Filomat 39:6 (2025), 2017–2027
https://doi.org/10.2298/FIL2506017S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on the boundedness of Marcinkiewicz integral operator on continual Herz-Morrey spaces

Mehvish Sultan[a], **Babar Sultan**[b,*]

[a]*Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan*
[b]*Department of Mathematics, Quaid-I-Azam University, Islamabad, Pakistan*

Abstract. In this paper, the authors prove the boundedness of Marcinkiewicz integral operator under some proper assumptions on continual Herz-Morrey spaces $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)$ with variable exponents.

1. Introduction

The Marcinkiewicz integral is introduced as a tool analogous to the Littlewood-Paley f-function. In 1938, Marcinkiewicz [13] introduced an integral on one-dimensional Euclidean space $\mathbb{R}$, known today as the Marcinkiewicz integral. The integral is defined without going into the interior of the unit disk, and Marcinkiewicz conjectured that it is bounded on $L^p([0,2\pi])$ for any p in open interval $(1,\infty)$. In 1944, using a complex variable method, Zygmund [29] proved the Marcinkiewicz conjecture. The higher-dimensional Marcinkiewicz integral was introduced by mathematician Elias M. Stein [19] in 1958.

Let $\mathbb{S}^{n-1}$ represents the unit sphere in $\mathbb{R}^n$, equipped with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Phi(y')d\Phi(y') = 0, \tag{1}$$

where $y' = y/|y|$ and y is not zero. The Marcinkiewicz integral in the context of the Littlewood-Paley f-function on $\mathbb{R}^n$ is given as

$$M_\Phi(f)(x) = \left(\int_0^\infty |F_{\Phi,s}(f)(x)|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}},$$

2020 *Mathematics Subject Classification*. Primary 46E30; Secondary 47B38.
Keywords. Continual Herz spaces, continual Herz-Morrey spaces, Marcinkiewicz integral operator.
Received: 05 October 2024; Accepted: 07 October 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Babar Sultan
Email addresses: mehvishsultanbaz@gmail.com (Mehvish Sultan), babarsultan40@yahoo.com (Babar Sultan)
ORCID iDs: https://orcid.org/0009-0005-2379-0476 (Mehvish Sultan), https://orcid.org/0000-0003-2833-4101 (Babar Sultan)

where

$$F_{\Phi,s}(f)(x) = \int\limits_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} f(y)dy.$$

The Marcinkiewicz integral operator M_Φ is defined by,

$$M_\Phi(f)(x) = \left(\int\limits_0^\infty \left| \int\limits_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

Numerous research papers have been dedicated to investigating the boundedness of the higher-dimensional Marcinkiewicz integral operator M_Φ on various function spaces. It then refers the reader to specific papers for further details on the developments and applications of this operator [3–5, 8, 9, 12, 25, 26, 28].

The study of function spaces with variable exponents has gained significant attention and made substantial progress in various branches of mathematics, including real analysis, partial differential equations (PDEs), and applied mathematics. Herz spaces were initially introduced by Herz [7] in his paper to investigate the absolute convergence of Fourier transforms. Subsequently, these spaces have found significant applications in diverse branches of applied mathematics. Herz spaces play a role in characterizing multipliers on Hardy spaces [2], in regularity theory for elliptic equations in divergence form [17] and involved in the study of the summability properties of Fourier transforms [6].

Herz spaces where introduced with discrete type norm (homogeneous version)

$$\|f\|_{H^\alpha_{p,q}} := \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha p} \left(\int_{2^{k-1}<|x|<2^k} |f(x)|^q dx \right)^{\frac{p}{q}} \right\}^{\frac{1}{p}}.$$

One of the important problems on Herz spaces is boundedness of sublinear operators. In [10], author introduced the idea of Herz spaces with variable exponent $H^\alpha_{p,q}$ and obtained the boundedeness of sublinear operators in the setting of Herz spaces with variable exponent. The most general results were obtained in [1], where the variability of α was allowed and authors discussed the boundednss of a wide class of sublinear operators (maximal, potential and Calderón-Zugmund operators) on variable Herz spaces. We also note that Herz-Morrey spaces with variable exponent is the generalizations of Herz spaces with variable exponent. The class of Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was initially defined by the author [11], and the boundedness of sublinear operators on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was proved.

The idea of grand variable Herz-Morrey spaces further extended the framework of Herz-Morrey spaces by incorporating the variable exponent setting. The boundedness of Marcinkiewicz integral operator of variable order in grand variable Herz-Morrey spaces are obtained in [25]. Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces are proved in [23]. In [24], authors obtained the BMO estimate for the higher order commutators of Marcinkiewicz integral operator in grand variable Herz-Morrey spaces. For more generalized versions of the Herz spaces with variable exponents, please see the papers [20, 21]. The idea of continual Herz spaces was introduced by [18] by replacing the discrete norm with continual Lebesgue norm and author obtained the boundeness results for sublinear operator in these spaces, for more results on continual Herz spaces see [14, 16].

