Hölder-Rogers inequality to $J_2(x)$, we get

$$J_2(x)\le\left(\int_a^x g_{\gamma,|\omega|}^{\alpha,\beta}(x,t)d_qt\right)^{p/p'}\int_a^x g_{\gamma,|\omega|}^{\alpha,\beta}(x,t)|f(t)|^pd_qt=J_{21}^{p/p'}(x)\times J_{22}(x),$$

where

$$J_{21}(x):=\int_a^x g_{\gamma,|\omega|}^{\alpha,\beta}(x,t)d_qt,\quad J_{22}(x):=\int_a^x g_{\gamma,|\omega|}^{\alpha,\beta}(x,t)|f(t)|^pd_qt.$$

Let us now consider $J_{21}(x)$. We show that the following inequality is true:

$$J_{21}(x) \leq M, \tag{25}$$

where M is a constant defined by (23). Indeed, considering (19) and (12) and using formulas (4), (6), we have

$$J_{21}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q}|\omega|^n}{\Gamma_q(\alpha n+\beta+1)}(x-a)_q^{\alpha n+\beta}.$$

Since $|\gamma| < 1$, we have $(\gamma)_{n,q} < 1$. Taking this into account and using

$$(b-a)_q^{\delta} < (b-aq)_q^{\delta} \quad (\delta > 0),$$

we obtain

$$\begin{aligned} J_{21}(x) &\leq \sum_{n=0}^{\infty} \frac{|\omega|^n}{\Gamma_q(\alpha n+\beta+1)}(b-a)_q^{\alpha n+\beta} \leq \sum_{n=0}^{\infty} \frac{|\omega|^n}{\Gamma_q(\alpha n+\beta+1)}(b-aq)_q^{\alpha n+\beta} \\ &= (b-aq)^{\beta} \sum_{n=0}^{\infty} \frac{|\omega|^n (b-a^{\beta+1}q)_q^{\alpha n}}{\Gamma_q(\alpha n+\beta+1)} = (b-qa)_q^{\beta} e_{\alpha,\beta+1}[\omega(b-q^{\beta+1}a)_q^{\alpha};q] = M. \end{aligned}$$

We note that the conditions $\alpha, \beta > 0$ and $|\omega(b-q^{\beta+1}a)_q^{\alpha}| < (1-q)^{\alpha}$ justify the convergence of series of the function $e_{\alpha,\beta+1}[\omega(b-q^{\beta+1}a)_q^{\alpha};q]$.

Then, by virtue of (25) , we obtain the following inequality

$$J_2(x) \leq M^{\frac{p}{p'}} J_{22}(x).$$

Taking the last inequality into account from (24), we get

$$\left\| {}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega} f \right\|_{L_q^p[a,b]}^{p} \leq M^{\frac{p}{p'}} \int_a^b J_{22}(x) d_q x. \tag{26}$$

Substituting the expression of $J_{22}(x)$ into (26), changing the order of integration, and using (19),(12) and (5), we get

$$\begin{aligned} J_q(f) &\leq M^{\frac{p}{p'}} \int_a^b |f(t)|^p d_q t \int_{qt}^b g_{\gamma,|\omega|}^{\alpha,\beta}(x,t) d_q x = M^{\frac{p}{p'}} \int_a^b |f(t)|^p d_q t \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q}|\omega|^n}{\Gamma_q(\alpha n+\beta)} \int_{qt}^b (x-t)_q^{\alpha n+\beta-1} d_q x \\ &= M^{\frac{p}{p'}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q}|\omega|^n}{\Gamma_q(\alpha n+\beta+1)} \int_a^b |f(t)|^p (b-qt)_q^{\alpha n+\beta} d_q t \leq M^{\frac{p}{p'}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q}|\omega|^n (b-qa)_q^{\alpha n+\beta}}{\Gamma_q(\alpha n+\beta+1)} \int_a^b |f(t)|^p d_q t \\ &= M^{\frac{p}{p'}+1} \int_a^b |f(t)|^p d_q t = M^p \left\| f \right\|_{L_q^p[a,b]}^{p}. \end{aligned}$$

Proposition 3.4 is proved. □

Now, we present the definition of the Prabhakar fractional q-differential operator.

Definition 3.5. *Let $f \in L_q^1[a,b]$, ${}^{P}I_{q,a+}^{\alpha,n-\beta,-\gamma,\omega} f \in AC_q^n[a,b]$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$. Then the Prabhakar fractional q-differential operator ${}^{P}D_{a,a+}^{\alpha,\beta,\gamma,\omega}$ is defined by*

$$\left({}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x) := \left(D_{q,a+}^{n}\, {}^{P}I_{q,a+}^{\alpha,n-\beta,-\gamma,\omega} f\right)(x), \tag{27}$$

where $n = \lceil \beta \rceil$.

Theorem 3.6. *Let $\alpha, \beta, \gamma, \omega \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$. Then for any function $f \in L_q^1[a,b]$ the following equality is valid:*

$$\left({}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} {}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} f\right)(x) = f(x). \tag{28}$$

Proof. Using Definition 3.5 and formula (20) and also Lemma 2.8, we have

$${}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} \left({}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} f\right)(x) = D_{q,a+}^n \left({}^P I_{q,a+}^{\alpha,n-\beta,-\gamma,\omega} {}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} f\right)(x) = D_{q,a+}^n \left(I_{q,a+}^n f\right)(x) = f(x).$$

The proof is complete. □

In the classical case, the Prabhakar fractional integral operators' semigroup property is commutative, but this property is non-commutative in the q-calculus case. To deal with this problem we need to introduce the following operator which affects only one parameter of the Prabhakar fractional q-operators.

We introduce the operator $\Lambda_q^{n\gamma,\omega}$ defined by setting

$$\Lambda_q^{n\gamma,\omega}\omega := q^{n\gamma}\omega, \quad n \in \mathbb{N}.$$

For example, $\Lambda_q^{n\gamma,\omega} f(\delta,\omega) = f(\delta, q^{n\gamma}\omega)$.

Using this operator we present some other properties of q-Prabhakar operators.

Theorem 3.7. *Let $f \in L_q^1[a,b]$, ${}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f \in AC_q[a,b]$ and $\alpha, \beta, \gamma, \omega \in \mathbb{R}$ with $\alpha > 0$, $0 < \beta \le 1$. Then*

$$\left({}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} {}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x) = f(x) - g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q) \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f\right)(a+). \tag{29}$$

Proof. Let

$$\varphi(x) := \left({}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} {}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x). \tag{30}$$

Applying the Prabhakar fractional q-derivative ${}_q^P D_{x,a+}^{\alpha,\beta,\gamma,\omega}$ to the both sides of (30) and using Theorem 3.6, we obtain

$${}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} \varphi = {}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} f. \tag{31}$$

We apply $\Lambda_q^{2\gamma,\omega}$ operator to the equality (31). Then, taking into account $w' = q^\gamma \omega$, we get

$${}^P D_{q,a+}^{\alpha,\beta,\gamma,q^\gamma\omega'} \varphi = {}^P D_{q,a+}^{\alpha,\beta,\gamma,q^\gamma\omega'} f. \tag{32}$$

From the last equality, we conclude that $f - \varphi$ is an element of the kernel of the Prabhakar fractional q-differential operator, i.e.,

$$f - \varphi \in \ker({}^P D_{q,a+}^{\alpha,\beta,\gamma,q^\gamma\omega'}).$$

Introducing notation $\psi := f - \varphi$ and considering (27) and $0 < \beta \le 1$, we have

$$\left({}^P D_{q,a+}^{\alpha,\beta,\gamma,q^\gamma\omega'} \psi\right)(x) = 0 \Leftrightarrow D_q \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,q^\gamma\omega'} \psi\right)(x) = 0.$$

By the standard properties of the q-differential operator the last equality means that $\left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,q^\gamma\omega'} \psi\right)(x)$ must be a constant:

$$\left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,q^\gamma\omega'} \psi\right)(x) = a_0, \tag{33}$$

where a_0 is an arbitrary constant.