Motivated by the idea, in this paper we introduce the idea of continual Herz-Morrey spaces with variable exponents. We obtain the boundedness results for the Marcinkiewicz integral operator in these spaces.

We define a measurable set H in $\mathbb{R}^n$ and a measurable function $r(\cdot)$ defined on H and taking values in the interval $[1,\infty)$. Let us assume that $r_- := \operatorname{ess}\inf_{h\in H} r(h)$, and $r_+ := \operatorname{ess}\sup_{h\in H} r(h)$, then we have the following inequality

$$1 \leq r_-(H) \leq r(h) \leq r_+(H) < \infty. \tag{2}$$

With respect to classes of variable exponents used in this paper, we adopt the following notation:

(i) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ represents a class of functions that meet certain conditions. The class $\mathcal{P}(H)$ consists of those exponents which satisfy (2) and (5);

(ii) in the case H is unbounded, $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,\infty}(H)$ are subset of the class $\mathcal{P}(H)$ that consists of functions satisfying a certain condition, as indicated by equation (6). The values of the functions in this subset fall within the interval $[1,\infty)$, and satisfy both conditions (6) and (7) respectively;

(iii) Let $\mathbb{R}_{\mu+} := (\mu,\infty)$, where $\mu \geq 0$. Then $M_\infty(\mathbb{R}_{\mu+})$ is the class consists of functions defined on the domain $\mathbb{R}\mu+$ that have certain properties. Specifically, these functions are of the form $g(t) = \text{constant} + g_0(t)$, where $g_0(t)$ is a function belonging to the class $\mathcal{P}\infty(\mathbb{R}_{\mu+})$, where $H = \mathbb{R}_{\mu+}$, $\mu \geq 0$.

(iv) $\mathcal{P}^{\log}_\infty(H)$ is the subset consists of exponents that satisfy the condition (5);

(v) in the case $H = \mathbb{R}_+$ (the case $\mu = 0$), $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ is the class comprises functions defined on the positive real line $\mathbb{R}+$ that are also in the class $\mathcal{M}\infty(\mathbb{R}_+)$ and satisfy a decay condition at the origin ($y = 0$). The decay condition implies that these functions are bounded by a logarithmic term as y approaches zero. Specifically, for $|y| \leq \frac{1}{2}$, the function satisfies the inequality $|f(y) - f_0| \leq \frac{C}{\ln|y|}$ for some real numbers f_0 and C. We also write $f_0 = f(0)$, $f_\infty = f(\infty)$ in this case;

(vi) $\mathcal{P}_{0,\infty}(\mathbb{R}+)$ is a subclass of functions within the class $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that have values in the interval $[1,\infty)$. In other words, these are the functions from $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that satisfy the given conditions and have their values constrained within the specified range with values in $[1,\infty)$.
If $\mathbb{R}_{\mu+}$, $\mu \geq 0$ and $\frac{dt}{t}$ denotes the Haar measure, norm of Lebesgue spaces with Haar measure is defined as,

$$\|g\|_{L^{q(\cdot)}(\mathbb{R}_{\mu+};\frac{dt}{t})} = \inf\left\{\lambda > 0 : \int_\mu^\infty \left|\frac{g(t)}{\lambda}\right|^{q(t)} \frac{dt}{t} \leq 1\right\}.$$

Let $\frac{d\theta}{\theta}$ denote the Haar measure, then the Mellin convolution operator with homogeneous kernal of order 0 is defined as

$$\Gamma\omega(j) = \int_0^\infty \mathcal{K}(\frac{j}{\theta})\omega(\theta)\frac{d\theta}{\theta}. \tag{3}$$

In [18], Samko introduced the new class of function spaces known as continual Herz spaces with variable exponent. Boundedness of some operators including sublinear operators, Riesz potential operator, Marcinkiewicz integrals can be found in [14, 16].