Hence, applying the Prabhakar q-fractional differential operator ${}^P D_{q,a+}^{\alpha,1-\beta,-\gamma,\omega'}$ to the last equality and using Theorem 3.6, we find

$$\psi(x) = {}^P D_{q,a+}^{\alpha,1-\beta,-\gamma,\omega'}(a_0).$$

By Definitions 3.1 and 3.5 of the Prabhakar fractional q-differential and q-integral operators and also (12) expansion of the generalized q-Prabhakar function, we have

$$\begin{aligned}\psi(x) &= a_0 D_q \int_a^x (x-qt)_q^{\beta-1} \sum_{n=0}^{+\infty} \frac{(\gamma)_{n,q}(\omega')^n}{\Gamma_q(\alpha n+\beta)} \left(x-q^\beta t\right)_q^{\alpha n} d_q t \\ &= a_0 \sum_{n=0}^{+\infty} (\gamma)_{n,q}(\omega')^n D_q \left[\frac{1}{\Gamma_q(\alpha n+\beta)} \int_a^x (x-qt)_q^{\alpha n+\beta-1} d_q t\right].\end{aligned}$$

Hence, using formulas (6), (5) and taking (12) and (19) into account, we find

$$\psi(x) = a_0 g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q).$$

Since $\psi = f - \varphi$, we obtain

$$f(x) = \varphi(x) + a_0 g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q). \tag{34}$$

Hence, applying the Prabhakar fractional q-integral operator ${}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega}$ to the last equality, we find

$$({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f)(x) = ({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} \varphi)(x) + a_0 {}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q). \tag{35}$$

Applying formula (18), where β is replaced with $1-\beta$, γ with $-\gamma$, σ with γ, μ with β, and s with a/q, and considering that $e_{\alpha,\beta}^0(x) = 1$, it is easy to show that

$${}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q) = g_{0,\omega'}^{\alpha,1}(x, a/q) = e_{\alpha,\beta}^0[\omega'(x-a)^\alpha] = 1. \tag{36}$$

Then from (35), we obtain

$$\left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f\right)(x) = \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} \varphi\right)(x) + a_0. \tag{37}$$

Considering the notation (30) and using (20) and Definition 3.5, we have

$$\left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} \varphi\right)(x) = \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} {}^P I_{q,a+}^{\alpha,\beta,\gamma,\omega'} {}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x) = \left({}^P I_{q,a+}^{\alpha,1,0,\omega} {}^P D_{q,a+}^{\alpha,\beta,\gamma,\omega} f\right)(x).$$

Hence, applying fundamental theorem of q-calculus [7], we find

$$\left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} \varphi\right)(x) = \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f\right)(x) - \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f\right)(a).$$

Comparing this with (37), we conclude that

$$a_0 = \left({}^P I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} f\right)(a).$$

Substituting the expression obtained from a_0 into (34) and considering the notation (30), we get (29).

The proof of Theorem 3.7 is complete. □

4. The Cauchy-type problem associated with q-Prabhakar differential operator

Let us consider the following Cauchy-type problem with Prabhakar fractional q-differential operator:

$$\left({}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega}y\right)(x) = f(x,y), \tag{38}$$

$$\left({}^{P}I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega}y\right)(a+) = \xi_0, \tag{39}$$

where $\alpha, \beta, \gamma, \omega, \xi_0 \in \mathbb{R}$ are such that $\alpha > 0$, $0 < \beta \leq 1$, $\xi_0 \neq 0$.