Let χ_H is the characteristic function of a set H and $\mathcal{R}_m := \mathcal{R}(2^{m-1}, 2^m)$ with $\chi_{\tau,t}(y) = \chi_{\mathcal{R}_{\tau,t}}(y)$. We can define continual Herz-Morrey spaces with variable exponent $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)$ by its norm,

$$\|f\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)} := \|f\|_{L^{p(\cdot)}(B(0,\gamma\mu+\theta))} + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha(t)}\|f\chi_{\mathcal{R}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((\gamma\mu,k_0);\frac{dt}{t})} < \infty. \tag{4}$$

2. Preliminaries

We denote by $B(y,s)$, the all with center at y and radius s. We define the spherical layer $\mathcal{R}(\tau,t)$ as $\mathcal{R}(\tau,t) := B(0,t)\setminus B(0,\tau) = \{y \in \mathbb{R}^n : \tau < |y| < t\}$. Let $\mathbb{N}$ denote the set of natural numbers with $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $\mathbb{Z}$ denote the set of all integers. For two non-nagative functions f and g, $f \leq g$ we mean $f \leq Cg$., where C does not depend on variables involved.

Variable exponent Hölder's inequality are stated as

$$\|fg\|_{r(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{q(\cdot)},$$

where we define r as $\frac{1}{r(i)} = \frac{1}{p(i)} + \frac{1}{q(i)}$ for every $i \in H$, and $p, q, r \in \mathcal{P}(\mathbb{R}^n)$.

Let us recall the well-known log-Hölder continuity condition (or Dini-Lipschitz condition) for $r : H \mapsto (0, \infty)$: there is a positive constant C such that for all $x, y \in H$ with $|x - y| \leq \frac{1}{2}$,

$$|r(x) - r(y)| \leq \frac{C}{-\ln|x-y|}. \tag{5}$$

Further, we say that $p(\cdot)$ satisfies the decay condition if there exists $r_\infty := r(\infty) = \lim_{|x|\to\infty} r(x)$, and there is a positive constant $C_\infty > 0$ such that

$$|r(x) - r_\infty| \leq \frac{C_\infty}{\ln(e + |x|)}. \tag{6}$$

We will need also the log Hölder continuity condition at 0 for $r(\cdot)$: there are constants $C_0 > 0$ such that for all $|x| \leq \frac{1}{2}$,

$$|r(x) - r(0)| \leq \frac{C_0}{\ln|x|}. \tag{7}$$

The best possible constant C in 5 (resp. C_∞ in 6) is called log-Hölder continuity or log-Dini-Lipschitz constant (resp. decay constant) for the exponent $r(\cdot)$.

Lemma 2.1. *[18] Let $D > 1$ and $r \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{r(0)}} \leq \|\chi_{\mathcal{R}_{s,D_s}}\|_{r(\cdot)} \leq t_0 s^{\frac{n}{r(0)}}, \text{ for } 0 < s \leq 1 \tag{8}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{r_\infty}} \leq \|\chi_{\mathcal{R}_{s,D_s}}\|_{r(\cdot)} \leq t_\infty s^{\frac{n}{r_\infty}}, \text{ for } s \geq 1, \tag{9}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$. These are constants that depend on a certain parameter denoted as D. Specifically, they might vary according to the value of D. It's noted that these constants t_0 and t_∞ do not depend on another parameter s.

Lemma 2.2. *(see [18]) Let $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$. The operator Γ is bounded on $L^{p(\cdot)}(\mathbb{R}_+; \frac{dt}{t})$ if*

$$\int_0^\infty |\Gamma(t)|^s \frac{dt}{t} < \infty \ \text{ when } s = 1 \ \& \ s = s_0, \tag{10}$$

for $\frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{p_+}$.

Lemma 2.3. *(see [18]) For every measurable function Ω, the following relations*

$$\int_{2b<|x|<j} |\Omega(x)|dx = \frac{1}{\ln 2} \int_b^t \frac{d\theta}{\theta} \int_{max(2b,\theta)<|x|<min(j,2\theta)} |\Omega(x)|dx, \ \ j > 2\alpha > 0, \tag{11}$$

and

$$\int_{|x|\geq 2j} |\Omega(x)|dx = \frac{1}{\ln 2} \int_t^\infty \frac{d\theta}{\theta} \int\limits_{max(\theta,2j)<|x|<2\theta} |\Omega(x)|dx, \ \ j > 0, \tag{12}$$

hold if the integrals on the left-hand side of above relations exists.

These Lemmas are already proved in [18].

Lemma 2.4. *Let $1 \leq p_- \leq p(x) \leq p_+ < \infty$ holds,*

$$\|g\|_{H^{p(\cdot),q,\alpha(\cdot)}_{\mu,\delta}} \approx \|g\|_{L^{p(\cdot)}(B(0,\gamma\mu+\theta))} + \|t^{\alpha_\infty}\|g\chi_{\mathcal{R}_{\gamma t,\delta t}}\|_{p(\cdot)}\|_{L^q(\mathbb{R}_{+,}\mu;\frac{dt}{t})}, \mu > 0, \tag{13}$$

and

$$\|g\|_{H^{p(\cdot),q,\alpha(\cdot)}_{0,\delta}} \approx \|t^{\alpha(0)}(1+t)^{\alpha_\infty-\alpha(0)}\|g\chi_{R_{\gamma t,\delta t}}\|_{p(\cdot)}\|_{L^{q(\cdot)}(\mathbb{R}_+;\frac{dt}{t})}, \tag{14}$$

then equivalences of norms given above are valids, if $\alpha \in M^{\log}_{\infty}(\mathbb{R}_{+,\mu})$ for (13) *and $\alpha \in M^{\log}_{0,\infty}(\mathbb{R}_+)$ in the case of* (14).

Lemma 2.5. *Let $4 \leq \mathbb{R} < \infty$ and $0 < \rho < 2$*

$$\|f\|_{L^{p(\cdot)}(B')} \leq C'\|j^{\alpha(j)}\|f\chi_{j,2j}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dj}{j})},$$

where $B' = B(0,R) \setminus B(0, 2+\theta)$ and $C' = C(\rho, R)$.

Let $\mathbb{S}^{n-1}$ represents the unit sphere in $\mathbb{R}^n$, equipped with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Phi(y')d\Phi(y') = 0, \tag{15}$$

where $y' = y/|y|$ and y is not zero. The Marcinkiewicz integral in the context of the Littlewood-Paley f-function on $\mathbb{R}^n$ is given as

$$M_\Phi(f)(x) = \left(\int_0^\infty |F_{\Phi,s}(f)(x)|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Phi,s}(f)(x) = \int_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} f(y)dy.$$

The Marcinkiewicz integral operator M_Φ is defined by,

$$M_\Phi(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}}.$$

By using extrapolation argument, cf. [27], we get

$$\|\mu_\Phi f\|_{L^{p(\cdot)}} \leq \|f\|_{L^{p(\cdot)}}.$$

3. Boundedness of the Marcinkiewicz integral operator

Lemma 3.1. *[15] Let $a > 0$, $s \in [1,\infty]$, $0 < d \leq s$ and $-n + (n-1)\frac{d}{s} < u < \infty$, then*

$$\left(\int_{|y|\leq a|x|} |y|^u |\Phi(x-y)|^d dy\right)^{1/d} \leq |x|^{(u+n)/d} \|\Phi\|_{L^s(\mathbb{S}^{n-1})}.$$

In the next theorem we prove the boundedness of the Marcinkiewcz integral operator on continual Herz-Morrey spaces $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)$, with all the parameter $p(\cdot)$, $q(\cdot)$ $\alpha(\cdot)$ are variables conditioned that the Marcinkiewcz integral operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 3.2. *Suppose that $p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, $q \in \mathcal{P}^{\log}_\infty(\mu,\infty)$ with $1 < p_- < p_+ < \infty$, $1 < q_- \le q_+ < \infty$ and $\alpha \in M_\infty(\mathbb{R}_{\mu+})$. Let $\mu > 0$, $0 < \gamma < \delta < \infty$ and Φ be homogeneous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q'^-$. If*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty} - \frac{n}{s}, \tag{16}$$

then every Marcinkiewicz integral operator i.e is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded from the continual Herz-Morrey spaces $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma',\delta')}(\mathbb{R}^n)$ to $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)$ for any $\delta < \delta' < \infty$ and $0 < \gamma' < \gamma$.

Proof. Without loss of generality, we choose $\mu = 2$, $\gamma = 1$, and $\delta = 2$. We want to show that

$$\left\|M_\Phi g\right\|_{L^{p(\cdot)}(B(0,2+\theta))} + \mathcal{N}^{p,q,\alpha}_{1,2}(M_\Phi g) \le \left\|g\right\|_{L^{p(\cdot)}(B(0,2+\theta))} + \mathcal{N}^{p,q,\alpha}_{\gamma',\delta'}(g),$$

with $\gamma' < 1$, $\delta' > 2$,

$$\mathcal{N}^{p,q,\alpha}_{\gamma,\delta}(g) := \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left\|t^{\alpha_\infty} \left\|g\chi_{\mathcal{R}_{\gamma t,\delta t}}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((2,k_0),dt/t)}.$$

The estimation for $\|M_\Phi g\|_{L^{p(\cdot)}(B(0,2+\theta))}$ can be calculated by using the boundedness of the Marcinkiewicz on Lebesgue spaces with variable exponents. So we have

$$\|M_\Phi g\|_{L^{p(\cdot)}(B(0,2+\theta))} \le \|g\|_{L^{p(\cdot)}(B(0,2+\theta))} \le \|g\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