We prove the existence and uniqueness of the solution to the problem (38)-(39).

Theorem 4.1. *Let $f(\cdot,\cdot) : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in L_q^1[a,b]$ for all $y \in L_q^1[a,b]$.*

Then y satisfies the relations (38) and (39), if and only if, y satisfies the following q-Volterra integral equation:

$$y(x) = {}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega'} f(x,y) + \xi_0 g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q). \tag{40}$$

Proof. First, we prove the necessity. We assume that $y \in L_q^1[a,b]$ satisfies (38)-(39). Since $f(x,y) \in L_q^1[a,b]$, (38) means that there exists a Prabhakar fractional q-differential ${}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega}y \in L_q^1[a,b]$. on $[a,b]$ so we can apply the operator ${}^{P}I_{q,a+}^{\alpha,\beta,\gamma,\omega'}$ to the equation (38). Then, considering the formula (29) and the condition (39), we get the integral equation (40).

Now, we prove the sufficiency. Let $y \in L_q^1[a,b]$ satisfy equation (40). Applying the operator ${}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega}$ to both sides of (40) and using (28), we get

$$\left({}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega}y\right)(x) - f(x,y) = \xi_0 {}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega} g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q). \tag{41}$$

We show that the right-hand side of (41) is equal to zero. Using (27) and (36), we find

$${}^{P}D_{q,a+}^{\alpha,\beta,\gamma,\omega} g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q) = D_q {}^{P}I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega} g_{\gamma,\omega'}^{\alpha,\beta}(x, a/q) = D_q(1) = 0.$$

Now, we show that the relation in (39) is also held. For this, we apply the operator ${}^{P}I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega}$ to both sides of (40). Considering (20) and (36), we get

$$\left({}^{P}I_{q,a+}^{\alpha,1-\beta,-\gamma,\omega}y\right)(x) - \int_a^x f(t, y(t)) d_q t = \xi_0. \tag{42}$$

By putting $x = +a$ in (42) we obtain the relation in (39).

Theorem 4.1 is proved. □

5. Existence and uniqueness of the solution to the Cauchy-type problem

In this section, we prove the existence and uniqueness of the solution to the problem (38)-(39). The result is obtained under the conditions of Theorem 4.1 and Lemma 3.4.

Theorem 5.1. *Let G be an open set in $\mathbb{R}$. Let $f(\cdot,\cdot) : [a,b] \times G \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in L_q^1[a,b]$ for all $y \in G$, and for all $x \in (a,b]$ and for all $y_1, y_2 \in G$, it satisfies*

$$\left|f(x,y_1) - f(x,y_2)\right| \leq A \left|y_1 - y_2\right|, \tag{43}$$

where $A > 0$ does not depend on $x \in [a,b]$ and $y_1, y_2 \in L_q^1[a,b]$.

Then there exists a unique solution $y \in L_q^1[a,b]$ to the problem (38)-(39).

Proof. According to Theorem 4.1, the problem (38)-(39) is equivalent to the integral equation (40). So, to prove the existence and uniqueness of the solution to the problem (38)-(39) it is sufficient to show the existence and uniqueness of the solution to the integral equation (40). To do this we rewrite the integral equation (39) in the following operator form

$$y(x) = (Ty)(x), \tag{44}$$

where

$$(Ty)(x) := y_0(x) + \int_a^x (x - qt)_q^{\beta-1} e_{\alpha,\beta}^{\gamma} \left[\omega'(x - q^{\beta}t)_q^{\alpha}\right] f[t, y(t)] d_q t \tag{45}$$

and

$$y_0(x) := \xi_0 (x - a)_q^{\beta-1} e_{\alpha,\beta}^{\gamma} \left[\omega'(x - q^{\beta-1}a)_q^{\alpha}; q\right].$$

First, we prove the existence of a unique solution $y(x)$ in the space $L_q^1[a, b]$. Our proof is based on the Banach fixed point theorem. We should note that $L_q^1[a, b]$ is a complete metric space [4].