For estimating the term $\mathcal{N}\left(M_\Phi g\right)^{p^*,q,\alpha}_{1,2}$:

Let $f_0(x) = g(x)\chi_{B(0,\frac{1}{2})}(x)$, $f_t(x) = g(x)\chi_{B(0,\gamma' t)\setminus B(0,\frac{1}{2})}(x)$, $g_t(x) = g(x)\chi_{B(\delta' t)\setminus B(0,\gamma' t)}$, $h_t(x) = g(x)\chi_{\mathbb{R}^n\setminus B(0,\delta' t)}$, then the function $g(x)$ will be splitted as

$$g(x) = f_0(x) + f_t(x) + g_t(x) + h_t(x).$$

Now we have pointwise inequality,

$$\left|M_\Phi g(x)\right| \le \left|M_\Phi f_0(x)\right| + \left|M_\Phi f_t(x)\right| + \left|M_\Phi g_t(x)\right| + \left|M_\Phi h_t(x)\right|.$$

Now we will find the estimate for $M_\Phi f_0$:

Let $y \in B(0,\frac{1}{2})$ with $x \in R_{t,2t}$ and $t > 2$ we get

$$\frac{t}{2} \le t - \frac{1}{2} < |x| - |y| \le |x-y|,$$

which implies

$$\begin{aligned}
&|M_\Phi(f_0)(x).\chi_{\mathcal{R}_{t,2t}}| \\
&\le C \int_{B(0,1/2)} \frac{|\Phi(x-y)|}{|x-y|^n}|g(y)|dy.\chi_{\mathcal{R}_{t,2t}} \\
&\le Ct^{-n} \int_{B(0,1/2)} \Phi(x-y)||g(y)|dy.\chi_{\mathcal{R}_{t,2t}} \\
&\le Ct^{-n}\|f_0\|_{L^{p(\cdot)}}\left(\|(\Phi(x-\cdot))\chi_{B(0,1/2)}\|_{L^{p'(\cdot)}}\right).\chi_{\mathcal{R}_{t,2t}}.
\end{aligned}$$

We define the relation $\frac{1}{p'(x)} = \frac{1}{P'(x)} + \frac{1}{s}$. By using Lemma 3.1 and generalized Hölder's inequality we have

$$\begin{aligned}\|(\Phi(x-\cdot))\chi_{B(0,1)}\|_{L^{p'(\cdot)}} &\leq \|(\Phi(x-\cdot))\chi_{B(0,1)}\|_{L^s(\mathbb{R}^n)}\|\chi_{B(0,1)}\|_{L^{P'(\cdot)}}\\ &\leq \left(\int\limits_{|y|<1} |\Phi(x-y)|^s dy\right)^{1/s}.\end{aligned}$$

Let $z = x - y$ and $|z| < |x| + |y| \leq 2t + 1 \leq 3t$. We have

$$\begin{aligned}\|(\Phi(x-\cdot))\chi_{B(0,1)}\|_{L^{p'(\cdot)}} &\leq \left(\int\limits_{|z|<3t} |\Phi(z)|^s dz\right)^{1/s}\\ &\leq t^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&\|M_\Phi f_0(x).\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\\ &\leq Ct^{-n}\|f_0\|_{L^{p(\cdot)}}\left(t^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\right)\\ &\leq Ct^{\frac{n}{s}-n}\|f_0\|_{L^{p(\cdot)}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\\ &\leq Ct^{\frac{n}{s}-n}\|\chi_{B_{2t}}\|_{L^{p(\cdot)}}\|f_0\|_{L^{p(\cdot)}}.\end{aligned}$$

$$\begin{aligned}N(M_\Phi f_0)_{p,q,\alpha} &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha_\infty-n+\frac{n}{s}}\|\chi_{B_{2t}}\|_{L^{p(\cdot)}}\|_{L^q((2,k_0);\frac{dt}{t})}\|f_0\|_{L^{p(\cdot)}}\\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha_\infty-\frac{n}{p'_\infty}+\frac{n}{s}}\|_{L^q((2,k_0);\frac{dt}{t})}\|f_0\|_{L^{p(\cdot)}}\\ &\leq C\|f_0\|_{L^{p(\cdot)}},\end{aligned}$$

where we used the fact that $\alpha_\infty - \frac{n}{p'_\infty} + \frac{n}{s} < 0$ and finiteness of L^q norm of power function we got last inequality.

Now we will find the estimate for $M_\Phi f_t$:

Let $x \in \mathcal{R}_{t,2t}$ and considering $(1-\gamma')t < |x-y|$ we get

$$|M_\Phi f_t(x)| \leq C \int\limits_{B(0,\gamma' t)\setminus B(0,1/2)} \frac{|\Omega(x-y)|}{|x-y|^n}|g(y)|dy.$$

By using Hölder's inequality we obtain,

$$\begin{aligned}|M_\Phi f_t(x)| &\leq \frac{C}{t^n}\int\limits_{1/2<|y|<t} |\Omega(x-y)||g(y)|dy\\ &\leq \frac{C}{t^n}\int_{1/2}^{t}\frac{d\rho}{\rho}\int\limits_{\frac{\rho}{2}<|y|<\rho}|\Omega(x-y)||g(y)|dy\\ &\leq \frac{C}{t^n}\int_{1/2}^{t}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\left(\|(\Phi(x-\cdot))\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p'(\cdot)}}\right).\end{aligned}$$

We define $q(\cdot)$ by the relation $\frac{1}{p'(x)} = \frac{1}{P'(x)} + \frac{1}{s}$. By using Lemma 3.1 and generalized Hölder's inequality we have

$$\begin{aligned}\|\Phi(x-\cdot)\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p'(\cdot)}} \leq& \|\Phi(x-\cdot)\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{s}(\mathbb{R}^n)}\|\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{P'(\cdot)}}\\ \leq& \left(\int\limits_{\frac{\rho}{2}|y|<\rho} |\Phi(x-y)|^s dy\right)^{1/s} \rho^{\frac{n}{p'_\infty}}.\end{aligned}$$

Let $z = x - y$ and $|z| < |x| + |y| \leq 2t + \rho \leq 3t$, we have

$$\begin{aligned}\|\Phi(x-\cdot)\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p'(\cdot)}} \leq& \left(\int\limits_{|z|<3t} |\Phi(z)|^s dz\right)^{1/s} \rho^{\frac{n}{p'_\infty}}\\ \leq& t^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}-\frac{n}{s}}.\end{aligned}$$

$$\begin{aligned}&\|M_\Phi f_t(x).\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\\ &\leq Ct^{-n}\int\limits_{1/2}^{t}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\left(\|\Phi(x-\cdot)\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{p'(\cdot)}\|\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\right)\\ &\leq Ct^{-n}\int\limits_{1/2}^{t}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\left(t^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}-\frac{n}{s}}\|\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\right)\\ &\leq Ct^{-n}\int\limits_{1/2}^{t}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\|\chi_{B_{2t}}\|_{L^{p(\cdot)}}t^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}-\frac{n}{s}}.\end{aligned}$$

$$\begin{aligned}&t^{\alpha_\infty}\|M_\Phi f_t(x).\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\\ &\leq Ct^{\alpha_\infty-n+\frac{n}{p_\infty}+\frac{n}{s}}\int\limits_{1/2}^{t}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\rho^{\frac{n}{p'_\infty}-1-\frac{n}{s}}d\rho\\ &\leq Ct^{\alpha_\infty-\frac{n}{p'_\infty}+\frac{n}{s}}\int\limits_{1/2}^{t}\|g\chi_{\mathcal{R}_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}\rho^{\frac{n}{p'_\infty}-1-\frac{n}{s}}d\rho\\ &\leq \int\limits_{1/2}^{t}\left(\frac{t}{\rho}\right)^{\alpha_\infty-\frac{n}{p'_\infty}+\frac{n}{s}}\omega(\rho)\frac{d\rho}{\rho},\end{aligned}$$

where $\omega(\rho) = \rho^{\alpha_\infty}\|g\chi_{R_{\frac{\rho}{2},\rho}}\|_{L^{p(\cdot)}}$. Define $\Gamma(t)$ as

$$\Gamma(t) = \begin{cases} t^{\alpha_\infty-\frac{n}{p'_\infty}+\frac{n}{s}}, & t > 1,\\ 0, & 0 < t < 1, \end{cases} \tag{17}$$

by defining, the operator $\Gamma\omega(t) = \int\limits_0^\infty K(\frac{t}{\rho})\omega(\rho)\frac{d\rho}{\rho}$, where $\omega(\rho) = \rho^{\alpha_\infty}\|\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}$. Left hand side of above equation is a Hardy type operator and by applying Lemma 2.2 we get

$$\mathcal{N}_{1,2}^{p,q,\alpha}(M_\Phi f_t) \leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|\Gamma\omega\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})} \leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|\omega\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})} \leq \|g\|_{H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

Now we will find the estimate for $M_\Phi g_t$:

As the operator M_Φ is bounded on the space $L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$ so we obtain

$$\|(M_\Phi g_t)\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}} \leq C\|g_t\|_{L^{p(\cdot)}} = C\|g\chi_{R_{\gamma' t,\delta' t}}\|_{L^{p(\cdot)}}$$

which implies

$$\mathcal{N}_{1,2}^{p,q,\alpha}(M_\Phi g_t) \leq C\|g\chi_{R_{\gamma' t,\delta' t}}\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

Now we will find the estimate for $M_\Phi h_t(x)$:

By using the fact that $|x-y| \geq \rho/2$ since $x \in \mathcal{R}_{t,2t}$, and Hölder's inequality, we get

$$|M_\Phi h_t(x)| \leq C \int_{|y|>8t} \frac{|\Omega(x-y)|}{|x-y|^n}|g(y)|dy.$$

$$\begin{aligned}
&|M_\Phi h_t(x)|\\
&\leq \frac{C}{t^n}\int_{|y|>8t}|\Omega(x-y)||g(y)|dy\\
&\leq \int_{4t}^{\infty}\frac{d\rho}{\rho}\int_{\rho<|y|<2\rho}\frac{|\Omega(x-y)|}{|x-y|^n}|g(y)|dy\\
&\leq \int_{4t}^{\infty}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\rho^{-n}\Big(\|(\Phi(x-\cdot))\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p'(\cdot)}}\Big).
\end{aligned}$$

We define $q(\cdot)$ by the relation $\frac{1}{p'(x)} = \frac{1}{P'(x)} + \frac{1}{s}$. By using Lemma (3.1) and generalized Hölder's inequality we have

$$\begin{aligned}
\|\Phi(x-\cdot)\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p'(\cdot)}} &\leq \|\Phi(x-\cdot)\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^s(\mathbb{R}^n)}\|\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{P'(\cdot)}}\\
&\leq \left(\int_{\rho|y|<2\rho}|\Phi(x-y)|^s dy\right)^{1/s}\rho^{\frac{n}{p'_\infty}}.
\end{aligned}$$

Let $z = x-y$ and $|z| < |x| + |y| \leq 2t + \rho \leq \frac{\rho}{2} + \rho \leq \frac{3\rho}{2}$. We have

$$\begin{aligned}
\|\Phi(x-\cdot)\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p'(\cdot)}} &\leq \left(\int_{|z|<\frac{3\rho}{2}}|\Phi(z)|^s dz\right)^{1/s}\rho^{\frac{n}{p'_\infty}}\\
&\leq \rho^{\frac{n}{s}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}-\frac{n}{s}}\\
&\leq \|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}}.
\end{aligned}$$

$$
\begin{aligned}
&\|M_\Phi h_t(x).\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}} \\
&\leq C\int_{4t}^{\infty}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\rho^{-n}\Big(\|(\Phi(x-\cdot))\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p'(\cdot)}}\|\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\Big) \\
&\leq C\int_{4t}^{\infty}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\rho^{-n}\Big(\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}-\frac{n}{s}}\|\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}}\Big) \\
&\leq C\int_{4t}^{\infty}\frac{d\rho}{\rho}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\rho^{-n}\|\chi_{B_{2t}}\|_{L^{p(\cdot)}}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\rho^{\frac{n}{p'_\infty}} \\
&\leq C\int_{4t}^{\infty}\rho^{-n+\frac{n}{p'_\infty}}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\|\chi_{B_{2t}}\|_{L^{p(\cdot)}}\frac{d\rho}{\rho}.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&t^{\alpha_\infty}\|M_\Phi h_t(x).\chi_{\mathcal{R}_{t,2t}}\|_{L^{p(\cdot)}} \\
&\leq Ct^{\alpha_\infty+\frac{n}{p_\infty}}\int_{4t}^{\infty}\rho^{-n+\frac{n}{p'_\infty}}\|g\chi_{\mathcal{R}_{\rho,2\rho}}\|_{L^{p(\cdot)}}\frac{d\rho}{\rho} \\
&\leq \int_{t}^{\infty}\left(\frac{t}{\rho}\right)^{\alpha_\infty+\frac{n}{p_\infty}}\omega(\rho)\frac{d\rho}{\rho},
\end{aligned}
$$

where $\omega(\rho)=\rho^{\alpha_\infty}\|g\chi_{R_{\rho,2\rho}}\|_{L^{p(\cdot)}}\chi_{(2,\infty)}(\rho)$. We reaches at Hardy type inequality, now the inequality $\alpha_\infty+n/p_\infty>0$ and Lemma 2.2 implies

$$
\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\|t^{\alpha_\infty}\|\chi_{R_{t,2t}}M_\Phi h_t(x)\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})}\leq C\|g\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.
$$

□

4. Conclusion

In this paper, we introduced the idea of continual Herz-Morrey spaces with variable exponents and obtained the boundedness of Marcinkiewicz integral operator on these spaces.