Select $h \in (a, b]$ such that

$$\delta_1 = A(h - qa)_q^{\beta} e_{\alpha,\beta+1}[\omega'(h - q^{\beta+1}a)_q^{\alpha}; q] < 1, \tag{46}$$

where $A > 0$ is the Lipschitz constant in (43). Clearly $y_0 \in L_q^1[a, h]$. Also, by Lemma 3.4 $(Ty)(x) \in L_q^1[a, h]$. Therefore, T maps $L_q^1[a, h]$ to itself. Moreover, from (43),(45) and Proposition 3.4, for any $y_1, y_2 \in L_q^1[a, h]$, we have

$$\begin{aligned}
\left\|Ty_1 - Ty_2\right\|_{L_q^1[a,h]} &\leq \left\|{}_q^P I_{x,a+}^{\alpha,\beta,\gamma,\omega} f(x, y_1(x)) - {}_q^P I_{x,a+}^{\alpha,\beta,\gamma,\omega} f(x, y_2(x))\right\|_{L_q^1[a,h]} \\
&\leq (h - qa)_q^{\beta} e_{\alpha,\beta+1}[\omega'(h - q^{\beta+1}a)_q^{\alpha}; q] \times \left\|f(x, y_1(x)) - f(x, y_2(x))\right\|_{L_q^1[a,h]} \\
&\leq A(h - qa)_q^{\beta} e_{\alpha,\beta+1}[\omega'(h - q^{\beta+1}a)_q^{\alpha}; q] \times \left\|y_1(x) - y_2(x)\right\|_{L_q^1[a,h]} \leq \delta_1 \left\|y_1(x) - y_2(x)\right\|_{L_q^1[a,h]}.
\end{aligned}$$

Our assumption (46) allows us to apply the Banach's fixed point theorem to obtain a unique solution $y^* \in L_q^1[a, h]$ to equation (44) in the interval $(a, h]$. According to this theorem, y^* will be obtained as a limit of a convergent sequence $(T^m y_0)(x)$:

$$\lim_{m\to\infty} \left\|T^m \overline{y}(x) - y^*\right\|_{L_q^1[a,h]} = 0$$

in the space $L_q^1[a, h]$, where $\overline{y}(x)$ is an arbitrary function in $L_q^1[a, h]$.

Since $\xi_0 \neq 0$ and $y_0 \in L_q^1[a, h]$ we can take $y_0(x)$ as $\overline{y}(x)$:

$$\overline{y}(x) := y_0(x).$$

Consequently, the sequence $T^m y_0(x)$ for $m \in \mathbb{N}$ is defined by the following recurrence relation

$$T^m y_0(x) = y_0(x) + \int_a^x (x - qt)_q^{\beta-1} e_{\alpha,\beta}^{\gamma} \left[\omega'(x - q^{\beta}t)_q^{\alpha}\right] f[t, T^{m-1} y_0(t)] d_q t.$$

If we denote $y_m(x) = (T^m y_0)(x)$, then the last relation takes the form

$$y_m(x) = y_0(x) + \int_a^x (x - qt)_q^{\beta-1} e_{\alpha,\beta}^{\gamma} \left[\omega'(x - q^{\beta}t)_q^{\alpha}\right] f[t, y_{m-1}(t)] d_q t, \quad m \in \mathbb{N}.$$

This means that the successive approximation method can be used to find a unique solution of (38)–(39). □

Conclusion

Introducing the Prabhakar fractional q-integral and q-differential operators, and exploring their properties and potential applications have been targeted. Prabhakar fractional q-integral and q-differential operators are comparatively less studied and hence, first, we have introduced the definitions of these operators, showed some new aspects of q-cases, and studied the existence and uniqueness of the solution of the Cauchy-type problem involving the Prabhakar fractional q-differential operator.

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