5. Data Availability

No data were used to support this study.

6. Conflicts of Interest

The authors declare no conflict of interest.

7. Funding

This work did not receive any external funding.

8. Authors Contribution

Contributions from all authors were equal and significant. The original manuscript was read and approved by all authors.

References

[1] A. Almeida, D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394**(2) (2012), 965–974.

[2] A. Baernstein II and E. T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*, Amer. Math. Soc. **318** (1985).

[3] Y. Ding, S. Lu, Q. Xue,*Marcinkiewicz integral on Hardy spaces*, Integral Equ. Oper. Theory. **42** (2002), 174–182.

[4] Y. Ding, S. Lu, P. Zhang, Weighted weak type estimates for commutators of the Marcinkiewicz integrals, Sci China Ser A, 47 (2004), 83–95.

[5] Y. Ding, Q. Xue, K.Yabuta, *A remark to the L^2 boundedness of parametric Marcinkiewicz integral*, J. Math. Anal. Appl. **387** (2012), 691–697.

[6] H. G. Feichtinger and F. Weisz, *Herz spaces and summability of Fourier transforms*, Math. Nachr. 281 (2008), 309–324.

[7] C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. **18** (1968), 283–324.

[8] G. Hu, S. Lu, D. Yang, *$L^p(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness for the Marcinkiewicz integral on product spaces*, Sci. China Ser. A. **46** (2003), 75–82.

[9] G. Hu, Y. Meng, D. Yang, *Estimates for Marcinkiewicz integrals in BMO and Campanato spaces*, Glasgow Math. J. **49** (2007), 167–187.

[10] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **13**(36) (2010), 33–50.

[11] M. Izuki, *Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents*, Math. Sci. Res. J. **13**(10) (2009), 243–253.

[12] S. Lu, *Marcinkiewicz integral with rough kernels*, Front. Math. China. **3** (2008), 1–14.

[13] J. Marcinkiewicz, *Sur quelques inégrales du type de Dini*, Ann. So. Polon. Math. **17** (1938), 42–50.

[14] A. Meskhi, H. Rafeiro, M. A. Zaighum, *On the boundedness of Marcinkiewicz integrals on continual variable exponent Herz spaces*, Georgian Math. J. **26**(1) (2019), 105–116.

[15] B. Muckenhoupt, R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*, Trans. Am. Maths Soc. **161** (1971), 249–258.

[16] H. Rafeiro and S. Samko, *Riesz potential operator in continual variable exponents Herz spaces*, Math. Nachr., **288**(4) (2015), 465–475.

[17] M. A. Ragusa, *Homogeneous Herz spaces and regularity results*, Nonlinear Anal. **71** (2009), 1–6.

[18] S. Samko, *Variable exponent Herz spaces*, Mediterr. J. Math. **10**(4) (2013), 2007–2025.

[19] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[20] B. Sultan and M. Sultan, *Boundedness of commutators of rough Hardy operators on grand variable Herz spaces*, Forum Math. **36**(3) (2024), 717–733.

[21] B. Sultan, M. Sultan, and I. Khan, *On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces*, Commun. Nonlinear Sci. Numer. Simul. **126** (2023). https://doi.org/10.1016/j.cnsns.2023.107464

[22] B. Sultan, M. Sultan, *Boundedness of higher order commutators of Hardy operators on grand Herz-Morrey spaces*, Bull. Sci. Math. **190** (2024).

[23] B. Sultan, M. Sultan, A. Khan, T. Abdeljawad, *Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces*, J. Inequal. Appl. **2024** (2024), 93.

[24] B. Sultan, M. Sultan, F. Gurbuz, *BMO estimate for the higher order commutators of Marcinkiewicz integral operator on grand variable Herz-Morrey spaces*, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat. **72** (4) (2023), 1–19.

[25] M. Sultan, B. Sultan, A. Khan, T. Abdeljawad, *Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces*, AIMS Math. **8**(9) (2023), 22338–22353.

[26] A. Torchinsky, S. Wang, *A note on the Marcinkiewicz integral*. Colloq Math **60-61**(1) (1990), 235–243.

[27] H. Wang, *Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent*, Czech Math J. **66** (2016), 251–269.

[28] H. Wu, *On Marcinkiewicz integral operators with rough kernels*, Integral Equ. Oper. Theory. **25**(2005), 285–298.

[29] A. Zygmund, *On certain integrals*. Trans. Amer. Math. Soc. **55** (1944), 170–204.